Noncommutative Complex Scalar Field and Casimir Effect

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Abstract

Using noncommutative deformed canonical commutation relations, a model describing a noncommutative complex scalar field theory is proposed. The noncommutative field equations are solved, and the vacuum energy is calculated to the second order in the parameter of noncommutativity. As an application to this model, the Casimir effect, due to the zero point fluctuations of the noncommutative complex scalar field, is considered. It turns out that in spite of its smallness, the noncommutativity gives rise to a repulsive force at the microscopic level, leading to a modified Casimr potential with a minimum at the point $a_{\min} = \sqrt{\frac{5}{84}}\pi\theta$.

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1. INTRODUCTION

In the last few years, many efforts have been made to understand the nature of space-time at the microscopic level by using new ideas like quantum groups, deformation theory, noncommutative geometry etc... This may shed a light on the real microscopic geometry and structure of our universe. One approach, is the study of noncommutative space-time and its implications to quantum field theories and other area of theoretical physics, the motivation for this kind of investigation is that the effects of noncommutativity of space may appear at very short distances of the order of the Planck length, or at very high energies [1]-[10]. The discovery of noncommutative geometry has allowed the exploration of new directions in theoretical physics, in particular, two-dimensional noncommutative harmonic oscillators are an extremely active area of research [26] [11]-[20].

Our paper is organized as follows: In Section 2, we consider a noncommutative action for a complex scalar field with self interaction, in section 3, we derive and solve the free noncommutative field equations, in section 4, we consider the noncommutative Casimir effect. Finally, in section 5, we draw our conclusions.

2. NONCOMMUTATIVE ACTION

From quantum field theory we know that charged particles with spin zero are described by a complex scalar field $\Phi(x)$, we also know that the equation of motion can be derived from variation of an action $S = \int d^4x \mathfrak{L}$, where $\mathfrak{L}$ is a Lagrangian density given by [21]-[25]

$$\mathfrak{L} = -\left(\partial_\mu \Phi\right)^* \left(\partial^\mu \Phi\right) - m^2 \Phi^* \Phi - g\left(\Phi^* \Phi\right)^2 \tag{1}$$

here m is the mass of the charged particles, and g is a positive parameter. The metric signature will be assumed to be $- + +...$, in what follows, we take $\hbar = c = 1$.

The complex scalar field can be quantized using the canonical quantization rules, for this we express it in terms of its real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right)$, where φ_1, φ_2 are real scalar fields; in terms of these real scalar fields the Lagrangian density reads

$$\mathfrak{L} = -\frac{1}{2}\left(\partial_\mu \varphi_a\right)^2 - \frac{1}{2}m^2\left(\varphi_a\right)^2 - \frac{1}{4}g\left(\varphi_a\varphi_a\right)^2 = -\frac{1}{2}\left(\partial_\mu \varphi_a\right)^2 - \frac{1}{2}\mu^2\left[\varphi\right]\left(\varphi_a\right)^2 \tag{2}$$

where $\mu^2\left[\varphi\right] = m^2 + \frac{1}{2}g\left(\varphi_a\right)^2$.

Let π_a be the canonical conjugate to φ_a

$$\pi_a = \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}_a} = \dot{\varphi}_a \tag{3}$$

The Hamiltonian density reads then

$$\mathcal{H} = \pi_a \dot{\varphi}_a - \mathfrak{L} = \frac{1}{2}(\pi_a)^2 + \frac{1}{2}\left(\overrightarrow{\nabla}\varphi_a\right)^2 + \frac{1}{2}\mu^2[\varphi](\varphi_a)^2 \tag{4}$$

To quantize the system, we split the Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$ into a free and interaction terms [21]

$$\mathcal{H}_0 = \frac{1}{2}(\pi_a)^2 + \frac{1}{2}\left(\overrightarrow{\nabla}\varphi_a\right)^2 + \frac{1}{2}m^2(\varphi_a)^2 \tag{5}$$

$$\mathcal{H}_{int} = \frac{1}{4}g(\varphi_a\varphi_a)^2 \tag{6}$$

then we pass to the interaction picture. In the interaction picture the equation of motion are given by

$$\dot{\varphi}_a(x) = \frac{\delta H_0}{\delta \pi_a(x)} \quad , \quad \dot{\pi}_a(x) = -\frac{\delta H_0}{\delta \varphi_a(x)} \tag{7}$$

where $H_0 = \int d^3\overrightarrow{x}\mathcal{H}_0$ is the free Hamiltonian.

In the canonical quantization the canonical variables φ_a and the canonical conjugates π_a are assumed to be operators satisfying the canonical commutation relations

$$\begin{aligned}
&[\varphi_a(t,\overrightarrow{x}), \pi_b(t,\overrightarrow{y})] = i\delta_{ab}\delta^3(\overrightarrow{x}-\overrightarrow{y}) \\
&[\varphi_a(t,\overrightarrow{x}), \varphi_b(t,\overrightarrow{y})] = 0 \\
&[\pi_a(t,\overrightarrow{x}), \pi_b(t,\overrightarrow{y})] = 0
\end{aligned} \tag{8}$$

It is well known, since the birth of quantum field theory in the papers of Born, Dirac, Fermi, Heisenberg, Jordan, and Pauli, that the free field behaves like an infinite number of coupled harmonic oscillators [21], using this analogy between free fields and an infinite number of coupled harmonic oscillators, one can impose non commutativity on the configuration space of dynamical fields φ_a, to do this we recall that the two-dimensional harmonic oscillator noncommutative configuration space can be realized as a space where the coordinates $\widehat{x}_a$, and the corresponding noncommutative momentum $\widehat{p}_a$, are operators satisfying the commutation relations

$$[\widehat{x}_a, \widehat{x}_b] = i\theta^2 \varepsilon_{ab} \qquad [\widehat{p}_a, \widehat{p}_b] = 0 \qquad [\widehat{x}_a, \widehat{p}_b] = i\delta_{ab} \tag{9}$$

where θ is a parameter with dimension of length, and ε_{ab} is an antisymmetric constant matrix.

It is well known that this noncommutative algebra can be mapped to the commutative Heisenberg-Weyl algebra [26]-[28]

$$[x_a, x_b] = 0 \qquad [p_a, p_b] = 0 \qquad [x_a, p_b] = i\delta_{ab} \tag{10}$$

through the relations

$$\widehat{x}_a = x_a - \frac{1}{2}\theta^2 \varepsilon_{ab} p_b \qquad \widehat{p}_a = p_a \tag{11}$$

To impose non commutativity on the configuration space of dynamical fields φ_a, we assume that the noncommutative canonical variables $\widehat{\varphi}_a$ and the noncommutative canonical conjugates $\widehat{\pi}_a$ satisfy the noncommutative commutation relations

$$\begin{aligned}
\left[\widehat{\varphi}_a\left(t, \overrightarrow{x}\right), \widehat{\pi}_b\left(t, \overrightarrow{y}\right)\right] &= i\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right)\delta_{ab} \\
\left[\widehat{\varphi}_a\left(t, \overrightarrow{x}\right), \widehat{\varphi}_b\left(t, \overrightarrow{y}\right)\right] &= i\theta\varepsilon_{ab}\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right) \\
\left[\widehat{\pi}_a\left(t, \overrightarrow{x}\right), \widehat{\pi}_b\left(t, \overrightarrow{y}\right)\right] &= 0
\end{aligned} \tag{12}$$

where θ is the parameter of noncommutativity, which is assumed to be a constant, and ε_{ab} is a 2×2 real antisymmetric matrix

$$\varepsilon_{12} = -\varepsilon_{21} = 1 \tag{13}$$

The noncommutative Hamiltonian density is assumed to have the form

$$\widehat{\mathcal{H}} = \frac{1}{2}\left(\widehat{\pi}_a\right)^2 + \frac{1}{2}\left(\overrightarrow{\nabla}\widehat{\varphi}_a\right)^2 + \frac{1}{2}\mu^2\left[\widehat{\varphi}\right]\left(\widehat{\varphi}_a\right)^2 \tag{14}$$

It is easy to see that the noncommutative commutation relations (12) can be mapped to the canonical commutation relations (8) if the noncommutative variables $\widehat{\varphi}_a$ and $\widehat{\pi}_a$ are related to the canonical variables φ_a and π_a by the relations

$$
\begin{aligned}
\widehat{\varphi}_a &= \varphi_a - \frac{1}{2}\theta\varepsilon_{ab}\pi_b \\
\widehat{\pi}_a &= \pi_a
\end{aligned} \tag{15}
$$

Using these transformations, the noncommutative Hamiltonian density eq(14) can be rewritten, up to a total derivative term and up to second order in the parameter θ, as

$$
\widehat{\mathcal{H}}=\frac{1}{2}\pi^{\sim}\mathbb{M}\pi - \frac{1}{8}\theta^2\pi^{\sim}\mathbb{D}\pi + \theta\pi^{\sim}\mathbb{N}\varphi+\frac{1}{2}\varphi^{\sim}\mathbb{B}\varphi + O\left(\theta^3\right) \tag{16}
$$

where

$$
\begin{aligned}
\mathbb{M}&= \mathbb{I}+\frac{1}{4}\theta^2\left(m^2\mathbb{I} - \varepsilon\widehat{\sigma}\varepsilon\right) = \mathbb{M}^{\sim} \\
\widehat{\sigma}_{ab} &= \frac{\delta^2}{\delta\varphi_a\delta\varphi_b}\left[\frac{1}{4}g\left(\varphi^{\sim}\varphi\right)^2\right] = \varphi^{\sim}\varphi\delta_{ab} + 2\varphi_a\varphi_b \\
\widehat{\sigma} &= \left(\varphi^{\sim}\varphi\right)\mathbb{I} + 2M\left[\varphi\right] \quad , \quad M_{ab}\left[\varphi\right] \ = \varphi_a\varphi_b \\
\mathbb{D}&=\overrightarrow{\nabla}^2\mathbb{I} = \mathbb{D}^{\sim} \\
\mathbb{N}&=\frac{1}{2}\left(m^2 - \overrightarrow{\nabla}^2 + g\left(\varphi_a\right)^2\right)\varepsilon = -\mathbb{N}^{\sim} \\
\mathbb{B}&=\left(m^2 - \overrightarrow{\nabla}^2 + \frac{1}{2}g\left(\varphi_a\right)^2\right)\mathbb{I} = \mathbb{B}^{\sim}
\end{aligned} \tag{17}
$$

with $\mathbb{I}$ denotes the 2×2 unit matrix, and $\mathbb{A}^{\sim}$ denotes the transpose of the operator $\mathbb{A}$.

From now on we keep only the modifications due to the noncommutativity up to second order in the parameter θ.

The relation between π_a and $\dot{\varphi}_a$ is given by

$$
\dot{\varphi}_a\left(x\right) = \frac{\delta\widehat{H}}{\delta\pi_a\left(x\right)} \tag{18}
$$

where $\widehat{H} = \int d^3x\widehat{\mathcal{H}}$. Using the expression of $\widehat{\mathcal{H}}$ and the symmetry properties of the operators $\mathbb{M}=\mathbb{M}^{\sim}$ and $\mathbb{D}=\mathbb{D}^{\sim}$, one gets

$$
\dot{\varphi}_a\left(x\right) = \mathbb{M}_{ab}\pi_b\left(x\right) - \frac{1}{4}\theta^2\mathbb{D}_{ab}\pi_b\left(x\right) + \theta\mathbb{N}_{ab}\varphi_b\left(x\right) \tag{19}
$$

From this relation we get the following iterative expression of π_a

$$\pi_a = \mathbb{K}_{ab}\left(\dot{\varphi}_b - \theta\mathbb{N}_{bc}\varphi_c\right) + \frac{1}{4}\theta^2\mathbb{K}_{ab}\mathbb{D}_{bc}\pi_c \tag{20}$$

where $\mathbb{K}$ is the inverse of the matrix $\mathbb{M}$

$$\mathbb{K} = \mathbb{M}^{-1} = \mathbb{I} - \frac{1}{4}\theta^2\left(m^2\mathbb{I} - \varepsilon\widehat{\sigma}\varepsilon\right) \tag{21}$$

Using the expression of the matrix $\mathbb{K}$ one gets, by iteration, the following expression of π_a

$$\pi_a = \left(\mathbb{I} + \frac{1}{4}\theta^2\overline{\mathbb{D}}\right)_{ab}\dot{\varphi}_b - \theta\mathbb{N}_{ab}\varphi_b \tag{22}$$

where $\overline{\mathbb{D}} = \mathbb{D} - \left(m^2\mathbb{I} - \varepsilon\widehat{\sigma}\varepsilon\right) = \overrightarrow{\nabla}^2\mathbb{I} - \left(m^2\mathbb{I} - \varepsilon\widehat{\sigma}\varepsilon\right) = \overline{\mathbb{D}}^{\sim}$.

We note that the noncommutative Hamiltonian density can be derived from the following noncommutative Lagrangian density

$$\widehat{\mathfrak{L}} = \frac{1}{2}\dot{\varphi}_a\left(\mathbb{I} + \frac{1}{4}\theta^2\overline{\mathbb{D}}\right)_{ab}\dot{\varphi}_b + \theta\varphi_a\mathbb{N}_{ab}\dot{\varphi}_b - \frac{1}{2}\varphi_a\left(\mathbb{B} + \theta^2\mathbb{N}^2\right)_{ab}\varphi_b \tag{23}$$
$$\widehat{\mathfrak{L}} = \frac{1}{2}\dot{\varphi}^{\sim}\left(\mathbb{I} + \frac{1}{4}\theta^2\overline{\mathbb{D}}\right)\dot{\varphi} + \theta\varphi^{\sim}\mathbb{N}\dot{\varphi} - \frac{1}{2}\varphi^{\sim}\left(\mathbb{B} + \theta^2\mathbb{N}^2\right)\varphi$$

via the usual Legendre transformation $\widehat{\mathfrak{L}} = \pi_a\dot{\varphi}_a - \widehat{\mathcal{H}}$. To get this expression we have used the symmetry properties of the operators $\overline{\mathbb{D}}$, $\mathbb{N}$ and $\mathbb{B}$.

3. NONCOMMUTATIVE FIELD EQUATIONS

Let us now consider the free theory, $g = 0$, the noncommutative free Hamiltonian density reads

$$\widehat{\mathcal{H}} = \frac{1}{2}\pi^{\sim}\mathbb{M}\pi - \frac{1}{8}\theta^2\pi^{\sim}\mathbb{D}\pi + \theta\pi^{\sim}\mathbb{N}\varphi + \frac{1}{2}\varphi^{\sim}\mathbb{B}\varphi \tag{24}$$

where

$$\mathbb{M} = \left(1 + \frac{1}{4}\theta^2 m^2\right)\mathbb{I} = \mathbb{M}^{\sim} \quad , \quad \mathbb{D} = \overrightarrow{\nabla}^2\mathbb{I} = \mathbb{D}^{\sim} \tag{25}$$
$$\mathbb{N} = \frac{1}{2}\left(m^2 - \overrightarrow{\nabla}^2\right)\varepsilon = -\mathbb{N}^{\sim}$$
$$\mathbb{B} = \left(m^2 - \overrightarrow{\nabla}^2\right)\mathbb{I} = \mathbb{B}^{\sim}$$

The noncommutative field equations are given by

$$\dot{\varphi}_a(x) = \frac{\delta \widehat{H}}{\delta \pi_a(x)} \tag{26}$$

$$\dot{\pi}_a(x) = -\frac{\delta \widehat{H}}{\delta \varphi_a(x)} \tag{27}$$

From the first equation we get

$$\pi_a = \left(\mathbb{I}+\frac{1}{4}\theta^2\overline{\mathbb{D}}\right)_{ab}\dot{\varphi}_b - \theta \mathbb{N}_{ab}\varphi_b \tag{28}$$

with $\overline{\mathbb{D}} = \left(m^2 - \overrightarrow{\nabla}^2\right)\mathbb{I}$.

The second equation gives

$$\dot{\pi}_a = \theta \mathbb{N}_{ab}\pi_b - \mathbb{B}_{ab}\varphi_b \tag{29}$$

The noncommutative field equations eq(28) and eq(29) may be written in the form

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]\varphi(x) = 0 \tag{30}$$

where

$$\begin{aligned} \mathcal{A} &= \left[1 - \frac{1}{4}\theta^2\left(m^2 - \overrightarrow{\nabla}^2\right)\right]\mathbb{I} = \mathcal{A}^{\sim} \\ \mathcal{C} &= \left[1 - \frac{1}{4}\theta^2\left(m^2 - \overrightarrow{\nabla}^2\right)\right]\left(m^2 - \overrightarrow{\nabla}^2\right)\mathbb{I} = \mathcal{C}^{\sim} \\ \mathcal{B} &= \theta\left(m^2 - \overrightarrow{\nabla}^2\right)\varepsilon = -\mathcal{B}^{\sim} \end{aligned} \tag{31}$$

It is easy to see that the field equations eq(30) may be derived from the Lagrangian eq(23)

$$\widehat{\mathbb{L}} = \int d^3\overrightarrow{x}\left[\frac{1}{2}\dot{\varphi}^{\sim}\mathcal{A}\dot{\varphi} + \frac{1}{2}\varphi^{\sim}\mathcal{B}\dot{\varphi} - \frac{1}{2}\varphi^{\sim}\mathcal{C}\varphi\right] \tag{32}$$

To get the general solution of eq(30) one begins by looking for solutions of the form [22]

$$u_A\left(t, \overrightarrow{x}\right) = \chi_A\left(\overrightarrow{x}\right)e^{-i\omega_A t}\zeta_A \tag{33}$$

known as mode functions, where χ_A are the eigenvectors of the operator $-\overrightarrow{\nabla}^2$ with eigenvalues σ_A

$$-\overrightarrow{\nabla}^2\chi_A\left(\overrightarrow{x}\right) = \sigma_A\chi_A\left(\overrightarrow{x}\right) \tag{34}$$

and ζ_A are 2×1 constant columns.

Insertion of eq(33) into eq(30) leads to the eigenvector-eigenvalue problem

$$\left[\left(1-\frac{1}{4}\theta^2\overline{\sigma}_A\right)\left(\overline{\sigma}_A-\omega_A^2\right)+i\theta\overline{\sigma}_A\varepsilon\omega_A\right]\zeta_A=0 \tag{35}$$

where we have used the abbreviation $\overline{\sigma}_A = m^2 + \sigma_A$.

This eigenvector-eigenvalue problem has a non trivial solution if and only if the frequencies ω_A are roots of the equation

$$\det\left[\left(1-\frac{1}{4}\theta^2\overline{\sigma}_A\right)\left(\overline{\sigma}_A-\omega_A^2\right)+i\theta\overline{\sigma}_A\varepsilon\omega_A\right]=0 \tag{36}$$

which can be written in the equivalent form

$$\left(1-\frac{1}{4}\theta^2\overline{\sigma}_A\right)^2\left(\overline{\sigma}_A-\omega_A^2\right)^2-\theta^2\overline{\sigma}_A^2\omega_A^2=0 \tag{37}$$

Hence, the frequencies ω_A are the positive roots of the equations

$$\omega_A^2 \pm \theta\overline{\sigma}_A\omega_A - \overline{\sigma}_A = 0 \tag{38}$$

The solutions are given by

$$\begin{aligned}\omega_A^{(+)} &= \frac{1}{2}\left[-\theta\overline{\sigma}_A+\sqrt{4\overline{\sigma}_A+\theta^2\overline{\sigma}_A^2}\right]\simeq\sqrt{\overline{\sigma}_A}-\frac{1}{2}\theta\overline{\sigma}_A+\frac{1}{8}\theta^2\overline{\sigma}_A^{\frac{3}{2}}\\ \omega_A^{(-)} &= \frac{1}{2}\left[+\theta\overline{\sigma}_A+\sqrt{4\overline{\sigma}_A+\theta^2\overline{\sigma}_A^2}\right]\simeq\sqrt{\overline{\sigma}_A}+\frac{1}{2}\theta\overline{\sigma}_A+\frac{1}{8}\theta^2\overline{\sigma}_A^{\frac{3}{2}}\end{aligned} \tag{39}$$

Because the mode functions

$$u_A^{(\pm)}\left(t,\overrightarrow{x}\right)=\chi_A\left(\overrightarrow{x}\right)e^{-i\omega_A^{(\pm)}t}\zeta_A^{(\pm)} \tag{40}$$

form a complete set the general solution of eq(30) may be expanded in terms of them:

$$\varphi\left(x\right)=\sum_A\left[u_A^{(+)}\left(x\right)a_A+\overline{u}_A^{(+)}\left(x\right)\overline{a}_A\right]+\sum_A\left[u_A^{(-)}\left(x\right)b_A+\overline{u}_A^{(-)}\left(x\right)\overline{b}_A\right] \tag{41}$$

for some time independent complex numbers a_A, b_A and their complex conjugates $\overline{a}_A$, $\overline{b}_A$, where $\overline{u}_A^{(\pm)}$ are the complex conjugates of the mode functions $u_A^{(\pm)}$. Starting from the equations satisfied by the mode functions $u_A^{(\pm)}$

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2}+\mathcal{B}\frac{\partial}{\partial t}-\mathcal{C}\right]u_A^{(\pm)}\left(x\right)=0 \tag{42}$$

one can see, after some algebraic operations [22], that these mode functions satisfy the Wronskian relations

$$
\begin{aligned}
&-i\int d^3\overrightarrow{x}\,u_A^{(+)*}\overleftrightarrow{\mathbb{W}}u_B^{(+)}=\delta_{AB} \qquad && +\ i\int d^3\overrightarrow{x}\,u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(+)}=\delta_{AB} \\
&-\,i\int d^3\overrightarrow{x}\,u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(+)}=0 && +\,i\int d^3\overrightarrow{x}\,u_A^{(+)*}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(+)}=0
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&-i\int d^3\overrightarrow{x}\,u_A^{(-)*}\overleftrightarrow{\mathbb{W}}u_B^{(-)}=\delta_{AB} \qquad && +\ i\int d^3\overrightarrow{x}\,u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(-)}=\delta_{AB} \\
&-\,i\int d^3\overrightarrow{x}\,u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(-)}=0 && +\,i\int d^3\overrightarrow{x}\,u_A^{(-)*}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(-)}=0
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&-i\int d^3\overrightarrow{x}\,u_A^{(+)*}\overleftrightarrow{\mathbb{W}}u_B^{(-)}=0 \qquad && +\ i\int d^3\overrightarrow{x}\,u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(-)}=0 \\
&-\,i\int d^3\overrightarrow{x}\,u_A^{(-)*}\overleftrightarrow{\mathbb{W}}u_B^{(+)}=0 && +\,i\int d^3\overrightarrow{x}\,u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(+)}=0
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&-i\int d^3\overrightarrow{x}\,u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(-)}=0 \qquad && +\,i\int d^3\overrightarrow{x}\,u_A^{(+)*}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(-)}=0 \\
&-i\int d^3\overrightarrow{x}\,u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(+)}=0 && +\,i\int d^3\overrightarrow{x}\,u_A^{(-)*}\overleftrightarrow{\mathbb{W}}\overline{u}_B^{(+)}=0
\end{aligned}
\tag{46}
$$

where

$$\overleftrightarrow{\mathbb{W}}(x)=-\mathcal{A}(x)\frac{\overrightarrow{\partial}}{\partial t}+\mathcal{A}(x)\frac{\overleftarrow{\partial}}{\partial t}+\mathcal{B}(x) \tag{47}$$

is the Wronskian operator corresponding to the differential operator [22]

$$\mathbb{F}=-\mathcal{A}\frac{\partial^2}{\partial t^2}+\mathcal{B}\frac{\partial}{\partial t}-\mathcal{C} \tag{48}$$

The Wronskian operator $\overleftrightarrow{\mathbb{W}}$ has the following symmetry and reality properties:

$$\overleftrightarrow{\mathbb{W}}^{\sim}=-\overleftrightarrow{\mathbb{W}}\quad,\quad\overleftrightarrow{\mathbb{W}}^{*}=-\overleftrightarrow{\mathbb{W}} \tag{49}$$

Here $\overline{\mathcal{O}}$, $\mathcal{O}^*$and $\mathcal{O}^\sim$denote the complex conjugate, the Hermitian conjugate and the transpose of the matrix (or the operator) $\mathcal{O}$, respectively.

In order that these Wronskian relations must hold, the operators $\mathcal{A}$ and $\mathcal{C}$ must be positive definite operators, but the eigenvalues of the operators $\mathcal{A}$ and $\mathcal{C}$ are given by

$$\mathcal{A}u_A^{(\pm)}(x) = \left[1 - \frac{1}{4}\theta^2\left(m^2 - \overrightarrow{\nabla}^2\right)\right]u_A^{(\pm)}(x) = \left(1 - \frac{1}{4}\theta^2\overline{\sigma}_A\right)u_A^{(\pm)}(x) \tag{50}$$
$$\mathcal{C}u_A^{(\pm)}(x) = \left[1 - \frac{1}{4}\theta^2\left(m^2 - \overrightarrow{\nabla}^2\right)\right]\left(m^2 - \overrightarrow{\nabla}^2\right)u_A^{(\pm)}(x) = \left(1 - \frac{1}{4}\theta^2\overline{\sigma}_A\right)\overline{\sigma}_A u_A^{(\pm)}(x)$$

so these eigenvalues are not positive for all indices A, to solve this problem we use the fact that $\theta \sim 10^{-13}m$ [26]-[29], so $\left(1 - \frac{1}{4}\theta^2\overline{\sigma}_A\right) > 0$ for all indices A such that $\overline{\sigma}_A < \frac{4}{\theta^2} \sim 10^{26}$, to make the spectrum of the operators $\mathcal{A}$ and $\mathcal{C}$ bounded we impose the following boundary conditions on the eigenfunctions $\chi_A(\overrightarrow{x})$ of the operator $-\overrightarrow{\nabla}^2$

$$\left|\frac{\partial}{\partial x_j}\chi_A(x_1, ..., x_j..., x_D)\right|_{\overrightarrow{x}=\overrightarrow{a}} \leq \frac{\alpha}{\theta} \qquad j = 1, 2, ..., D \tag{51}$$

at some arbitrary point $\overrightarrow{x} = \overrightarrow{a}$, and α is some constant with dimension $(\text{length})^{-\frac{3}{2}}$. Note that in the classical limit where $\theta \to 0$ this condition is trivially satisfied.

As an example we consider the free scalar field confined in a D-dimensional rectangular box of volume $V = L^D$ and impose periodic boundary conditions on the walls of the box, the normalized eigenfunctions $\chi_A(\overrightarrow{x})$ of the operator $-\overrightarrow{\nabla}^2$, are [39]

$$\sqrt{\frac{1}{V}}\exp\left[\sum_{k=1}^{D}\frac{2\pi i n_k}{L}x_k\right] \qquad \text{with} \quad n_k = 0, \pm1, \pm2, ..., \text{ for each } k = 1, 2, ..., D \tag{52}$$

in this case the boundary conditions eq(51) read

$$\left|\frac{\partial}{\partial x_j}\chi_A(x_1, ..., x_j..., x_D)\right|_{\overrightarrow{x}=\overrightarrow{a}} = \sqrt{\frac{1}{V}}\left|\frac{2\pi n_j}{L}\right| \leq \frac{\alpha}{\theta} \qquad j = 1, 2, ..., D \tag{53}$$
$$\left|\frac{2\pi n_j}{L}\right| \leq \frac{\alpha\sqrt{V}}{\theta} \qquad j = 1, 2, ..., D$$

if we choose $\alpha = \frac{1}{\sqrt{DV}}$ we get

$$\sigma_A \equiv \sigma_{n_1 n_2 ... n_D} = \sum_{k=1}^{D}\left(\frac{2\pi n_k}{L}\right)^2 \leq \frac{\alpha^2 DV}{\theta^2} \tag{54}$$
$$\frac{1}{4}\theta^2\overline{\sigma}_A = \frac{1}{4}\theta^2\left[m^2 + \sum_{k=1}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right] \leq \frac{1}{4}\theta^2 m^2 + \frac{\alpha^2 DV}{4} < 1$$

where we have used the fact that θ is an infinitesimal parameter such that $\theta^2 m^2 < 1$. Hence $\mathcal{A}$ and $\mathcal{C}$ are positive definite operators.

As a second example, we consider the free scalar field confined in a D-dimensional rectangular box of volume $V = L^D$ and impose Dirichlet boundary conditions on the walls of the box, the normalized eigenvectors χ_A of $-\overrightarrow{\nabla}^2$ with Dirichlet boundary conditions on the walls of the box

$$\begin{aligned} \chi_A\left(0, x_2, x_3, ..., x_D\right) &= \chi_A\left(L, x_2, x_3, ..., x_D\right) = 0 \\ \chi_A\left(x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_D\right) &= \chi_A\left(x_1, ..., x_{k-1}, L, x_{k+1}, ..., x_D\right) \quad , k = 2, ..., D \end{aligned} \tag{55}$$

are given by [39]

$$\begin{aligned} -\overrightarrow{\nabla}^2 \chi_A\left(\overrightarrow{x}\right) &= \sigma_A \chi_A\left(\overrightarrow{x}\right) \\ \chi_A\left(\overrightarrow{x}\right) &= \sqrt{\frac{2}{V}} \sin\left(\frac{\pi n_1}{L} x_1\right) \exp\left[\sum_{k=2}^{D} \frac{2\pi i n_k}{L} x_k\right] \end{aligned} \tag{56}$$

with $n_1 = 1, 2, ...$ and $n_k = 0, \pm 1, \pm 2, ...$ for $k = 2, 3, ..., D$.

The eigenvalues are given by

$$\sigma_A \equiv \sigma_{n_1 n_2 ... n_D} = \left(\frac{\pi n_1}{L}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \tag{57}$$

in this case the boundary conditions eq(51) read

$$\left|\frac{\partial}{\partial x_j} \chi_A\left(x_1, ..., x_j ..., x_D\right)\right|_{\overrightarrow{x} = \overrightarrow{a}} = \sqrt{\frac{2}{V}} \left|\sin\left(\frac{\pi n_1}{L} a_1\right)\right| \left|\frac{2\pi n_j}{L}\right| \leq \frac{\alpha}{\theta} \quad , \qquad j = 2, ..., D \tag{58}$$

$$\left|\frac{2\pi n_j}{L}\right| \leq \frac{1}{\left|\sin\left(\frac{\pi n_1}{L} a_1\right)\right|} \frac{\alpha\sqrt{V}}{\sqrt{2}\theta} \quad , \quad \frac{L}{a_1} \notin \mathbb{N} \quad , \quad j = 1, 2, ..., D$$

$$\left|\frac{\partial}{\partial x_1} \chi_A\left(x_1, ..., x_j ..., x_D\right)\right|_{\overrightarrow{x} = \overrightarrow{a}} = \sqrt{\frac{2}{V}} \left|\frac{n_1 \pi}{L} \cos\left(\frac{\pi n_1}{L} a_1\right)\right| \leq \frac{\alpha}{\theta} \tag{59}$$

$$\left|\frac{n_1 \pi}{L}\right| \leq \frac{1}{\left|\cos\left(\frac{\pi n_1}{L} a_1\right)\right|} \frac{\alpha\sqrt{V}}{\sqrt{2}\theta} \quad , \quad \frac{L}{a_1} \notin \mathbb{N} \quad , \quad j = 1, 2, ..., D \tag{60}$$

if we choose $\alpha = \sqrt{\frac{a_1}{L(D-1)V}}$ and $a_1 \approx 0$ we get

$$\sigma_A \equiv \sigma_{n_1 n_2 \dots n_D} = \left(\frac{\pi n_1}{L}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \leq \left[\frac{1}{\left|\cos\left(\frac{\pi n_1}{L} a_1\right)\right|^2} + \frac{D-1}{\left|\sin\left(\frac{\pi n_1}{L} a_1\right)\right|^2}\right] \frac{\alpha^2 V}{2\theta^2} \quad (61)$$

$$\sigma_A \equiv \leq \left[2\frac{1+\cos^2\left(\frac{\pi n_1}{L} a_1\right)(D-2)}{\sin\left(2\frac{n_1 \pi}{L} a_1\right)}\right] \frac{\alpha^2 V}{2\theta^2} \approx \frac{D-1}{\frac{n_1 \pi}{L} a_1} \frac{\alpha^2 V}{2\theta^2} \leq \frac{(D-1)\,L}{\pi a_1} \frac{\alpha^2 V}{2\theta^2}$$

$$\frac{1}{4}\theta^2 \overline{\sigma}_A = \frac{1}{4}\theta^2 \left[m^2 + \left(\frac{\pi n_1}{L}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2\right] \leq \frac{1}{4}\theta^2 m^2 + \frac{(D-1)\,L}{a_1} \frac{\alpha^2 V}{8\pi} < 1$$

where we have used the fact that θ is an infinitesimal parameter such that $\theta^2 m^2 < 1$. Hence $\mathcal{A}$ and $\mathcal{C}$ are positive definite operators.

From now on, we make the assumption that $\left(1 - \frac{1}{4}\theta^2 \overline{\sigma}_A\right) > 0$ for all indices A.

Quantization of the noncommutative complex scalar field theory is straightforward via the Peierls bracket (see [22] for more details). In the quantum theory, the field φ becomes a Hermitian operator, and the operator version of eq(41)

$$\varphi(x) = \sum_A \left[u_A^{(+)}(x)\, a_A + \overline{u}_A^{(+)}(x)\, a_A^*\right] + \sum_A \left[u_A^{(-)}(x)\, b_A + \overline{u}_A^{(-)}(x)\, b_A^*\right] \quad (62)$$

holds for some constant operators a_A, b_A and their Hermitian conjugates a_A^*, b_A^*. By using the Wronskian relations eq(43) $-$eq(46) we get

$$\begin{aligned} a_A &= -i \int d^3\overrightarrow{x}\, u_A^{(+)*}(x)\, \overleftrightarrow{\mathbb{W}} \varphi(x) \qquad a_A^* = +\, i \int d^3\overrightarrow{x}\, u_A^{(+)\sim}(x)\, \overleftrightarrow{\mathbb{W}} \varphi(x) \\ b_A &= -i \int d^3\overrightarrow{x}\, u_A^{(-)*}(x)\, \overleftrightarrow{\mathbb{W}} \varphi(x) \qquad b_A^* = +\, i \int d^3\overrightarrow{x}\, u_A^{(-)\sim}(x)\, \overleftrightarrow{\mathbb{W}} \varphi(x) \end{aligned} \quad (63)$$

The quantum theory is obtained by setting

$$\left[\varphi_a(x), \varphi_b(y)\right] = i\widetilde{G}_{ab}(x,y) \quad (64)$$

where $\widetilde{G}$ is the commutator matrix

$$\begin{aligned} \widetilde{G}(x,y) = &-i \sum_A u_A^{(+)}(x)\, u_A^{(+)*}(y) + i \sum_A \overline{u}_A^{(+)}(x)\, u_A^{(+)\sim}(y) \\ &- i \sum_A u_A^{(-)}(x)\, u_A^{(-)*}(y) + i \sum_A \overline{u}_A^{(-)}(x)\, u_A^{(-)\sim}(y) \end{aligned}$$

Using the Wronskian relations eq(43) −eq(46) one can see that the commutator matrix $\widetilde{G}$ is the unique function that solves the Cauchy problem:

$$\varphi(x) = \int d^3\overrightarrow{y}\,\widetilde{G}(x,y)\,\overleftrightarrow{\mathbb{W}}(y)\,\varphi(y) \quad \text{at the same time } t = x^0 = y^0 \tag{65}$$

Moreover the commutator matrix $\widetilde{G}$ satisfies the equation

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]\widetilde{G}(x,y) = 0 \tag{66}$$

Using eq(64) and the Wronskian relations eq(43) −eq(46) we get the commutation relations

$$\begin{aligned}
[a_A, a_B^*] = \delta_{AB} \quad &, \quad [a_A, a_B] = [a_A^*, a_B^*] = 0 \\
[b_A, b_B^*] = \delta_{AB} \quad &, \quad [b_A, b_B] = [b_A^*, b_B^*] = 0 \\
[a_A, b_B^*] = [a_A, b_B] = &\ [a_A^*, b_B^*] = [a_A^*, b_B] = 0
\end{aligned} \tag{67}$$

The noncommutative Hamiltonian operator is given by eq(24)

$$\widehat{H} = \int d^3\overrightarrow{x}\,\widehat{\mathcal{H}} = \int d^3\overrightarrow{x}\left[\frac{1}{2}\pi^{\sim}\mathbb{M}\pi - \frac{1}{8}\theta^2\pi^{\sim}\mathbb{D}\pi + \theta\pi^{\sim}\mathbb{N}\varphi + \frac{1}{2}\varphi^{\sim}\mathbb{B}\varphi\right] \tag{68}$$

It is easy to show, by substituting the expression of π eq(28)

$$\pi = \left(\mathbb{I} + \frac{1}{4}\theta^2\overline{\mathbb{D}}\right)\dot{\varphi} - \theta\mathbb{N}\varphi = \mathcal{A}\dot{\varphi} - \frac{1}{2}\mathcal{B}\varphi \tag{69}$$

into eq(68), that the noncommutative Hamiltonian operator can be written as

$$\widehat{H} = \frac{1}{2}\int d^3\overrightarrow{x}\,\varphi^{\sim}(x)\,\overleftrightarrow{\mathbb{W}}(x)\,\dot{\varphi}(x) \tag{70}$$

using the Wronskian relations eq(43) −eq(46), and the expression of φ eq(62), the noncommutative Hamiltonian operator $\widehat{H}$ of the system can be expressed as

$$\widehat{H} = \sum_A \left(\omega_A^{(+)} a_A^* a_A + \omega_A^{(-)} b_A^* b_A\right) + \frac{1}{2}\sum_A\left(\omega_A^{(+)} + \omega_A^{(-)}\right) \tag{71}$$

where the commutation relations have been used to get this form.

The noncommutative vacuum energy E_{vac} reads

$$E_{vac} = \langle vac|\,\widehat{H}\,|vac\rangle = \frac{1}{2}\sum_A\left(\omega_A^{(+)} + \omega_A^{(-)}\right) \tag{72}$$

Hence

$$E_{vac}=\frac{1}{2}\sum_{A}\left(\omega_{A}^{(+)}+\omega_{A}^{(-)}\right)=\sum_{A}\left(\sqrt{\sigma_{A}}+\frac{1}{8}\theta^{2}\sigma_{A}^{\frac{3}{2}}\right) \tag{73}$$

$$E_{vac}=\sum_{A}\left(\sqrt{m^{2}+\sigma_{A}}+\frac{1}{8}\theta^{2}\left(m^{2}+\sigma_{A}\right)^{\frac{3}{2}}\right)$$

where the summation over A is constrained by the condition eq(51). The noncommutative vacuum energy E_{vac}, in the case were the free scalar field is confined in a D-dimensional rectangular box of volume $V=L^{D}$ with periodic boundary conditions on the walls of the box, can be written as

$$E_{vac}=\sum_{n_{1},n_{2},\ldots,n_{D}}\left(\left[m^{2}+\sum_{k=1}^{D}\left(\frac{2\pi n_{k}}{L}\right)^{2}\right]^{\frac{1}{2}}+\frac{1}{8}\theta^{2}\left[m^{2}+\sum_{k=1}^{D}\left(\frac{2\pi n_{k}}{L}\right)^{2}\right]^{\frac{3}{2}}\right) \tag{74}$$

where the summation over $n_{1},n_{2},...,n_{D}$ is constrained by the condition eq(53)

$$|n_{k}|\leq\frac{L}{2\pi\sqrt{D}\theta}\qquad k=1,2,...,D$$

In the limit $L\rightarrow\infty$ we can approximate the sums that occur in eq(74) with (divergent) integrals

$$E_{vac}=V\int\frac{d^{D}\overrightarrow{p}}{(2\pi)^{D}}\left(\left[\overrightarrow{p}^{2}+m^{2}\right]^{\frac{1}{2}}+\frac{1}{8}\theta^{2}\left[\overrightarrow{p}^{2}+m^{2}\right]^{\frac{3}{2}}\right)$$

Although these integrals are mathematically meaningless, one can use some sort of regularization technique that makes the integrals finite. Using the $\zeta-$function regularization (see the definitions and intermediate stages of the calculation in Section 4) [39], we get the following expression for the vacuum energy E_{vac}

$$E_{vac}=\left[\frac{V\left[m^{2}\right]^{\frac{D+1}{2}}}{(4\pi)^{\frac{D}{2}}}\left[l^{2}m^{2}\right]^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s-D-1}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}+\frac{1}{8}\theta^{2}\frac{V\left[m^{2}\right]^{\frac{D+3}{2}}}{(4\pi)^{\frac{D}{2}}}\left[l^{2}m^{2}\right]^{-\frac{3s}{2}}\frac{\Gamma\left(\frac{3s-D-3}{2}\right)}{\Gamma\left(\frac{3s-3}{2}\right)}\right]_{s\rightarrow0} \tag{75}$$

If D is even the right-hand side of eq(75) is analytic at $s=0$ with the result

$$\begin{aligned}E_{vac}&=\frac{V\left[m^{2}\right]^{\frac{D+1}{2}}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(-\frac{D+1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}\left[1+\frac{1}{8}\theta^{2}\frac{3m^{2}}{D+1}\right]\\&=\frac{V\left[m^{2}\right]^{\frac{D+1}{2}}}{(4\pi)^{\frac{D}{2}}}\frac{(-2)^{\frac{D}{2}}}{1.3.5...(D+1)}\left[1+\frac{1}{8}\theta^{2}\frac{3m^{2}}{D+1}\right]\end{aligned} \tag{76}$$

where we have used the following properties of the $\Gamma-$function [39] [23]

$$z\Gamma(z) = \Gamma(z+1)$$
$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-2)^n\sqrt{\pi}}{1.3.5...(2n-1)} \quad , \quad n = 1,2,3,...$$

When D is odd the right-hand side of eq(75) is not analytic at $s = 0$, it has simple poles at $s = 0$, one simple pole from $\Gamma\left(\frac{s-D-1}{2}\right)$ and another simple pole from $\Gamma\left(\frac{3s-D-3}{2}\right)$. If we expand eq(75) about the pole, in the case where $D = 3$, we find

$$E_{vac} = -\frac{V}{2}\left(\frac{m^2}{4\pi}\right)^2\left\{\left[1+\frac{m^2}{48}\theta^2\right]\frac{2}{s}-\frac{1}{2}\left[1+\frac{5m^2}{24}\theta^2\right]-\left[1+\frac{m^2}{16}\theta^2\right]\ln\frac{l^2m^2}{4}\right\}$$

to get this expression the following formula has been used [39] [23]

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon}-\gamma+1+\frac{1}{2}+...+\frac{1}{n}\right)+O(\epsilon)$$

where n is a positive integer or zero, and γ is the Euler constant.

The vacuum energy, when D is odd, is divergent, this is just one example of a variety of ultraviolet divergences that are encountered in quantum field theory, they arise in a continuum theory due to the infinite number of degrees of freedom that exist even in a finite volume, they can be reabsorbed into a rescaling of the fields and into a rescaling of coupling constants. These ultraviolet divergences can be eliminated by hand since only energy differences can be observed, they are only important if we worry about gravitational phenomena, since in general relativity any form of energy contributes to the gravitational interaction [21] [24].

4. NONCOMMUTATIVE CASIMIR EFFECT

The Casimir effect is a non-classical electromagnetic, attractive force which is concerned with vacuum fluctuations in the electromagnetic field between two uncharged parallel conducting plates [30]. The size of this force was first predicted and calculated in 1948 by Casimir, who found that there is an attractive force per unit area between two parallel, uncharged, perfectly conducting plates separated by a distance a

$$F_{Casimir} = -\frac{\hbar c\pi^2}{240a^4}$$

This was first looked for by Sparnaay (1958), and recently has been confirmed by Lamoreaux, Mohideen and Roy, and recently by Chan, Aksyuk, Kleiman, Bishop, and Capasso [30][32]-[38].

Casimir effect is of great interest for both the theoretical and the experimental sides, it finds applications in various physical phenomena, for example in quantum reflection of atoms on different surfaces and Bose-Einstein condensation [31].

In this section we will consider the complex scalar field analogue of the Casimir effect, for this we consider a massive complex scalar field in a D-dimensional rectangular box, satisfying Dirichlet boundary conditions at $x_1 = 0$ and $x_1 = a$, but is unconfined in the remaining directions, let $L_1 = a$, $L_2 = L_3 = ... = L_D = L$ be the sides of the box, and $V = L_1 L_2 L_3 ... L_D$ its volume, ultimately we will let L becomes infinitely large [39].

The normalized eigenvectors χ_A of $-\overrightarrow{\nabla}^2$ with Dirichlet boundary conditions on the walls of the box

$$\begin{aligned} \chi_A(0, x_2, x_3, ..., x_D) &= \chi_A(a, x_2, x_3, ..., x_D) = 0 \qquad (77) \\ \chi_A(x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_D) &= \chi_A(x_1, ..., x_{k-1}, L, x_{k+1}, ..., x_D) \quad , k = 2, ..., D \end{aligned}$$

are given by [39]

$$\begin{aligned} -\overrightarrow{\nabla}^2 \chi_A(\overrightarrow{x}) &= \sigma_A \chi_A(\overrightarrow{x}) \qquad (78) \\ \chi_A(\overrightarrow{x}) &= \sqrt{\frac{2}{V}} \sin\left(\frac{\pi n_1}{a} x_1\right) \exp\left[\sum_{k=2}^{D} \frac{2\pi i n_k}{L} x_k\right] \end{aligned}$$

with $n_1 = 1, 2, ...$ and $n_k = 0, \pm 1, \pm 2, ...$ for $k = 2, 3, ..., D$.

The eigenvalues are given by

$$\sigma_A \equiv \sigma_{n_1 n_2 ... n_D} = \left(\frac{\pi n_1}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \qquad (79)$$

The noncommutative vacuum energy E_{vac} is given by

$$\begin{aligned} E_{vac} &= \sum_A \left(\sqrt{m^2 + \sigma_A} + \frac{1}{8}\theta^2 \left(m^2 + \sigma_A\right)^{\frac{3}{2}}\right) \qquad (80) \\ E_{vac} &= E_{vac}^{(C)} + E_{vac}^{(NC)} \end{aligned}$$

where $E_{vac}^{(C)}$ is the classical vacuum energy

$$E_{vac}^{(C)} = \sum_A \sqrt{m^2 + \sigma_A} = \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} ... \sum_{n_D=-\infty}^{\infty} \sqrt{m^2 + \left(\frac{\pi n_1}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2} \tag{81}$$

and $E_{vac}^{(NC)}$ is the pure noncommutative vacuum energy

$$\begin{aligned} E_{vac}^{(NC)} &= \frac{1}{8}\theta^2 \sum_A \left(m^2 + \sigma_A\right)^{\frac{3}{2}} \\ &= \frac{1}{8}\theta^2 \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} ... \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{\pi n_1}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2\right]^{\frac{3}{2}} \end{aligned} \tag{82}$$

To deal with the infinite sum of zero point energies in eq(81) and eq(82), we must introduce a regularization method to extract finite expression [39][40][41][23]. One elegant way for doing this is to use $\zeta-$function regularization [39], the idea of the method is to define the divergent sum $\sum_A E_A$ over zero-point energies in eq(81) and eq(82) by the analytic continuation of a convergent sum. First, we consider the infinite sum in eq(81), we define the energy $\zeta-$function by

$$E(s) = \sum_A E_A \left(l E_A\right)^{-s} \tag{83}$$

where $E_A = \sqrt{m^2 + \sigma_A}$, s is a complex variable and l is a constant with units of length, introduced to keep(lE_A) dimensionless. This ensures that $E(s)$ has dimensions of energy for all values of s.

The classical vacuum energy can be written as

$$E_{vac}^{(C)} = \lim_{s\to 0} E(s) = E(0) \tag{84}$$

where the energy $\zeta-$function $E(s)$ is given by

$$E(s) = l^{-s} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} ... \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{n_1 \pi}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2\right]^{\frac{1-s}{2}} \tag{85}$$

In the limit $L \to \infty$, we can replace the sums over $n_2, n_3, ..., n_D$ with integrals, so the energy $\zeta-$function becomes

$$E\left(s\right)=l^{-s}\frac{V}{a}\sum_{n_1=1}^{\infty}\int\frac{d^{D-1}\overrightarrow{p}}{\left(2\pi\right)^{D-1}}\left[\left(\frac{n_1\pi}{a}\right)^2+\overrightarrow{p}^2+m^2\right]^{\frac{1-s}{2}} \tag{86}$$

Using the identity

$$a^{-z}=\frac{1}{\Gamma\left(z\right)}\int_0^{\infty}dtt^{z-1}e^{-at} \tag{87}$$

which holds for $\mathrm{Re}(z)>0$ and $\mathrm{Re}(a)>0$, where $\Gamma\left(z\right)$ is Gamma function

$$\Gamma\left(z\right)=\int_0^{\infty}dtt^{z-1}e^{-t} \tag{88}$$

defined for $\mathrm{Re}(z)>0$, we obtain the following expression for the energy $\zeta-$function

$$\begin{aligned}E\left(s\right)&=l^{-s}\frac{V}{a}\sum_{n_1=1}^{\infty}\frac{1}{\Gamma\left(\frac{s-1}{2}\right)}\int\frac{d^{D-1}\overrightarrow{p}}{\left(2\pi\right)^{D-1}}\int_0^{\infty}dtt^{\frac{s-3}{2}}\exp\left(-\left[\left(\frac{n_1\pi}{a}\right)^2+\overrightarrow{p}^2+m^2\right]t\right) \\ &=l^{-s}\frac{V}{a}\sum_{n_1=1}^{\infty}\frac{1}{\Gamma\left(\frac{s-1}{2}\right)}\int_0^{\infty}dtt^{\frac{s-3}{2}}\exp\left(-\left[\left(\frac{n_1\pi}{a}\right)^2+m^2\right]t\right)\int\frac{d^{D-1}\overrightarrow{p}}{\left(2\pi\right)^{D-1}}\exp\left(-\overrightarrow{p}^2t\right)\end{aligned} \tag{89}$$

Finally, integrating over $\overrightarrow{p}$, and using the identity (87) one finds the following expression

$$E\left(s\right)=l^{-s}\frac{V}{\left(4\pi\right)^{\frac{D-1}{2}}a}\frac{\Gamma\left(\frac{s-D}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\sum_{n_1=1}^{\infty}\left[\left(\frac{n_1\pi}{a}\right)^2+m^2\right]^{\frac{D-s}{2}} \tag{90}$$

When $m\to 0$, the energy $\zeta-$function becomes

$$\begin{aligned}E\left(s\right)&=l^{-s}\frac{V}{\left(4\pi\right)^{\frac{D-1}{2}}a}\frac{\Gamma\left(\frac{s-D}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\sum_{n_1=1}^{\infty}\left(\frac{n_1\pi}{a}\right)^{D-s} \\ E\left(s\right)&=l^{-s}\frac{V}{\left(4\pi\right)^{\frac{D-1}{2}}a}\left(\frac{\pi}{a}\right)^{D-s}\frac{\Gamma\left(\frac{s-D}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\zeta\left(s-D\right)\end{aligned} \tag{91}$$

where $\zeta\left(s\right)=\sum_{n=1}^{\infty}n^{-s}$ is the Riemann $\zeta-$function.

Let us now consider the interesting case where $D=3$, in this case the energy $\zeta-$function takes the form

$$\begin{aligned}E\left(s\right)&=\frac{V}{4\pi a}\left(\frac{\pi}{a}\right)^3\left(l\frac{\pi}{a}\right)^{-s}\frac{\Gamma\left(\frac{s-3}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\zeta\left(s-3\right) \\ E\left(s\right)&=\frac{V\pi^2}{4\pi a^4}\frac{2}{s-3}\left(l\frac{\pi}{a}\right)^{-s}\zeta\left(s-3\right)\end{aligned} \tag{92}$$

In the limit $s \to 0$, we get

$$E(0) = -\frac{\pi^2}{6a^4}\zeta(-3) = -\frac{\pi^2}{720a^4} \tag{93}$$

Hence the vacuum energy is

$$E_{vac}^{(C)} = -\frac{\pi^2 A}{720a^3} \tag{94}$$

where $A = L_1 L_2 = L^2$ is the area of the parallel (uncharged conducting) plates.

By the same steps we will now calculate the noncommutative vacuum energy $E_{vac}^{(NC)}$, let $\mathcal{E}(s)$ be the energy $\zeta-$function

$$\mathcal{E}(s) = l^{-3s} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} ... \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{n_1 \pi}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{\frac{3(1-s)}{2}} \tag{95}$$

then

$$E_{vac}^{(NC)} = \lim_{s \to 0} \mathcal{E}(s) = \mathcal{E}(0) \tag{96}$$

In the limit $L \to \infty$, we can replace the sums over $n_2, n_3, ..., n_D$ with integrals, so the energy $\zeta-$function becomes

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a} \sum_{n=1}^{\infty} \int \frac{d^{D-1}\overrightarrow{p}}{(2\pi)^{D-1}} \left[\left(\frac{n\pi}{a}\right)^2 + \overrightarrow{p}^2 + m^2 \right]^{\frac{3(1-s)}{2}} \tag{97}$$

using the relation(87), one gets

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a} \sum_{n=1}^{\infty} \frac{1}{\Gamma\left(\frac{3}{2}(s-1)\right)} \int_0^{\infty} dt t^{\frac{(3s-5)}{2}} \int \frac{d^{D-1}\overrightarrow{p}}{(2\pi)^{D-1}} e^{-\left[\left(\frac{n\pi}{a}\right)^2 + \overrightarrow{p}^2 + m^2\right]t} \tag{98}$$

integrating over $\overrightarrow{p}$, and using the identity (87) one finds

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a(4\pi)^{\frac{D-1}{2}}} \frac{1}{\Gamma\left(\frac{3}{2}(s-1)\right)} \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{a}\right)^2 + m^2 \right]^{-\frac{(3s-D-2)}{2}} \int_0^{\infty} dt t^{\frac{(3s-D-2)}{2}-1} e^{-t}$$

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a(4\pi)^{\frac{D-1}{2}}} \frac{\Gamma\left(\frac{(3s-D-2)}{2}\right)}{\Gamma\left(\frac{3s-3}{2}\right)} \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{a}\right)^2 + m^2 \right]^{-\frac{(3s-D-2)}{2}} \tag{99}$$

When $m \to 0$, the energy $\zeta-$function becomes

$$\mathcal{E}\left(s\right)=\frac{V}{a\left(4\pi\right)^{\frac{D-1}{2}}}\left(\frac{\pi}{a}\right)^{D+2}\frac{\Gamma\left(\frac{(3s-D-2)}{2}\right)}{\Gamma\left(\frac{3s-3}{2}\right)}\left(\frac{l\pi}{a}\right)^{-3s}\zeta\left(3s-D-2\right) \tag{100}$$

In the case where $D=3$, the energy $\zeta-$function takes the form

$$\mathcal{E}\left(s\right)=\frac{V\pi^{4}}{4a^{6}}\frac{2}{3s-5}\left(\frac{l\pi}{a}\right)^{-3s}\zeta\left(3s-5\right) \tag{101}$$

In the limit $s\rightarrow 0$, we get

$$\mathcal{E}\left(0\right)=-\frac{\pi^{4}}{4a^{6}}\frac{2}{5}\zeta\left(-5\right)=+\frac{\pi^{4}}{2520a^{6}} \tag{102}$$

Hence the noncommutative vacuum energy is

$$E_{vac}^{(NC)}=\mathcal{E}\left(0\right)=\frac{\pi^{4}A}{2520a^{5}} \tag{103}$$

The total vacuum energy $E_{vac}=E_{vac}^{(C)}+E_{vac}^{(NC)}$, is given by

$$\begin{aligned} E_{vac}&=E_{vac}^{(C)}+E_{vac}^{(NC)}=-\frac{\hbar c\pi^{2}A}{720a^{3}}+\frac{1}{8}\theta^{2}\frac{\hbar c\pi^{4}A}{2520a^{5}} \\ E_{vac}&=-\frac{\hbar c\pi^{2}A}{720a^{3}}\left(1-\frac{\pi^{2}\theta^{2}}{28a^{2}}\right) \end{aligned} \tag{104}$$

The Casimir force reads

$$F_{Casimir}=-\frac{\partial E_{vac}}{\partial a}=-\frac{\hbar c\pi^{2}A}{240a^{6}}\left(a^{2}-\frac{5}{84}\pi^{2}\theta^{2}\right) \tag{105}$$

where the first term represents the classical attractive Casimir force, while the second term represents the noncommutative Casimir force, which is repulsive.

From eq(105) we see that the total vacuum energy E_{vac} has a minimum at

$$a_{\min}=\sqrt{\frac{5}{84}}\pi\theta \quad , \quad \theta\neq 0 \tag{106}$$

At the equilibrium point $a_{\min}$, the total vacuum energy E_{vac} takes the value

$$E_{vac}^{\min}=E_{vac}\left(a_{\min}\right)=-\frac{\hbar c\pi^{2}A}{720a_{\min}^{3}}\left(1-\frac{1}{28}\frac{\pi^{2}\theta^{2}}{a_{\min}^{2}}\right)=-\left(3.8497\times 10^{-28}\,\mathrm{J\,m}\right)\frac{A}{\theta^{3}} \tag{107}$$

It is well known that the motion near the equilibrium may be approximately described as harmonic oscillations, indeed near the equilibrium we may write $a = a_{\min} + \delta$, if we expand the total vacuum energy E_{vac} in a Taylor series

$$E_{vac}(a) = E_{vac}(a_{\min}) + E'_{vac}(a_{\min})\,\delta + \frac{1}{2}E'_{vac}(a_{\min})\,\delta^2 + ... \tag{108}$$

we get

$$E_{vac}(a) = -\frac{\hbar c\pi^2 A}{720a^3}\left(1 - \frac{1}{28}\frac{\pi^2\theta^2}{a^2}\right) \simeq -\frac{\hbar c\pi^2 A}{1800a_{\min}^3} + \frac{1}{2}\left(\frac{\hbar c\pi^2 A}{120a_{\min}^5}\right)\delta^2 \tag{109}$$

Hence the equation of motion near the equilibrium may be derived from the following (harmonic oscillator) Lagrangian

$$L = \frac{1}{2}\rho A\dot{\delta}^2 - \frac{1}{2}\left(\frac{\hbar c\pi^2 A}{120a_{\min}^5}\right)\delta^2 = \frac{1}{2}\rho A\dot{\delta}^2 - \frac{1}{2}\rho A\omega^2\delta^2 \tag{110}$$

where ρ is the density of the parallel plate, and ω is the angular frequency of vibration

$$\omega = \sqrt{\frac{\hbar c\pi^2}{120\rho a_{\min}^5}} = \frac{3.9499 \times 10^{-13}}{\sqrt{\rho}}\frac{1}{\theta^{\frac{5}{2}}} \tag{111}$$

5. CONCLUSION

Thought this work we have considered a noncommutative complex scalar field theory with self interaction, by imposing non commutativity to the canonical commutation relations. The noncommutative field equations are derived and solved, the vacuum energy is calculated to the second order in the parameter of non commutativity. As an example, we have considered the Casimir effect, due to the zero point fluctuations of the noncommutative complex scalar field. It turns out that in spite of its smallness, the noncommutativity gives rise to a repulsive force at the microscopic level, leading to an effective Casimr potential with a minimum at the point $a_{\min} = \sqrt{\frac{5}{84}}\pi\theta$.

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