A FUNDAMENTAL PRINCIPLE OF RELATIVITY

By Douglas L. Weller

ABSTRACT

The laws of physics hold equally in reference frames that are in motion with respect to each other. This premise of Albert Einstein's theory of relativity is a fairly easy concept to understand in the abstract, however the mathematics—particularly the tensor calculus used by Einstein to describe general relativity—used to flesh out this premise can be very complex, making the subject matter difficult for the non-specialist to intuitively grasp. Here is set out a fundamental principle of relativity that can be used as a tool to understand and explain special and general relativity. The fundamental principle of relativity is used to independently derive the Lorentz factor, the Minkowski metric and the Schwarzschild metric. The fundamental principle is also used to derive metric tensors for systems with multiple point masses and to explain Newtonian kinetic energy, gravitational potential energy and mass-energy equivalence in the context of special and general relativity.

INTRODUCTION

According to Einstein's theories of relativity, the laws of physics hold equally well even when measured from different reference frames that are in motion with respect to each other. Special relativity deals with the subset of reference frames that are in uniform motion with respect to each other. General relativity encompasses all reference frames, even reference frames that are variously accelerated with respect to each other. While the notion of special and general relativity is straightforward in the abstract, the mathematics used to describe relativity, particularly general relativity, can be very complex making study of relativity intimidating to non-specialists.

This paper sets out a tool that can be used to understand and teach relativity in an intuitive way, without requiring an extensive knowledge of tensor calculus. Specifically, a velocity through the time dimension is defined so that total velocity through the time and space dimensions is always equal to the speed of light c. Using this very simple tool, called here a "fundamental principle of relativity," many of the common equations of special and general relativity can be derived and intuitively understood. For example, this paper uses the fundamental principle to derive the Lorentz factor, the Minkowski metric, the Schwarzschild metric and metric tensors for systems with multiple point masses, and to explain Newtonian kinetic energy, gravitational potential energy, mass-energy equivalence, and discontinuities in the geometry of space-time. The fundamental principle of relativity is also shown to be directly related to the conservation of momentum and energy.

I. FUNDAMENTAL PRINCIPLE OF RELATIVITY DEFINED

When travel in all dimensions is taken into account, everything travels at the speed of light c. This fundamental principle of relativity is herein first examined in the context of special relativity, that is, for the case where motion through space is not accelerated. For example, as measured from any reference frame an unaccelerated particle will have a velocity ($\vec{v}_S$) through the space dimensions and a velocity ($\vec{v}_\tau$) through the time dimension so that,

$$c = \left|\vec{v}_S + \vec{v}_\tau\right|. \tag{1}$$

A. Velocity through Space

The velocity of the particle through space ($\vec{v}_S$), is what is ordinarily considered velocity. That is, for a space-time reference frame defined by the reference coordinates (*x, y, z, t*), velocity ($\vec{v}_S$) is defined as

$$\vec{v}_S = \vec{v}_x + \vec{v}_y + \vec{v}_z, \tag{2}$$

and the magnitude v_S of this velocity is defined as

$$v_S = \left|\vec{v}_S\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,. \tag{3}$$

B. Velocity through Time

What is meant by velocity through time can be understood in light of Hermann Minkowski's fundamental axiom for space-time set out in an address[1] given in September 1908:

> *The substance at any world-point may always, with the appropriate determination of space and time, be looked upon as at rest.*

Minkowski's fundamental axiom for the space-time continuum indicates that for the substance at a world point (e.g., a particle) there exists a local reference frame, with its own local space and time coordinates, in which the substance is at rest with respect to the local space coordinates (but not with respect to the local time coordinate).

For example, assume the local reference frame for a particle has the local space coordinates (ξ,η,ς) and the local time coordinate τ. For the particle, with respect to the local reference frame, $\frac{d\xi}{d\tau} = \frac{d\eta}{d\tau} = \frac{d\varsigma}{d\tau} = 0$.

The magnitude v_τ of the velocity of time $\vec{v}_\tau$ is defined to be equal to the speed of light c times the rate of passage of local time as measured by coordinate τ with respect to the rate of the passage of time as measured by reference time coordinate t, that is,

$$\left|\vec{v}_\tau\right| = v_\tau = c\frac{d\tau}{dt}. \tag{4}$$

When the reference coordinates and the local coordinates are stationary with respect to each other, $d\tau = dt$ so that $v_\tau = v_t = c\frac{dt}{dt} = c$.

II. THE FUNDAMENTAL PRINCIPLE AND SPECIAL RELATIVITY

The special theory of relativity is based on the postulate—which Einstein[2] called the "special principle of relativity"—that the laws of physics hold equally well even when measured from reference frames that are in uniform motion with respect to each other. Thus the laws of physics detected by making measurements from any selected reference coordinates (x, y, z, t) will also hold true when measurements are made from any selected local coordinates (ξ,η,ς,τ), as long as any motion between the reference frame for coordinates (ξ,η,ς,τ) and the reference frame for coordinates (x, y, z, t) is uniform.

The foundational equations for special relativity can be very simply derived from the fundamental principle, and are essentially just different mathematical expressions of equation (1), as will be shown below.

In special relativity, coordinates are customarily chosen (although this is not mathematically necessary) so that a separate time coordinate measures passage of time in a time dimension that is perpendicular to the space dimensions. This allows equation (1) to be rewritten as

$$c = \left|\vec{v}_S + \vec{v}_\tau\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \left(c\frac{d\tau}{dt}\right)^2}\,, \tag{5}$$

or more simply,

$$c^2 = \left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \left(c\frac{d\tau}{dt}\right)^2. \tag{6}$$

From equation (6) it is very straightforward to obtain the Lorentz factor and the Minkowski metric, which are the foundational equations for special relativity.

A. The Lorentz Factor

Equation (6) can be rewritten as $c^2 = v_S{}^2 + \left(c\frac{d\tau}{dt}\right)^2$ and thus $\frac{c^2 - v_S{}^2}{c^2} = \left(\frac{d\tau}{dt}\right)^2$. Solving for $\frac{dt}{d\tau}$ yields

$$\frac{dt}{d\tau} = \frac{c}{\sqrt{c^2 - v_S{}^2}} = \frac{1}{\sqrt{1 - v_S{}^2/c^2}}\,, \tag{7}$$

which is the Lorentz factor[3] used in special relativity to describe the relationship between dt and $d\tau$.

B. The Minkowski Metric

Equation (6) can be rewritten as $dt^2c^2 = dt^2\left[\left(\frac{dx}{dt}\right)+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2+\left(c\frac{d\tau}{dt}\right)^2\right]$, which reduces to $c^2dt^2 = c^2d\tau^2 + dx^2 + dy^2 + dz^2$, typically expressed as the Minkowski metric in the form

$$c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{8}$$

When the spherical coordinate system $(r,\ \theta,\ \varphi)$ are used,

$v_S{}^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2$ and therefore

$c = \left|\vec{v}_S + \vec{v}_\tau\right| = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 + \left(c\frac{d\tau}{dt}\right)^2}$ and

$c^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 + \left(c\frac{d\tau}{dt}\right)^2$. To obtain the Minkowski metric in spherical coordinates, this can be rewritten as $dt^2c^2 = dt^2\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 + \left(c\frac{d\tau}{dt}\right)^2\right]$, which reduces to the Minkowski metric in spherical coordinates:

$$c^2d\tau^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - \left(r^2\sin^2\theta\right)d\varphi^2. \tag{9}$$

III. THE FUNDAMENTAL PRINCIPLE AND NEWTONIAN PHYSICS

The fundamental principle explains the existence of kinetic energy and gravitational potential energy in Newtonian Physics.

A. Kinetic Energy

According to the fundamental principle a particle with a mass *m* (called hereafter particle *m*) has a velocity and thus a momentum not only in space but also in time. That can be expressed as

$$Momentum = m\vec{v}_S + m\vec{v}_\tau . \tag{10}$$

When particle *m* is at rest with respect to space coordinates (*x, y, z*), then $v_S = 0$ and $v_\tau = c$, as can be readily understood from equation (1).

When particle *m* is moving in space with respect to reference space coordinates (*x, y, z*) there is a change in momentum from the rest state not only in the space component of momentum $m\vec{v}_S$ but also in the time component of momentum $m\vec{v}_\tau$. From rest to any space velocity v_S, there is a change in the value of the space component of momentum from 0 to the value mv_S in the direction of travel through space. That is the change in momentum, $mv_S - 0 = mv_S$, is the value Newtonian Physics recognizes as the momentum of particle *m*.

When particle *m* moves from rest to any space velocity v_S, there is a change in the value of the time component of momentum from *mc* to the value mv_τ in the direction of time. This value for change in momentum $mc - mv_\tau$, is the momentum in the time dimension that is “sacrificed” to achieve space velocity v_S and which is restored to the time dimension when the space velocity is returned to 0.

The value $mc - mv_\tau$ is readily calculated from equation (1), which can be rewritten as $c = \sqrt{{v_S}^2 + {v_\tau}^2}$ and $v_\tau = \sqrt{c^2 - {v_S}^2}$ so that

$$mc - mv_\tau = m\left(c - \sqrt{c^2 - {v_S}^2}\right), \tag{11}$$

for coordinate systems where the time coordinate t is perpendicular to the space coordinates (x, y, z). For the case where $c >> v_S$, $\sqrt{c^2 - {v_S}^2}$ essentially reduces to $c - \frac{{v_S}^2}{2c}$ (neglecting the fourth and higher magnitudes) and therefore $mc - mv_\tau \approx mc - m\left(c - \frac{{v_S}^2}{2c}\right) = m\frac{{v_S}^2}{2c}$, which can be rewritten as

$$mc^2 - mcv_\tau = c(mc - mv)_\tau \approx \frac{1}{2}m{v_S}^2 = E_K. \tag{12}$$

Equation (12) shows that the speed of light times the loss of momentum in the time dimension is equal to the value Newtonian Physics recognizes as the kinetic energy (E_K) of particle m.

B. Gravitational Potential Energy and Escape Velocity

In Newtonian physics, gravity stores gravitational potential energy just like motion stores kinetic energy. This indicates that what Newtonian physics calls gravitational potential energy (like kinetic energy) can also be understood as a loss of momentum in the time dimension times the speed of light c. Since gravitational potential energy has the same relationship to gravitational escape velocity ($\vec{v}_G$) as kinetic energy does to velocity in space

$\vec{v}_S$, when making calculations on gravitational effects on time dilation, gravitational escape velocity ($\vec{v}_G$), can be used in place of space ($\vec{v}_S$) in equation (1), i.e.,

$$c = \left|\vec{v}_G + \vec{v}_\tau\right|. \tag{13}$$

From a location with zero gravity to a location where the gravitational escape velocity equals v_G, there is a change in the value of the time component of momentum from *mc* to the value mv_τ in the direction of time for a particle of mass *m*. This value for change in momentum $mc - mv_\tau$, is the momentum in the time dimension that is lost to the gravity field and which is restored to the time dimension when the particle completely escapes the gravity field

The value $mc - mv_\tau$ is readily calculated from equation (13) which can be rewritten as $v_\tau = \sqrt{c^2 - v_G{}^2}$ so that

$$mc - mv_\tau = m\left(c - \sqrt{c^2 - v_G{}^2}\right), \tag{14}$$

for coordinate systems where the time coordinate *t* is perpendicular to the direction of gravitational escape velocity. Using the first approximation for the case where $c >> v_G$, equation (14) $\sqrt{c^2 - v_G{}^2}$ reduces to $c - \frac{v_G{}^2}{2c}$ so $mc^2 - mcv_\tau \approx \frac{1}{2}mv_G{}^2$, the value Newtonian Physics recognizes as the gravitational potential energy of particle *m* at locations where the gravitational escape velocity is equal to v_G.

In Newtonian physics, for a point mass of mass *M*, the gravitational escape velocity v_G at a radial distance *r* from the center of the mass *M* based on a gravitational constant *G*, is

$$v_G = \sqrt{\frac{2GM}{r}}\ . \tag{15}$$

It is clear from equation (15) that as the radial distance r decreases, the gravitational velocity increases. There is a critical radius, called the Schwarzschild radius R, where the gravitational velocity equals the speed of light c. That is, when $R = r$, then $v_G = c$. Making these replacements into equation (15) yields

$$c = \sqrt{\frac{2GM}{R}}\ . \tag{16}$$

Combining equation (15) and equation (16) allows a definition of the magnitude of gravitational velocity, based on the ratio of the Schwarzschild radius R to the radial distance r, used throughout the remainder of this paper, that is,

$$v_G = c\sqrt{\frac{R}{r}}\ . \tag{17}$$

IV. THE FUNDAMENTAL PRINCIPLE AND MASS-ENERGY EQUIVALENCE

In 1905, Albert Einstein[4] derived an energy equivalent for mass by calculating the difference in energy (ΔL) between light in a local reference frame defined by local space coordinates (ξ, η, ς) and the same light in the reference frame defined by space coordinates (x, y, z). The difference in energy ΔL was set equal to the Newtonian value for kinetic energy of a mass to produce Einstein's value for mass-energy equivalence.

To obtain ΔL, Einstein defined a value L to represent energy of light in the local reference frame defined by local coordinates (ξ,η,ς). Using the principle of the constancy of light, Einstein calculated the difference in energy (ΔL) to be

$$\Delta L = L\left(\frac{1}{\sqrt{1 - {v_S}^2 / c^2}} - 1\right).^{5} \tag{18}$$

From equation (7), $\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - {v_S}^2 / c^2}}$), so that equation (18) reduces to $\Delta L = \frac{dt}{d\tau}L - L$. From equation (4), the velocity of light in the local reference frame defined by space coordinates (ξ,η,ς) is $v_\tau = c\frac{d\tau}{dt}$. In the reference frame defined by space coordinates (*x, y, z*) the velocity of light is $v_t = c\frac{dt}{dt} = c$. Therefore, equation (18) reduces further to

$$\Delta L = \frac{dt}{d\tau}L - L = \frac{v_t}{v_\tau}L - L = \frac{c}{v_\tau}L - L. \tag{19}$$

According to equation (19), when L represents the energy of light in the local reference frame defined by local coordinates (ξ,η,ς), the same light has the energy $\frac{c}{v_\tau}L$ in the reference frame defined by space coordinates (*x, y, z*) so that the energy of light in each reference frame is inversely proportional to the velocity of time (v_τ) in that reference frame.

Einstein assumes that $c >> v_S$ ("neglecting magnitudes of fourth and higher orders"), and thus simplifies equation (18) to

$$\Delta L \approx \frac{1}{2}\frac{L}{c^2}{v_S}^2. \tag{20}$$

Einstein compares this value to the Newtonian value for kinetic energy (E_K), $E_K = \frac{1}{2}mv_S{}^2$ to derive the energy equivalence of mass. That is, $\Delta L = E_K$ so that $\frac{1}{2}\frac{L}{c^2}v_S{}^2 = \frac{1}{2}mv_S{}^2$ and so that $L = mc^2$ the equation for the energy equivalence of mass, normally written in the form $E = mc^2$.

A. Calculating Mass-Energy Equivalence without using Approximations

Einstein's 1905 paper, written in the context of Newtonian physics, utilizes two "Newtonian" approximations. The approximation represented by equation (20) is used to calculate the difference in energy of light between two time frames. Einstein also implicitly utilizes the approximation represented by equation (12) to obtain the Newtonian value for kinetic energy resulting from the motion of a mass.

Einstein's method of calculating mass-energy equivalence can also be performed without approximations. In this case, the value for E_K, calculated from equation (11) is

$$E_K = mc^2 - mcv_\tau = mc^2 - mc\sqrt{c^2 - v_S{}^2} = mc^2\left(1 - \sqrt{1 - v_S{}^2/c^2}\right). \tag{21}$$

Setting, as did Einstein, $E_K = \Delta L$, and using equation (21) to give the value for E_K and equation (18) to give the value for ΔL yields

$$\Delta L = E_K \Rightarrow L\left(\frac{1}{\sqrt{1 - v_S{}^2/c^2}} - 1\right) = mc^2\left(1 - \sqrt{1 - v_S{}^2/c^2}\right). \tag{22}$$

Simplifying this equations leads to

$$L\left(\frac{1-\sqrt{1-v_S{}^2/c^2}}{\sqrt{1-v_S{}^2/c^2}}\right)=mc^2\left(1-\sqrt{1-v_S{}^2/c^2}\right), \tag{23}$$

and

$$L=mc^2\left(\frac{1-\sqrt{1-v_S{}^2/c^2}}{1-\sqrt{1-v_S{}^2/c^2}}\right)\sqrt{1-v_S{}^2/c^2}\,, \tag{24}$$

and

$$L=mc\sqrt{c^2-v_S{}^2}\,, \tag{24a}$$

and finally

$$L=mcv_\tau, \tag{25}$$

which is the speed of light c times the momentum of the mass in the dimension of time (mv_τ). When the local coordinates equal the reference coordinates, $v_\tau=c\dfrac{dt}{dt}=c$ and therefore $L=mc^2$, the same value obtained by Einstein.

Obtaining the value $L=mc^2$ when no approximations are used confirms that the value for kinetic energy in relativistic physics (i.e., without approximations) is equal to mc^2-mcv_τ, as set out in equation (12). Comparing equation (12) with equation (25) indicates that the energy equivalence of mass m in the reference frame defined by local coordinates (ξ,η,ς,τ) is mcv_τ so that kinetic energy, mc^2-mcv_τ, is the difference in the energy equivalence of mass m in the reference frame defined by coordinates (x, y, z, t) and the energy equivalence of mass m in the reference frame defined by local coordinates (ξ,η,ς,τ).

V. THE FUNDAMENTAL PRINCIPLE AND GENERAL RELATIVITY

General relativity extends special relativity by removing the requirement that motion between coordinates be *uniform*.[6] That is, the laws of physics detected by making measurements from any selected reference coordinates (*x, y, z, t*) will also hold true when measurements are made from any other selected local coordinates ($\xi, \eta, \varsigma, \tau$) even when accelerated with respect to the reference coordinates (*x, y, z, t*).

In the discussion above on gravity in Newtonian physics, gravitational acceleration was shown to result in a change of momentum in the time dimension, even without a corresponding change in the momentum in the space dimensions. Because gravitational escape velocity $\vec{v}_G$ affects the velocity of time in the same way velocity in space $\vec{v}_S$ affects the velocity of time, equation (1) can be modified to take into account the effects of gravity as follows:

$$c = \left|\vec{v}_\tau + \vec{v}_S + \vec{v}_G\right|. \tag{26}$$

As shown below, the Schwarzschild metric is just an alternative mathematical expression of equation (26).

A. Deriving the Schwarzschild Metric

When Karl Schwarzschild derived the Schwarzschild metric, describing the gravitational field of a point mass of mass M, he made three assumptions[7] that necessarily resulted in the Schwarzschild metric being mathematically equivalent to equation (26). First, when he described the gravitational field, he formed the components of the gravitational field using a gravitational constant, as is done in Newtonian physics.[8] As a result, the definitions of gravitational velocity v_G given in equations (15) and (17) are valid in the Schwarzschild metric. Second, he constrained the Schwarzschild metric to resolve to the Minkowski metric at infinite radial distance from the point mass,[9] i.e., when $v_G = 0$. As a result, the definitions of velocity of space v_S and the velocity of time v_τ in the Minkowski metric are incorporated into the Schwarzschild metric. Third, Schwarzschild followed Einstein's field equations in which, in accordance with what Einstein calls the laws of momentum and energy, gravity does not change the total momentum as calculated across the four dimensions of space-time.[10] Thus any increase in the momentum represented by an increase in v_G, must come at the expense of a corresponding decrease in v_S and/or v_τ. The result of these three assumptions is that the Schwarzschild metric, by necessity, conforms to equation (26).

The direction of gravitational velocity $\vec{v}_G$ is dependent upon the reference coordinates selected to express the Schwarzschild metric, and particularly upon the reference time coordinate. As will be seen below, with the appropriate selection of time coordinate, gravitational velocity $\vec{v}_G$ can be regarded as occurring in virtually any direction.

A very simple form of the Schwarzschild metric results when the direction of gravitational velocity $\vec{v}_G$ is treated as if it were in a direction orthogonal to the dimensions of time and space.

1. When gravitational velocity is orthogonal to time and space

For coordinates (i.e., reference frames) where $\vec{v}_\tau$, $\vec{v}_S$ and $\vec{v}_G$ are orthogonal with respect to each other then $\left|\vec{v}_S + \vec{v}_G + \vec{v}_\tau\right| = \sqrt{{v_S}^2 + {v_G}^2 + {v_\tau}^2}$ so that equation (26) can be rewritten as $c^2 = v_S{}^2 + v_G{}^2 + v_\tau{}^2$. From equation (4) $v_\tau = c\dfrac{d\tau}{dt}$, thus $c^2 = v_S{}^2 + v_G{}^2 + c^2\left(\dfrac{d\tau}{dt}\right)^2$. Solving for $\dfrac{d\tau}{dt}$ yields

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_S{}^2}{c^2} - \frac{v_G{}^2}{c^2}}\,. \tag{27}$$

From equation (17) $v_G = c\sqrt{\dfrac{R}{r}}$, so that equation (27) is equivalent to

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_S{}^2}{c^2} - \frac{R}{r}}\,. \tag{28}$$

Equation (28) can be rearranged as $d\tau^2 = dt^2 - dt^2\dfrac{v_S{}^2}{c^2} - dt^2\dfrac{R}{r}$ which can be simplified to

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R}{r}) - dt^2 v_S{}^2\,. \tag{29}$$

When using the spherical coordinate system $v_S{}^2 = \left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{d\theta}{dt}\right)^2 + \left(r^2\sin^2\theta\right)\left(\dfrac{d\varphi}{dt}\right)^2$.

Making the substitution into equation (29) yields a version of Schwarzschild metric in spherical coordinates measured from a reference frame where all dimensions are orthogonal:

$$c^2 d\tau^2 = \left(1 - \frac{R}{r}\right)c^2 dt^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right) d\varphi^2\,. \tag{30}$$

When using the Cartesian coordinate system, $v_S{}^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2$. When the point mass is located at the Cartesian coordinate (x_1, y_1, z_1), then $r=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$ Making the substitution into equation (29) yields the Schwarzschild metric in Cartesian coordinates measured from a reference frame where all dimensions are orthogonal.

$$c^2 d\tau^2 = c^2\left(1-\frac{R}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}\right)dt^2 - dx^2 - dy^2 - dz^2. \qquad (31)$$

When the center of the point mass is at the Cartesian coordinate $(0,0,0)$, equation (31) reduces to

$$c^2 d\tau^2 = c^2\left(1-\frac{R}{\sqrt{x^2+y^2+z^2}}\right)dt^2 - dx^2 - dy^2 - dz^2. \qquad (32)$$

It is possible to specifically define a new orthogonal dimension $\vec{g}$ so that $\frac{dg}{dt}=v_G=c\sqrt{\frac{R}{r}}$ and thus $dg^2=c^2dt^2\frac{R}{r}$. Making this substitution into equation (32) yields

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - dg^2. \qquad (33)$$

Making the same substitution in equation (30) yields another five dimensional form of the Schwarzschild metric,

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - \left(r^2\sin^2\theta\right)d\varphi^2 - dg^2. \qquad (34)$$

2. When gravitational velocity overlaps space-time

When the dimensions of gravity, space and time are not all orthogonal, then $\left|\vec{v}_S + \vec{v}_G + \vec{v}_\tau\right|^2 = v_S{}^2 + v_G{}^2 + v_\tau{}^2 + P$, where P is a value that results from overlap in the directional components of $\vec{v}_\tau$, $\vec{v}_S$ and $\vec{v}_G$. Equation (26) can therefore be rewritten as $c^2 = v_S{}^2 + v_G{}^2 + v_\tau{}^2 + P$. From equation (4) $v_\tau = c\dfrac{d\tau}{dt}$ and from equation (17) $v_G = c\sqrt{\dfrac{R}{r}}$; therefore, $c^2 = v_S{}^2 + c^2\dfrac{R}{r} + c^2\left(\dfrac{d\tau}{dt}\right)^2 + P$. Solving for $\dfrac{d\tau}{dt}$ yields

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_S{}^2}{c^2} - \frac{R}{r} - \frac{P}{c^2}}\,. \tag{35}$$

Equation (35) can be rearranged as $d\tau^2 = dt^2 - dt^2\dfrac{v_S{}^2}{c^2} - dt^2\dfrac{R}{r} - dt^2\dfrac{P}{c^2}$ which can be simplified as

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R}{r} - \frac{P}{c^2}) - dt^2 v_S{}^2\,. \tag{36}$$

When using the spherical coordinate system $v_S{}^2 = \left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{d\theta}{dt}\right)^2 + \left(r^2 \sin^2\theta\right)\left(\dfrac{d\varphi}{dt}\right)^2$. Making the substitution into equation (36) yields a very generic version of the Schwarzschild metric, that is

$$c^2 d\tau^2 = \left(1 - \frac{R}{r} - \frac{P}{c^2}\right)c^2 dt^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right)d\varphi^2\,. \tag{37}$$

a. The Painlevé-Gullstrand coordinates

Since gravitational acceleration is in a radial direction towards the center of mass M, it is perhaps most intuitive and useful to select reference coordinates so that gravitational velocity $\vec{v}_G$ is regarded as motion that occurs in a radial direction towards the center of mass M.[11]

Specifically, when using the spherical coordinate system $\left|\vec{v}_S\right|^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2$, where $\frac{dr}{dt}$ is the portion of the velocity through space in the radial direction. In this case, gravitational velocity v_G also occurs in the radial direction where $\left|\vec{v}_G\right| = c\sqrt{\frac{R}{r}}$. This means $\left|\vec{v}_G + \vec{v}_S\right|^2 = \left(c\sqrt{\frac{R}{r}} + \frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2$ and thus $\left|\vec{v}_\tau + \vec{v}_S + \vec{v}_G\right|^2 = \vec{v}_\tau + {v_S}^2 + {v_G}^2 + 2\frac{dr}{dt}c\sqrt{\frac{R}{r}}$ and therefore

$$P = 2\frac{dr}{dt}c\sqrt{\frac{R}{r}}. \tag{38}$$

Making this substitution of P into equation (37) yields

$c^2d\tau^2 = \left[1 - \frac{R}{r} - \frac{2(dr/dt)c\sqrt{R/r}}{c^2}\right]c^2dt^2 - dr^2 - r^2d\theta^2 - \left(r^2\sin^2\theta\right)d\varphi^2$ which reduces to the Schwarzschild metric expressed in the Painlevé-Gullstrand coordinates[12] as set out below:

$$c^2d\tau^2 = \left(1 - \frac{R}{r}\right)c^2dt^2 - dr^2 - r^2d\theta^2 - \left(r^2\sin^2\theta\right)d\varphi^2 - 2c\sqrt{\frac{R}{r}}drdt. \tag{39}$$

b. The original Schwarzschild coordinates

Karl Schwarzschild originally derived the Schwarzschild metric using the Schwarzschild coordinates as reference coordinates.[13] As measured by the reference frame for the Schwarzschild coordinates, the gravitational dimension only partially overlaps the space dimensions.

For the Schwarzschild coordinates $\left|\vec{v}_\tau + \vec{v}_G + \vec{v}_S\right|^2 = {v_\tau}^2 + {v_G}^2 + {v_S}^2 + \left(\frac{dr}{dt}\right)^2 \frac{R}{r-R}$ and thus for the Schwarzschild coordinates

$$P = \left(\frac{dr}{dt}\right)^2 \frac{R}{r-R}. \qquad (40)$$

Substituting the "P" value for the Schwarzschild coordinates into equation (37) yields $c^2 d\tau^2 = \left\{1 - \frac{R}{r} - \frac{(dr/dt)^2\left[R/(r-R)\right]}{c^2}\right\} c^2 dt^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right) d\varphi^2$ which reduces to $c^2 d\tau^2 = \left(1 - \frac{R}{r}\right) c^2 dt^2 - dr^2\left(1 + \frac{R}{r-R}\right) - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right) d\varphi^2$ and thus to the Schwarzschild metric expressed in Schwarzschild coordinates as set out below:

$$c^2 d\tau^2 = c^2 (1 - \frac{R}{r}) dt^2 - \frac{dr^2}{(1 - R/r)} - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right) d\varphi^2. \qquad (41)$$

B. Coordinate Transformations

When measurements are made from different reference frames in (spatial, time and/or gravitational) motion with respect to each other, it is necessary to perform a coordinate transformation (sometimes called a Lorentz transformation) in order to compare measurements made by the coordinates used in each reference frame. The existence of valid coordinate transformations between two reference frames means that measurements made from each reference frame are equally valid.

For example, a coordinate transformation between the orthogonal coordinates used in equation (30) and the non-orthogonal coordinates used in equation (36) can be accomplished by the well-known method[14] of using the same space coordinates $(r,\ \theta,\varphi)$ for both the orthogonal coordinates and the non-orthogonal coordinates and on that basis determining the relationship between the time coordinate for the orthogonal coordinates and the time coordinate for the non-orthogonal coordinates.

For example, let the time coordinate for the orthogonal coordinates be labeled t_O. Let the time coordinate for the non-orthogonal coordinates be labeled t_N. Since $c^2 d\tau^2$ is an invariant that appears in both equation (30) and equation (36), the following is true:

$$\begin{aligned} c^2 d\tau^2 &= \left(1-\frac{R}{r}\right)c^2 dt_O{}^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right)d\varphi^2 \\ &= \left(1-\frac{R}{r}-\frac{P}{c^2}\right)c^2 dt_N{}^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2\theta\right)d\varphi^2 . \end{aligned} \tag{42}$$

This means that $\left(1-\frac{R}{r}\right)c^2 dt_O{}^2 = \left(1-\frac{R}{r}-\frac{P}{c^2}\right)c^2 dt_N{}^2$ which results in the relationship between dt_O and dt_N being expressed as

$$dt_O = dt_N \sqrt{1-\frac{rP}{c^2(r-R)}} \tag{43}$$

and

$$dt_G = dt_N \sqrt{\frac{r-R}{r-R-rP/c^2}} . \tag{44}$$

To transform the orthogonal coordinates to the Schwarzschild coordinates, to the Painlevé-Gullstrand coordinates, or to any other version of the non-orthogonal coordinates, it is only necessary to use the appropriate value of P.

For example, let the time coordinate for the Schwarzschild coordinates be labeled t_S (i.e., $dt_S = dt_N$). From equation (40) above, $P = \left(\frac{dr}{dt_S}\right)^2 \frac{R}{r-R}$. Substituting the value of P for the Schwarzschild coordinates into equation (43) yields the following relationship between the Schwarzschild time coordinate and the orthogonal time coordinate:

$$dt_O = \sqrt{dt_S{}^2 - \frac{Rr}{c^2(r-R)^2} dr^2} . \tag{45}$$

VI. SPACE-TIME METRICS FOR MULTIPLE POINT MASSES

To derive a multiple body space-time metric, consider a substantial point with a velocity through space v_S and a velocity of local time v_τ measured with respect to a reference frame. The substantial point is in the midst of multiple bodies consisting of a mass M_1 with a Schwarzschild radius R_1 and a center located at a coordinate location (x_1, y_1, z_1), a mass M_2 with a Schwarzschild radius R_2 and a center located at a coordinate location (x_2, y_2, z_2), … and a mass M_n with a Schwarzschild radius R_n and a center located at a coordinate location (x_n, y_n, z_n). There is a gravitational velocity v_{G1} associated with mass M_1, a gravitational velocity v_{G2} associated with mass M_2 … and a gravitational velocity v_{Gn} associated with mass M_n.

The fundamental principle of relativity for this case can be expressed in an equation as

$$c = \left|\vec{v}_\tau + \vec{v}_S + \vec{v}_{G1} + \vec{v}_{G2} + ... + \vec{v}_{Gn}\right|. \quad (46)$$

Selecting the reference frame (assuming one exists) so that all velocities are orthogonal (i.e., so that $c^2 = v_\tau{}^2 + v_S{}^2 + v_{G1}{}^2 + v_{G2}{}^2 + ... + v_{Gn}{}^2$), and remembering $v_\tau = c\dfrac{d\tau}{dt}$ leads to

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_S{}^2}{c^2} - \frac{v_{G1}{}^2}{c^2} - \frac{v_{G2}{}^2}{c^2} - ... - \frac{v_{Gn}{}^2}{c^2}}. \quad (47)$$

Equation (47) can be rearranged to form the generic space-time metric for multiple point masses set out below:

$$c^2 d\tau^2 = c^2 dt^2 - dt^2 v_S{}^2 - dt^2 v_{G1}{}^2 - dt^2 v_{G2}{}^2 - ... - dt^2 v_{Gn}{}^2. \quad (48)$$

When using the Cartesian coordinate system, $v_S{}^2 = \dfrac{dx^2}{dt^2} + \dfrac{dy^2}{dt^2} + \dfrac{dz^2}{dt^2}$. From equation (17), $v_{Gi}{}^2 = c^2 \dfrac{R_i}{r_i}$. When the center of mass M_i is at the Cartesian coordinate $(x_i,\ y_i,\ z_i)$, $r_i = \sqrt{\left(x - x_i\right)^2 + \left(y - y_i\right)^2 + \left(z - z_i\right)^2}$. Making the appropriate substitutions into equation (48) yields the following multiple body metric tensor equation

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R_1}{r_1} - \frac{R_2}{r_2} - ... - \frac{R_n}{r_n}) - dx^2 - dy^2 - dz^2 \quad (49)$$

where

$$r_1 = \sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2 + \left(z - z_1\right)^2}$$

$$r_2 = \sqrt{\left(x - x_2\right)^2 + \left(y - y_2\right)^2 + \left(z - z_2\right)^2},$$

$$\ldots$$

$$r_n = \sqrt{\left(x - x_n\right)^2 + \left(y - y_n\right)^2 + \left(z - z_n\right)^2}\,.$$

Equation (49) sets out a multiple body metric described in a four dimensional space-time continuum. Utilizing the relationship $\frac{dg_i}{dt} = v_{Gi} = c\sqrt{\frac{R_i}{r_i}}$ and thus $dg_i^{\,2} = c^2 dt^2 \frac{R_i}{r_i}$ allows the multiple body metric to be described in an n+4 dimensional space-time-gravity continuum as follows:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - dg_1^{\,2} - dg_2^{\,2} - \ldots - dg_n^{\,2}\,. \qquad (50)$$

A. Non-Orthogonal Coordinates for Multiple Body Metrics

The space-time metric can also be derived when the gravitational velocities are non-orthogonal to each other and/or to the space and time dimensions.

In general, the multiple body metric for non-orthogonal gravitational velocities, can be derived from the equation $c^2 = \left|\vec{v}_S + \vec{v} + \vec{v}_{G1} + \vec{v}_{G2} + \ldots + \vec{v}_{Gn\tau}\right|^2 = v_S^{\,2} + v_\tau^{\,2} + v_{G1}^{\,2} + v_{G2}^{\,2} + \ldots + v_{Gn}^{\,2} + P$, where P is some value that is a result of the overlap of vectors. The relationship between dt and $d\tau$ that serves as the basis for the metric will have the form

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_S^{\,2}}{c^2} - \frac{v_{G1}^{\,2}}{c^2} - \frac{v_{G2}^{\,2}}{c^2} - \ldots - \frac{v_{Gn}^{\,2}}{c^2} - \frac{P^2}{c^2}}\,. \qquad (51)$$

The multiple body metric will have the form

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R_1}{r_1} - \frac{R_2}{r_2} - ... - \frac{R_n}{r_n} - \frac{P^2}{c^2}) - dx^2 - dy^2 - dz^2 . \qquad (52)$$

The relationship between the increment of the time coordinate of the orthogonal multiple body metric (dt_O) and the increment of the time coordinate of the non-orthogonal multiple body metric (dt_G) will be

$$dt_O = dt_G \sqrt{1 - \frac{P/c^2}{1 - R_1/r_1 - R_2/r_2 - ... - R_n/r_n}} . \qquad (53)$$

B. Example Use of a Multiple Body Metric

Consider the case of a first body M_1 around which a second body M_2 rotates (e.g., a planet rotating around a sun, or a moon rotating around a planet). The velocity of the second body M_2 is assumed to be much less than the speed of light. Let the center of the first body M_1 be located at coordinate (0, 0, 0), (i.e., $x_1 = y_1 = z_1 = 0$) and have a Schwarzschild radius R_1. Let the second body M_2 have a Schwarzschild radius R_2 and the location of the center of the second body M_2 be described by $x_2 = a \cos t$, $y_2 = b \sin t$, $z_2 = 0$, where a and b are constants. The two body metric tensor equation for these is

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - dg_1^{\ 2} - dg_2^{\ 2} . \qquad (54)$$

The two body metric for these two bodies in four dimensions is

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R_1}{r_1} - \frac{R_2}{r_2}) - dx^2 - dy^2 - dz^2 \qquad (55)$$

or

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R_1}{\sqrt{x^2 + y^2 + z^2}} - \frac{R_2}{\sqrt{(x - a\cos t)^2 + (y - b\sin t)^2 + (z)^2}}) - dx^2 - dy^2 - dz^2.$$

In order to add a third body that rotates around the first body M_1, all that is needed is to add a gravitation coordinate g_3 for the third body and include it in the metric. For example, add a third body M_3 with a Schwarzschild radius R_3 and let the location of the third body $\mathbf{M_3}$ be described by $x_3 = 0$, $y_3 = b_1 \sin (b_2 t + b_3)$, $z_3 = a_1 \cos (b_2 t + b_3)$, where a_1, b_1, b_2 and b_3 are all constants.

The three body metric for these two bodies in seven dimensions is

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - dg_1{}^2 - dg_2{}^2 - dg_3{}^3. \tag{56}$$

The three body metric for these three bodies in four dimension is

$$c^2 d\tau^2 = c^2 dt^2 (1 - \frac{R_1}{r_1} - \frac{R_2}{r_2} - \frac{R_3}{r_3}) - dx^2 - dy^2 - dz^2 \tag{57}$$

where

$$r_1 = \sqrt{x^2 + y^2 + z^2} \ ,$$

$$r_2 = \sqrt{(x - a\cos t)^2 + (y - b\sin t)^2 + (z)^2} \ ,$$

$$r_3 = \sqrt{x^2 + (y - b_1 \sin(b_2 t + b_3))^2 + (z - a_1 \sin(b_2 t + b_3))^2} \ .$$

A metric for describing the gravity field in a vacuum for any system (e.g., an entire planetary system including sun, planets comets, moons, etc.) can be constructed by determining the locations of all the bodies to be included in the metric and then, based on the locations, forming a multiple body metric as set out in equation (48).

VII. CONSERVATION OF MOMENTUM AND ENERGY

The energy equivalence E of an isolated system consisting of mass m is given by the well known equation[15]

$$E = mc^2 . \tag{58}$$

This value for energy equivalence E can be apportioned, as suggested by the discussion above on kinetic energy as follows:

$$E = mc^2 = mcv_\tau + (mc^2 - mcv_\tau) = mcv_\tau + mc\left(c - \sqrt{c^2 - v_S{}^2}\right). \tag{59}$$

In equation (59), the value $(mc^2 - mcv_\tau)$ and the equivalent value $mc\left(c - \sqrt{c^2 - v_S{}^2}\right)$ represent kinetic energy E_K. The value mcv_τ represents the energy equivalence of mass m in local reference frames in which τ is the time coordinate.

Equation (59) explains how the law of energy conservation holds true for particles in motion. A particle of mass m in uniform (i.e., unaccelerated) motion with respect to a reference frame has a kinetic energy associated with it. The kinetic energy is a component of the total energy equivalence of mass m, i.e., $E = mc^2$, as measured from that reference frame. If another reference frame is chosen from which to make measurements, the amount of kinetic energy as measured from the new reference frame may change, but the total energy equivalence of the particle as measured from the new reference frame will still be $E = mc^2$, The apportionment of energy as set out in Equation (59) is based on kinetic energy as it appears in "Newtonian" physics. The equation $c^2 = v_\tau^{\ 2} + v_S^{\ 2}$, suggests a more mathematically simple way to apportion the energy equivalence $E = mc^2$, that is

$$E = mc^2 = mv_\tau^{\ 2} + mv_S^{\ 2}. \tag{60}$$

Defining a spatial component of energy equivalence E_S so that $E_S = mv_S^{\ 2}$ and defining a local time component of energy equivalence E_τ so that $E_\tau = mv_\tau^{\ 2}$ allows equation (11) to be rewritten as

$$E = mc^2 = E_\tau + E_S. \tag{61}$$

The momentum of a particle of mass *m* can also be apportioned, based on Equation (1), $c = \left|\vec{v}_\tau + \vec{v}_S\right|$, as

$$mc = \left|m\vec{v}_\tau + m\vec{v}_S\right|. \tag{62}$$

Equation (62) indicates that for uniform motion, the total momentum, taking into account momentum in the time dimension and in the space dimensions, is invariant always being equal to mc.

Equations (59) through (62) make it clear why the fundamental principle, as represented by equation (1), is true for special relativity. That is, the fundamental principle is merely a restatement of the laws of conservation of momentum and energy across the four dimensions of time and space.

A. Momentum and Energy in the Schwarzschild Metric

Equations illustrating conservation of momentum and energy can be generated for the Schwarzschild metric similar to those generated above for special relativity and the Minkowski metric.

Equation (26), $c = \left|\vec{v}_\tau + \vec{v}_S + \vec{v}_G\right|$, can be rewritten as

$$mc = \left|m\vec{v}_\tau + m\vec{v}_S + m\vec{v}_G\right|, \tag{63}$$

indicating that within the Schwarzschild metric, the total momentum for a particle of mass *m*, taking into account time, space and gravity components, is invariant, always being equal to mc.

For coordinates (i.e. reference frames) in which the time dimension and the gravity dimension are orthogonal to each other and to the space dimensions, equation (26) is equivalent to $c^2 = v_S{}^2 + v_G{}^2 + v_\tau{}^2$, which indicates the equivalent energy of a particle of mass *m* can be apportioned as

$$E = mc^2 = mv_\tau{}^2 + mv_S{}^2 + mv_G{}^2. \tag{64}$$

Defining a gravitational component of energy E_G so that $E_G = mv_G{}^2$ leads to

$$E = mc^2 = E_\tau + E_S + E_G. \tag{65}$$

For non-orthogonal coordinates, equation (26) is equivalent to $c^2 = v_K{}^2 + v_G{}^2 + v_\tau{}^2 + P$. In this case, $E = mc^2 = mv_K{}^2 + mv_G{}^2 + mv_\tau{}^2 + mP$ and thus $E = mc^2 = E_\tau + E_K + E_G + E_P$ where $E_P = mP$, E_P representing a component of energy resulting from the overlap of dimensions. Thus, equation (26), which is a mathematical equivalent of the Schwarzschild metric, is a restatement of the laws of conservation of momentum and energy across the four dimensions of time and space, taking into account the presence of a point mass of mass *M*.

Similar equations illustrating conservation of momentum and energy within systems with multiple bodies can be written, i.e., $mc = \left| m\vec{v}_\tau + m\vec{v}_S + m\vec{v}_{G1} + m\vec{v}_{G2} + ... + m\vec{v}_{Gn} \right|$, $mc^2 = mv_\tau{}^2 + mv_S{}^2 + mv_{G1}{}^2 + mv_{G2}{}^2 + ... + mv_{Gn}{}^2$, and so on.

VIII. THE DISCONTINUITY IN SPACE-TIME GEOMETRY

The conservation of momentum and energy discussed above provide a physical explanation for why velocity in space can never exceed the speed of light. In equation (60), kinetic energy, $E_K = mc^2 - mcv_\tau = mc\left(c - \sqrt{c^2 - v_S{}^2}\right)$, is a component of the energy equivalence of matter *m* and thus cannot exceed $E = mc^2$. When $v_S = c$, then $E_K = mc^2$, indicating v_S has reached its maximum value because the entire energy equivalence of mass *m* is used up as kinetic energy.

Likewise, equation (58) indicates that the momentum in the space dimension mv_S cannot exceed mc and equation (61) indicates the energy component, $E_S = mv_S{}^2$, can never exceed the energy equivalence of matter $E = mc^2$. Space-time locations where $mc \neq \left| m\vec{v}_\tau + m\vec{v}_S \right|$, $E = mc^2 \neq mcv_\tau + mc(c - v_\tau)$ and $E = mc^2 \neq mv_\tau{}^2 + mv_S{}^2$ are undefined in the Minkowski metric and are thus incompatible with special relativity where motion between the reference frame for coordinates $(\xi, \eta, \varsigma, \tau)$ and the reference frame for coordinates (*x, y, z, t*) is uniform.

The boundary in the Minkowski metric at $v_S = c$ (i.e., where $v_\tau = 0$, $c\frac{d\tau}{dt} = 0$ and $\frac{dt}{d\tau} = \infty$) can be referred to as a time singularity because at the boundary an infinite passage of reference time t is required for any finite passage of local time τ.

A. The Discontinuity in the Schwarzschild Metric

Conservation of momentum and energy explain why in the Schwarzschild metric neither v_S nor v_G can be greater than c.

In general, when for a particle of mass m, $|\vec{v}_G + \vec{v}_S| = c$, then $v_\tau = 0$ indicating that kinetic energy and gravitational energy have completely used up the energy equivalence of mass m.

The boundary in the Schwarzschild metric at $|\vec{v}_G + \vec{v}_S| = c$ (i.e., when $v_\tau = 0$, $c\frac{d\tau}{dt} = 0$ and $\frac{dt}{d\tau} = \infty$) is often referred to as a time singularity because at the boundary an infinite passage of reference time t is required for any finite passage of local time τ.

The boundary for the discontinuity in the multiple body metrics also occurs at $v_\tau = 0$, i.e., where $c = |\vec{v}_S + \vec{v}_{G1} + \vec{v}_{G2} + ... + \vec{v}_{Gn}|$.

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[8] See K. Schwarzschild, §4 (Schwarzschild's integration constant α incorporates a gravitational constant).

[9] See K. Schwarzschild, §1.

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