# Wick rotation with the condition of retaining the validity of the Lorentz group.

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**Abstract:** This document reports some of the important results of a theoretical work that performed the Wick rotation with the condition of retaining the validity of the Lorentz group.

**Keywords:** Wick rotation, Lorentz group, Special Relativity, photons, elementary particles, four-vectors, Higgs boson, space-time.

## (1) INTRODUCTION.

In order to develop a clear communication of this theoretical work, this document consists of three sections. This section 1 gives a brief introduction of the two concepts involved, also presents the formalism of the main hypotheses proposed and demonstrates the conservation of validity of the Lorentz group. The section 2 reported some of the consequences of adopting the model of particles implicit in the proposed hypothesis and finally, section 3 shows some final conclusions of this work. It is also important to note that this analysis is developed in the theory of special relativity (Lorentz group), pending its development for the transformations of the general theory of relativity.

### (1.1) Concepts.

(1.1.1) Wick rotation.

(A) It is a method of finding a solution of a mathematical problem in Minkowski space time from a solution to a related problem in Euclidean space by means of a transformation that substitutes an imaginary-number variable for a real-number variable. In this paper is studied the replacement of t by it. This is the time variable.

(B) In quantum field theory, is a rotation of the real time axis to the imaginary time axis in the complex-time plane [REF. 1].

(1.1.2) Lorentz group.

Is the set of all Lorentz transformations of Minkowski space-time.

### (1.2) formalism of the hypothesis.

Consider the following expression:

$$ds^2 = -c^2dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \qquad \textbf{(1)}$$

Where c is the speed of light in the empty space, $dx_1$, $dx_2$ and $dx_3$ are the differential coordinates of the axes 1, 2 and 3 of $M^3$.

This is the measurement of distances in Minkowski space-time [REF. 2], in tensor form is:

$$\eta_{\upsilon\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If applying the Wick rotation ($t_w$= it), with the special condition that the Lorentz group remains valid, then results in the following metric structure:

$$\eta_{UV} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R^2 \end{pmatrix}$$

Where is should consider the following points:

(A) The topology of space-time turns out to be $M^4xS^1$, this is a Cartesian product between a Euclidean space-time and a circle of radius R. That is, there is a cylindrical space-time.

(B) The radius R of the circle should depend on the rest mass of elementary particle is observed. That is, for each type of elementary particle, according to their rest mass, there must be a specific circle and therefore an own cylindrical surface, but maintaining in common the space-time $M^4$.

(C ) The algebraic form of this new metric is:

$$ds^2 = c^2dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \qquad \textbf{(2)}$$

Where $dx_4$=Rd$\phi$ and d$\phi$ is the infinitesimal angular displacement around the cylinder.
And where $-2\pi \leq \phi \leq 2\pi$.

(D) Any particle with rest mass $m_0$ must be treated as a photon that moves through the cylindrical surface defined by $M^4$ x $S^1$, where the radius of $S^1$ is a function of $m_0$. This is that the length of the trajectory of any elementary particle, in $M^3$x $S^1$ ($x_1$, $x_2$, $x_3$, R$\phi$) must be defined by:

$$\int (dx_1^2 + dx_2^2 + dx_3^2 + R^2\, d\phi^2)^{1/2} = c\int dt \qquad \textbf{(3)}$$

Where -Infinite < $\phi$ > + Infinite

And therefore in $M^4xS^1$ by:

$$(2)^{1/2}\int c\, dt = \int dL \qquad \textbf{(4)}$$

Note that for a photon of the three-dimensional space $M^3$ (the ordinary photon) it follow that d$\phi$ = 0, this is that they are fully immersed in $M^3$.

**(1.3) Conservation of validity of the Lorentz group in the Wick rotation, by means of this model.**

(1.3.1) Displacement

To proceed to check the validity of the Lorentz transformation for the displacement of a particle, in this model, consider the following points:

(1) The process of measuring the position of events in space, by the theory of relativity, does not consider the fourth spatial coordinate $dx_4 = R\, d\phi$. So it only sees events in $M^3$. This agrees with that $S^1$ is imperceptible.

(2) To measure the spatial displacement of one particle, should be register at least two sequential positions of the particle.

(3) From the expression (3) is can reach the velocity vector of four components:

$$\mathbf{C} = (v_1, v_2, v_3, v_4)$$

Where v = dx/dt.

From this it can deduce that the projection of this speed in $M^3$, is given by the first three components of the previous expression and whose magnitude is:

$$v_i = c \cos \beta$$

Where i = 1, 2, 3, and β is the angle between **C** and $\mathbf{V_i}$. The latter being the velocity vector in $M^3$, or the perceived speed of the particle in the three-dimensional space.

(4) Different speeds $v_i$ of an elementary particle, referring to an inertial reference frame O, is equivalent to the different velocities measured in the same particle in different frames of reference with speeds v respect to O.

Then, from points 1 and 2, it follows that the magnitude (with four components) of the projection in $M^3xS^1$ of the displacement **D** of a particle, is:

$$D^2 = \Delta x_1^2+\Delta x_2^2+\Delta x_3^2+ (2\pi R)^2$$

Where the fourth component is invariant through the measured displacements. Now, when using the displacement measurement to also calculate the velocity $v_i$ of the particle (see point 3), then it have the following special case for the expression of the magnitude of **D**:

$$D^2 = c^2\Delta t^2 = \Delta x_1^2+\Delta x_2^2+\Delta x_3^2+ (n2\pi R)^2 \qquad \mathbf{(5)}$$

Where n is an integer number that assigned discrete values to D and whose transcendence is explained in the section on the uncertainty principle. Now, if in the expression (5) is made an interchange in the positions of $(n2\pi R)^2$ and $D^2$ and if is take into consideration point 4, then it leads to the same Lorentz transformation for the observed displacement of a particle. It is also important to note that n2πR turns out to be the proper time of the particle.

(1.3.2) Matrix of transformation for the coordinates of reference frames with inertial motion in $M^3$.

Is must use the following assumptions:

(1) The inertial motion at speed v, of the reference frames, are limited to space $M^3$. In the special case to consider is limited to $x_1$.

(2) The event whose location in space-time is measured should be located in $M^3$ (perceived well by the compact nature of the fourth spatial dimension). This leads to exclude the coordinate $x_4 = R\phi$ for the location of the event.

(3) The elements involved in the measurement process can be modelled by representative elementary particles. But, by the point (1.2-d), these particles must be represented by equivalent photons in $M^4$ x $S^1$ whose path are defined by the expression (3) and (4).

(4) The origins of the reference frames coincide at the beginning.

(5) The physical agent used for the synchronization is light.

(6) According to the conclusion of Section (1.3.1), the proper time of the elementary particle is $n2\pi R$.

Considering the above, it can easily work with this model to arrive at the following matrix of coordinates transformation for the event location:

$$\begin{pmatrix} 1/\mathrm{sen}\beta & -\cos\beta/\mathrm{sen}\beta & 0 & 0 \\ -\cos\beta/\mathrm{sen}\beta & 1/\mathrm{sen}\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Where $\cos\beta = v/c$ and therefore sen $\beta = (1-v^2/c^2)^{1/2}$.

The angle $\beta$ is the formed between $x_1$ and straight path (this is possible when viewing the cylindrical surface $M^4xS^1$ as a plane) of the equivalent photons of the representatives elementary particles of the elements involved in the measurement process for the location of the event. Note that is the same Lorentz transformation for space-time coordinates.

(1.3.3) Momentum and energy.

Corresponding to the expression (3), the following vector of four components should be the projection in $M^3$ x $S^1$, of the magnitude of the momentum of the equivalent photon of an elementary particle with rest mass $m_0$:

$$E^2/c^2 = (m_0v_1\gamma)^2+(m_0v_2\gamma)^2+(m_0v_3\gamma)^2+(m_0c)^2 \tag{7}$$

Where $\gamma = 1/(1-v^2/c^2)^{1/2}= 1/\mathrm{sen}\beta$ , $E=m_0 c^2\gamma$ and the component in the fourth spatial axis compact is $m_0c$.

Considering that this reference frames do not moving along $S^1$ (perceived well by the compact nature of the fourth spatial dimension) then $m_0c$ is an invariant in the spatial inertial motion of reference frames in $M^3$, therefore the exchange of the positions of $(m_0c)^2$ and $E^2c^2$ in expression (7) implies the matrix (6), which is the same Lorentz transformation for the momentum and energy.

(1.3.4) Current density.

Corresponding with the expression (3) and (7), the following vector of four components should be the projection in $M^3xS^1$, of the magnitude of the current density for the equivalent photon of an elementary particle with rest mass $m_0$:

$$c^2\rho^2 = j_1^2 + j_2^2 + j_3^2 + c^2\rho_0^2 \qquad \textbf{(8)}$$

Because:

$$m_0c/p = c\rho_0/j$$

Where $p^2 = (m_0v_1\gamma)^2+(m_0v_2\gamma)^2+(m_0v_3\gamma)^2$ y $j^2 = j_1^2 + j_2^2 + j_3^2$.

Then $c\rho_0$ must be an invariant through the spatial inertial motion of the reference frames in $M^3$, therefore of the exchange the positions of $c^2\rho_0^2$ and $c^2\rho^2$ in expression (8) implies the matrix (6), which is the same Lorentz transformation for the current density.

## (2) DEVELOPMENT OF SOME RESULTS OF THIS MODEL FOR ELEMENTARY PARTICLES.

### (2.1) Calculation of R.

In order to define the value of R, is can display the paths of the equivalents photons on the cylindrical space $M^3xS^1$ (for simplicity is omit the time dimension), as if they were on an equivalent extended plane. Assuming this, then the periodicity on $S^1$ of the path of equivalent photon, is replaced by a continuous emission of equivalents photons of a single elementary particle. On this basis, one can easily calculate the wavelength (distance between two plane wave fronts) of these photons, resulting in:

$$\lambda = h\,(1-v^2/c^2)^{1/2}/m_0c \qquad \textbf{(9)}$$

Where h is Planck's constant.

This assumes[1] that $\lambda$= h/p, where p is the momentum of the photon. The perimeter of $S^1$ must match the wavelength of the equivalent photon of the elementary particle observed when at rest, so if is assuming that v = 0 then:

$$R = \lambda_{v=0}/2\pi$$

Or:

$$R = (1/2\pi)\,(h/m_0c) \qquad \textbf{(10)}$$

[1] Although not shown here, this relation and Planck's constant it can be derived rigorously within this proposed model.

**(2.2) wavelength de Broglie explained by this model.**

When is applying a simple geometric analysis on the extended plane scheme that are equivalent to $M^3xS^1$ and assuming an angle β between the direction of the equivalent photons and $M^3$ (this is that the elementary particle for which the photons are equivalent, is in motion), then can verify the formation of a rectangle triangle that is compound by three sides: the wavelength λ of the equivalents photons, a plane wave front of equivalents photons and making an angle β with the first, the wavelength of De Broglie $\lambda_{DB}$. The latter turns out to be the length between the planes wave fronts of the equivalents photons, but measured in $M^3$. It can be verified that from the analysis of this triangle is obtained:

$$\lambda_{DB} = \lambda / \cos\beta$$

And therefore

$$\lambda_{DB} = h\,(1-v^2/c^2)^{1/2} / (m_0 v)$$

Where v is the velocity of the particle.

Note that this expression is the same for the length of de Broglie wave.

**(2.3) Particle-antiparticle annihilation explained by this proposed model.**

Consider the following relations:

$$-m_0c/(E/c) = (-c\Delta\tau = -\Delta x_4) / c\Delta t = -c\rho_0 / c\rho = -\mathrm{sen}\beta$$

They suggest that the sign change in the load, involve changing the direction to the component of the fourth axis of the four-vector for the momentum (7) and also involve a change in direction for the displacement of the equivalent photon along $S^1$. Considering this and if there is to check what happens when an electron ($e^-$) and an positron ($e^+$) collide, these being of opposite electrical charges, and assume the motion of equivalent photons only in the $x_4=R\phi$ and $x_1$ axis and opposite (in the $x_4$-axis because are opposite charges) then is can obtain the following system of equations of momentum and energy before and after the collision:

$x_1$ axis: $$P_{e^-i1} - P_{e^+i1} = P_{e^-f1} - P_{e^+f1} = 0$$

$x_4$ axis: $$P_{e^-i4} - P_{e^+i4} = P_{e^-f4} - P_{e^+f4} = 0$$

Energy: $$\left((P_{e^-i1})^2+(P_{e^-i4})^2\right)^{1/2}c + \left((P_{e^+i1})^2+(P_{e^+i4})^2\right)^{1/2}c = \left((P_{e^-f1})^2+(P_{e^-f4})^2\right)^{1/2}c + \left((P_{e^+f1})^2+(P_{e^+f4})^2\right)^{1/2}c$$

Where $P_{e^-i1}$, $P_{e^+\mathbf{i1}}$, $P_{e^-f1}$ and $P_{e^+\mathbf{f1}}$ are the components in the $x_1$ axis of initial and final momentum of the equivalent photons of the positron and electron respectively; $P_{e^-i4}$, $P_{e^+\mathbf{i4}}$, $P_{e^-f4}$ and $P_{e^+\mathbf{f4}}$ are the initial and final momentum four-vector in the $x_4$ axis of the equivalent photons of the electron and positron respectively

It is noted that the system allows the following solution:

$$P_{e^-f4} = P_{e^+f4} = 0$$

This implies the following reaction:

$$e^- + e^+ \rightarrow \gamma + \gamma$$

Where $\gamma$ symbolizes the ordinary photon of $M^3$.

Which expressing that the result of the collision is two photons that is contained in ordinary three-dimensional space and keeping all the initial energy, which is experimentally found by the official physical.

**(2.4) Intrinsic magnetic moment of electron explained by this proposed model.**

Something interesting about the structure of this particle model is that the equivalent photon path of a particle with rest mass, through $S^1$, implies a fixed and given angular momentum that does not disappear even if this particle is resting in $M^3$. In the case of the electron, is can get the value of the magnetic moment when considering the following points:

(A) The magnetic moment $\mu$ for a current **I** around the area **A**, is given by:

$$\boldsymbol{\mu} = I\,\mathbf{A}$$

(B) The identity that is generally used for the total charge density $\rho$ is:

$$\rho = n\,e$$

Where n is the number of electrons per unit volume and e is the electric charge for the electron.

(C ) When the electron is at rest, the expression (8) takes the following form:

$$c\rho = c\rho_0$$

This is the current density for the equivalent photon, when the electron is at rest.

(D) The radius of rotation of the equivalent photon is the same radius R, whose value is given by the expression (10).

Then to mix these four expressions is obtained the following intrinsic magnetic moment for the electron:

$$\boldsymbol{\mu} = \frac{e\,h}{4\pi\, m_0}$$

This is the same expression for the magnetic moment for the intrinsic spin of the electron, predicted by relativistic quantum physics. Note that the sign of $\mu$ depends on the sign of the charge, this is that depends on whether the equivalent photon travels in a positive or negative in the $x_4$-axis. It also depends on whether the equivalent photon travels over the positive $x_4$-axis or over the negative $x_4$-axis. It was analyzed the case in which the electron is moving and is concluded that the magnetic moment remains constant.

**(2.5) Graphical interpretation of the proposed cylindrical universe, time dilation and the uncertainty principle explained by this model.**

(2.5.1) Graphical interpretation of the proposed cylindrical universe.

The following Figure (1) illustrates a segment of the space-time path of the equivalent photon of one elementary particle:

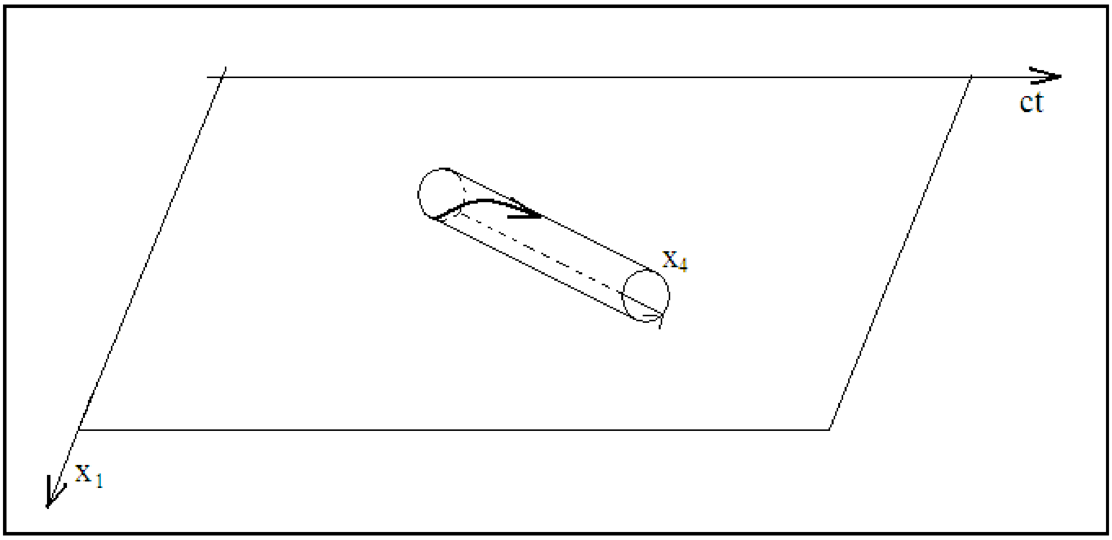


**FIGURE 1**

Comments:

(1) For this illustration assumes that the spatial displacement of the elementary particle is only in the $x_1$ axis. However, its equivalent photon must also have a path in $S^1$, that is to say the $x_4$ axis compact.

(2) Note that the connection of the cylindrical dimension $S^1$ with $M^4$ is a line (in the figure, the dotted line that goes from one extreme to the other cylinder).

(3) The path of the equivalent photon (drawn as an arrow) is performed on the cylindrical surface.

(2.5.2) Dilation of the proper time.

The fact that $n(2\pi R)$ is the proper time of an elementary particle (this consideration is was discussed in Section 1.3.1) corresponds to the fact that the frequency of crossings, of the equivalent photon of this particle, composes his own temporal flow. It follows that the dilatation of the proper time of a particle moving at velocity $v_i$, is due to the fact that the frequency of crossings decreases, because the angle $\beta$ takes values different from zero. In Figure (1), the crossing with $M^4$ will be when the world line coincides with the dashed line, where $S^1$ connects with $M^4$. In developing the relation between ct and the length of the path along $Rd\phi$, in expression (3), is obtained:

$$1/(1-v^2/c^2)^{1/2}$$

That is the same dilation factor of special relativity.

(2.5.3) Uncertainty Principle explained by this model.

Consider the following points:

(A) Because all elementary particle that representing any clock at rest, is associated with a wave period $T_0$ derived from the expression (9), it have:

$$T_0 = h / (m_0 c^2)$$

Is the sensitivity or the minimum uncertainty in the measurement of time, because in that period of time the clock matter is not making registers of events in ordinary three-dimensional space, because its equivalent photons are not having contact with $M^4$. In the case shown in Figure (1) is illustrates this latter, but for the equivalent photon of a particle moving in $x_1$.

(B) According to quantum physics, all energy measurement for a system can be reduced finally to the measurement of the frequency of waves associated with the quantum system [REF. 3].

In considering the above and if proceed to calculate the energy that is measured, from a reference frame tied to a clock at rest, for a wave that intersects the location of this clock, then is have the following results:

(1) The time (t) that is taken to measure the frequency of the wave, is given by $t = nT_0$ . Where n is a positive integer which is referred by section (1.3.1)

(2) The frequency (f wave) of the wave under study is:

$$f_{wave} = (s/n)\, m_0\, c^2 / h$$

Where $m_0$ is the rest mass of representative elementary particle of the clock and s refers to the number of cycles or waves recorded during the time t.

(3) It follows that the wave energy is given by:

$$E_{wave} = (s/n)\, E_0$$

This can be rewritten as:

$$E_{wave} = E_0/n + E_0/n + E_0/n + ... + E_0/n$$

That is, the sum of s times $E_0/n$.

(4) The previous series means that $E_0/n$ is the sensitivity or uncertainty in the measurement of energy of any wave, over a time $t = n\, T_0$.

(5) If this uncertainty in the measurement of energy is called $\Delta Em = E_0/n$ and identities is replaced conveniently, then it follows that:

$$\Delta Em\ t = h$$

Is the minimum uncertainty in the measurement of the energy of a wave and therefore any energy uncertainty ($\Delta E$) is :

$$\Delta E\ t > h$$

This is the same uncertainty principle for the measurement of the energy used by quantum physics. Note that if $T_0 = 0$, that is to say if the time registration was not discreet, this is continuous, then there would not be a minimum uncertainty in the measurement of the energy of any system.

## (3) CONCLUSIONS

(1) The cylindrical contour of the space-time $M^4xS^1$, of this proposed model, corresponds to the Higgs field in the interpretation of the physical reality of the standard model of elementary particles; however this proposed model predicts the total inexistence of the Higgs boson, because the rest mass of the particles is the product of displacement of the equivalent photons, through $S^1$ (note that ordinary photons do not take route in $S^1$ and therefore have no rest mass) and not the product of the interaction with bosons that provide mass. Also the relativistic mass increase is due to diagonal movement, on the aforesaid cylindrically contour, of the equivalent photons.

(2) This proposed model predicts that the movement within the ordinary three-dimensional space of an elementary particle with rest mass, is given by "leaps" (as a result of the equivalent photon travel in $S^1$), so that the length between these leaps is given by:

$$d = \frac{h\ v}{m_0\ c^2\ (1-v^2/c^2)^{1/2}}$$

This is that for one electron each of these spatial intervals is 1(nm) when is hold a speed corresponding to 99.99970611% of the light and which is related to the injection of a kinetic energy of 3.374x10 -11 joules. At this time there is enough technology to access to an experimental realization of this scale, however, must be fine-tuned the means used to show the traces of the electron with the intention that the diameter of these traces are not precisely equal or exceeds the length of the respective area of the period of absence of such a particle, which is exactly what it seek verification.

(3) The five-dimensional theory of Kaluza-Klein was abandoned mainly because it assigns values of rest mass $m_0$ extremely high to charged particles as the electron [REF. 4]. This is because, in this theory, these masses are inversely proportional to the value of the radius of the compact dimension. In this regard, it is interesting to note that in this proposed model of equivalent photon, the radius R of the compact dimension, given by the expression (10), also maintains an inverse relationship with $m_0$, however allows mass values in the order of those observed in reality, due to the factor h (Planck's constant), which is very small.

(4) Although this document does not show, can be reached to same model of equivalent photon from a different analysis done on special relativity and the hypothesis of de Broglie.

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