One More Step Towards Generalized Graph-Based Weakly Relational Domains

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Abstract. This paper proposes to extend graph-based weakly relational domains to a generalized relational context. Using a new definition of coherence, we show that the definition of a normal form for this domain is simplified. A transitive closure algorithm for combined relations is constructed and a proof of its correctness is given. Using the observed similarity between transitive closure of a combined relation and the normal form closure of a graph-based weakly relational domain, we extract a mathematical property that a relational abstract domain must satisfy in order to allow us to use an algorithm with the same form as the transitive closure algorithm to compute the normal form of a graph-based weakly relational domain.

Keywords. Abstract interpretation, Transitive closure, Weakly-relational domains, Verification

Introduction

With the accelerating invasion of *the real world* by computing machines, we increasingly depend on the correctness of the programs they execute. Since manual verification requires a significant amount of resources, we must resort to automatic verification.

Abstract interpretation is a mathematical model that can be used for software verification [1,2]. Graph-based relational abstract domains allow to apply simple, relational abstract domains to composite systems. For a motivating example and references to related work, see [3]. In this paper we explore to what extent graph-based relational abstract domains can be generalized.

To this end we first construct a transitive closure algorithm for combined relations[1]. We next formulate a constraint system in a general, relational context and propose a mathematical property that I believe will allow to adapt the transitive closure algorithm to construct a normal form closure algorithm for constraint systems. The normal form closure algorithm is an essential component to construct a graph-based weakly-relational domain.

[1] For a different algorithm for computing the transitive closure of combined relations, see [4].

1. A Recursive Transitive Closure Algorithm

1.1. Preliminaries

This section briefly reviews the basic definitions of relations. Readers familiar with relations may want to skip to definition 1.7.

Definition 1.1 (Relation)**.** *For two sets $\mathbb{A}$ and $\mathbb{B}$, every subset of $\mathbb{A} \times \mathbb{B}$ represents a* relation from $\mathbb{A}$ to $\mathbb{B}$*. A relation from $\mathbb{A}$ to $\mathbb{A}$ is a relation on $\mathbb{A}$.*

Definition 1.2 (Range and domain)**.** *For a relation $\mathbf{R}$ from a set $\mathbb{A}$ to a set $\mathbb{B}$, we define the* range *of $\mathbf{R}$, denoted as $\mathcal{R}\mathbf{R}$, as the set of elements $b \in \mathbb{B}$ for which an element $a \in \mathbb{A}$ can be found such that $(a, b) \in \mathbf{R}$. Similarly, we define the* domain *of $\mathbf{R}$, denoted as $\mathcal{D}\mathbf{R}$, as the set of elements $a \in \mathbb{A}$ for which an element $b \in \mathbb{B}$ can be found such that $(a, b) \in \mathbf{R}$.*

Relations will be denoted with a bold face font while sets that act as ranges or domains of relations will be denoted with a blackboard font.

Definition 1.3 (Join operation)**.** *Given three sets, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, a relation $\mathbf{R}_{\mathbb{A},\mathbb{B}}$ from $\mathbb{A}$ to $\mathbb{B}$ and a relation $\mathbf{R}_{\mathbb{B},\mathbb{C}}$ from $\mathbb{B}$ to $\mathbb{C}$. Applying the* join operation $\bullet$ *on the relations $\mathbf{R}_{\mathbb{A},\mathbb{B}}$ and $\mathbf{R}_{\mathbb{B},\mathbb{C}}$ results in a relation $\mathbf{R}_{\mathbb{A},\mathbb{B}} \bullet \mathbf{R}_{\mathbb{B},\mathbb{C}}$ from $\mathbb{A}$ to $\mathbb{C}$ where for each $a \in \mathbb{A}$ and $c \in \mathbb{C}$ we define that $(a, c) \in \mathbf{R}_{\mathbb{A},\mathbb{B}} \bullet \mathbf{R}_{\mathbb{B},\mathbb{C}}$* iff *an element $b \in \mathbb{B}$ can be found such that $(a, b) \in \mathbf{R}_{\mathbb{A},\mathbb{B}}$ and $(b, c) \in \mathbf{R}_{\mathbb{B},\mathbb{C}}$.*

Definition 1.4 (Relation exponentiation)**.** *For a natural number n, a set $\mathbb{A}$ and a relation $\mathbf{R}$ on $\mathbb{A}$ the relation $\mathbf{R}^n$ is defined as*

$$\mathbf{R}^n \triangleq \begin{cases} \mathbf{1}_{\mathbb{A}} & n = 0 \\ \mathbf{R}^{n-1} \bullet \mathbf{R} & n \geq 1 \end{cases} \tag{1}$$

where $\mathbf{1}_{\mathbb{A}}$ denotes the identity relation on $\mathbb{A}$, defined as $\mathbf{1}_{\mathbb{A}} = \{(a, a) | a \in \mathbb{A}\}$.

Definition 1.5 (Transitive closure)**.** *For a set $\mathbb{A}$ and a relation $\mathbf{R}$ on $\mathbb{A}$, the* transitive closure $\mathbf{R}^\star$ *of $\mathbf{R}$ is defined as*

$$\mathbf{R}^\star \triangleq \bigcup_{s=0}^{+\infty} \mathbf{R}^s \tag{2}$$

Note that this operation is idempotent:

Lemma 1.1. *For every relation $\mathbf{R}$, we have $\mathbf{R}^\star = (\mathbf{R}^\star)^\star$.*

Definition 1.6 (Transitively closed)**.** *A relation $\mathbf{R}$ is* transitively closed *iff $\mathbf{R} = \mathbf{R}^\star$.*

Since the transitive closure operation is idempotent, the transitive closure of any relation is transitively closed.

Definition 1.7 (Combined relation)**.** *For*

- *a finite set of indices $L = \{j_1, j_2, \ldots, j_n\}$ with $n \in \mathbb{N}$,*

- *a set* $\mathbb{S}_i$ *for each index* $i \in L$ *where all sets* $\mathbb{S}_i$ *are mutually disjoint, i.e.* $\mathbb{S}_i \cap \mathbb{S}_j = \emptyset$ *for each pair of indices* $(i, j) \in L^2$ *and*
- *a relation* $\mathbf{P}_{i,j} \subseteq \mathbb{S}_i \times \mathbb{S}_j$ *for each pair of indices* $(i, j) \in L^2$

the combined relation $\mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *is the union of the relations on the pairs of indices, i.e.*[2]

$$\mathbf{C}(L, \mathbb{S}, \mathbf{P}) \triangleq \bigsqcup_{(i,j)}^{L^2} \mathbf{P}_{i,j} \tag{3}$$

a relation on $\bigsqcup_{i}^{L} \mathbb{S}_i$.

If $\mathbf{p}$ is a combined relation with $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$ then we will also use the notation $\mathbf{p}_{i,j}$ to denote $\mathbf{P}_{i,j}$ for pairs of indices $(i, j) \in L^2$.

The sets and relations may be parametrised and infinite (or both).

1.2. Problem Statement

Problem 1.1. *Given a combined relation* $\mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *and an algorithm to compute the result of the operations (for every* $(i, j, k) \in L^3$*)*

- $\mathbf{R} \bullet \mathbf{Q}$ *for relations* $\mathbf{R}$ *from* $\mathbb{S}_i$ *to* $\mathbb{S}_j$ *and* $\mathbf{Q}$ *from* $\mathbb{S}_j$ *to* $\mathbb{S}_k$ *,*
- $\mathbf{R} \cup \mathbf{Q}$ *for relations* $\mathbf{R}$ *and* $\mathbf{Q}$ *from* $\mathbb{S}_i$ *to* $\mathbb{S}_j$ *and*
- $\mathbf{R}^\star$ *for a relation* $\mathbf{R}$ *on* $\mathbb{S}_i$

we want to find an efficient algorithm that computes a representation of the transitive closure $\mathbf{C}(L, \mathbb{S}, \mathbf{P})^\star$ *of the combined relation.*

1.3. Solution

This section provides a complete description of the main ideas that lead to a novel solution of the problem described in section 1.2. The main result of this section is summarized by algorithm 1. Technical details are deferred to the appendix.

Since an algorithm to compute the union and join of a combined relation can easily be constructed from the union operation and join operation of the constituent relations, the resulting algorithm to compute the transitive closure of the combined relation will allow to use the combined relation itself as a constituent relation for a larger problem. Our strategy will therefore be to first solve the case with only two indices and to subsequently apply the solution recursively.

1.3.1. The Special Case of Two Indices

Let us first consider the case with only two indices, $L = \{i, j\}$. To simplify the notation we use the shorthands $\mathbb{I}$ for $\mathbb{S}_i$ and $\mathbb{J}$ for $\mathbb{S}_j$. To analyse the problem, we will make use of the restriction of a relation:

[2] $\sqcup$ denotes the disjoint union operation. For a triple of sets $(\mathbb{A}, \mathbb{B}, \mathbb{C})$ the statement $\mathbb{A} \sqcup \mathbb{B} = \mathbb{C}$ is equivalent to $(\mathbb{A} \cup \mathbb{B} = \mathbb{C}) \wedge (\mathbb{A} \cap \mathbb{B} = \emptyset)$.

Definition 1.8 (Restriction)**.** *For a relation* $\mathbf{R}$ *from a set* $\mathbb{A}$ *to a set* $\mathbb{B}$*, we define the restriction* $\mathbf{R}|^{\mathbb{C}}_{\mathbb{D}}$ *of* $\mathbf{R}$ *to* $(\mathbb{C}, \mathbb{D})$ *with* $\mathbb{C} \subseteq \mathbb{A}$ *and* $\mathbb{D} \subseteq \mathbb{B}$ *as* $\mathbf{R} \cap \mathbb{C} \times \mathbb{D}$*.*

Note that any relation can be decomposed into a disjoint union of relations on a partition of its domain and range:

Lemma 1.2 (Disjunct union decomposition)**.** *For a relation* $\mathbf{R}$ *from a set* $\mathbb{A}$ *to a set* $\mathbb{B}$*, subsets* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *of* $\mathbb{A}$ *that partition*[3] $\mathbb{A}$ *and subsets* $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *of* $\mathbb{B}$ *that partition* $\mathbb{B}$ *we have* $\mathbf{R} = \mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_1} \sqcup \mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_2} \sqcup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_1} \sqcup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_2}$.

Let us use the more succinct notation $\mathbf{K}$ for $\mathbf{C}(L, \mathbb{S}, \mathbf{P})^\star$, the transitive closure of the combined relation. Since $\mathbf{K}$ must be defined through its constituent relations we will search for an expression for the relations $\mathbf{K}|^{\mathbb{I}}_{\mathbb{I}}$, $\mathbf{K}|^{\mathbb{I}}_{\mathbb{J}}$, $\mathbf{K}|^{\mathbb{J}}_{\mathbb{I}}$ and $\mathbf{K}|^{\mathbb{J}}_{\mathbb{J}}$ in terms of $\mathbf{P}_{i,i}$, $\mathbf{P}_{i,j}$, $\mathbf{P}_{j,i}$ and $\mathbf{P}_{j,j}$ so that lemma 1.2 allows us to use these relations as the constituent relations of the combined relation $\mathbf{K}$:

$$\mathbf{K} = \mathbf{K}|^{\mathbb{I}}_{\mathbb{I}} \sqcup \mathbf{K}|^{\mathbb{I}}_{\mathbb{J}} \sqcup \mathbf{K}|^{\mathbb{J}}_{\mathbb{I}} \sqcup \mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} = \bigsqcup_{(a,b)}^{\{\mathbb{I},\mathbb{J}\}^2} \mathbf{K}|^a_b \tag{4}$$

To find these expressions, note first that the following lemma provides an alternative way to characterise a relation as transitively closed:

Lemma 1.3. *For a relation* $\mathbf{R}$ *on a set* $\mathbb{A}$*, we have* $\mathbf{R}^\star = \mathbf{R}$ *iff* $\mathbf{R} \bullet \mathbf{R} = \mathbf{R}$ *and* $\mathbf{1}_{\mathbb{A}} \subseteq \mathbf{R}$*.*

So when is $\mathbf{K}$ transitively closed? Let us try to use lemma 1.3 on $\mathbf{K}$ to determine the required constraints:

$$\begin{aligned} \mathbf{K} \bullet \mathbf{K} &= \Big(\bigsqcup_{(a,b)}^{\{\mathbb{I},\mathbb{J}\}^2} \mathbf{K}|^a_b\Big) \bullet \Big(\bigsqcup_{(a,b)}^{\{\mathbb{I},\mathbb{J}\}^2} \mathbf{K}|^a_b\Big) \\ &= \bigcup_{(a,b,c,d)}^{\{\mathbb{I},\mathbb{J}\}^4} \mathbf{K}|^a_b \bullet \mathbf{K}|^c_d \end{aligned} \tag{5}$$

In the last transition we have made use of

Lemma 1.4. *The join operation* $\bullet$ *distributes over the union operation.*

Since $\mathbb{I} \cap \mathbb{J} = \emptyset$ and the join $\mathbf{R} \bullet \mathbf{Q}$ of relations for which $\mathcal{R}\mathbf{R} \cap \mathcal{D}\mathbf{Q} = \emptyset$ is the empty relation, we can eliminate the terms for which $b \neq c$ so that the expression simplifies to

$$\mathbf{K} \bullet \mathbf{K} = \bigcup_{(a,f,d)}^{\{\mathbb{I},\mathbb{J}\}^3} \mathbf{K}|^a_f \bullet \mathbf{K}|^f_d \tag{6}$$

a union of eight relations. If we now compare this expression to expression (4) of $\mathbf{K}$ obtained through lemma 1.2 we see that each relation in our expression for $\mathbf{K}$ can be

[3] a pair of sets $(\mathbb{A}, \mathbb{B})$ partitions a set $\mathbb{C}$ iff $\mathbb{A} \sqcup \mathbb{B} = \mathbb{C}$

identified with the union of two relations in our expression for $\mathbf{K} \bullet \mathbf{K}$ since the sets $\mathbb{I}$ and $\mathbb{J}$ partition $\mathbb{I} \cup \mathbb{J}$:

$$\begin{aligned}
\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} &= \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \cup \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} \\
\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} &= \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \cup \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \\
\mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} &= \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} \cup \mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \\
\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} &= \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \cup \mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}
\end{aligned} \tag{7}$$

Note that

Lemma 1.5. *For any transitively closed relation $\mathbf{R}$ on a set $\mathbb{A}$, every restriction $\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}$ with $\mathbb{B} \subseteq \mathbb{A}$ is transitively closed as well: $\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} = (\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}})^{\star}$.*

Since we will choose $\mathbf{K}$ such that it is transitively closed, lemma 1.3 and 1.5 allow us to rewrite $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ as $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and $\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$ as $\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$.

Note also that

Lemma 1.6. *The join operation $\bullet$ is associative.*

Since $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ is transitively closed, we know that $\mathbf{1}_{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and thus, using lemma 1.4, that $\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}$. The second equation in (7) further implies that $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}$. Combining both gives $\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} = \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}$. In the same way we can derive that $\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} = \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$. We can therefore conclude that $\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}$ has the form

$$\mathbf{K}|_{\mathbb{J}}^{\mathbb{I}} = \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{Y}_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \tag{8}$$

with $\mathbf{Y}_{\mathbb{J}}^{\mathbb{I}}$ a relation that must yet be determined. It can readily verified that the combination of lemma 1.3, 1.5 and 1.6 ensures that a relation of this form satisfies the second equation of (7).

Similarly, we can derive that $\mathbf{K}|_{\mathbb{I}}^{\mathbb{J}}$ has the form

$$\mathbf{K}|_{\mathbb{I}}^{\mathbb{J}} = \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{Y}_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \tag{9}$$

The remaining constraints of (7) that represent the relations on $\mathbb{I}$ and $\mathbb{J}$ can now be written as

$$\begin{aligned}
\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} &= \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \cup (\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{Y}_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{Y}_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}) \\
\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} &= \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \cup (\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{Y}_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{Y}_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}})
\end{aligned} \tag{10}$$

Because $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and $\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$ are transitively closed, we have $\mathbf{1}_{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and $\mathbf{1}_{\mathbb{J}} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$ so that we can derive from equation (10) that $\mathbf{Y}_{\mathbb{J}}^{\mathbb{I}} \bullet \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}} \bullet \mathbf{Y}_{\mathbb{I}}^{\mathbb{J}} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and $\mathbf{Y}_{\mathbb{I}}^{\mathbb{J}} \bullet \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}} \bullet \mathbf{Y}_{\mathbb{J}}^{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$.

Since we require that $\mathbf{K} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})^{\star}$ we must have $\mathbf{C}(L, \mathbb{S}, \mathbf{P}) \subseteq \mathbf{K}$ and by decomposing the relation on disjoint ranges and domains we see that $\mathbf{P}_{i,i} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$, $\mathbf{P}_{i,j} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{I}}$, $\mathbf{P}_{j,i} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{J}}$ and $\mathbf{P}_{j,j} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$. Combining this information with the equations

(8) and (9) further gives $\mathbf{P}_{i,j} \subseteq \mathbf{Y}^{\mathbb{I}}_{\mathbb{J}}$ and $\mathbf{P}_{j,i} \subseteq \mathbf{Y}^{\mathbb{J}}_{\mathbb{I}}$. Based on the derived forms (8), (9) and (10), and the fact that we expect that $\mathbf{K}$ is the smallest relation (w.r.t. inclusion) that satisfies the constraints we therefore propose

Theorem 1.1. *If we choose the expressions*

$$\begin{aligned}
\mathbf{K}|^{\mathbb{I}}_{\mathbb{I}} &= (\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}^{\star}_{j,j} \bullet \mathbf{P}_{j,i})^{\star} \\
\mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} &= (\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star} \\
\mathbf{K}|^{\mathbb{I}}_{\mathbb{J}} &= \mathbf{K}|^{\mathbb{I}}_{\mathbb{I}} \bullet \mathbf{P}_{i,j} \bullet \mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} \\
\mathbf{K}|^{\mathbb{J}}_{\mathbb{I}} &= \mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} \bullet \mathbf{P}_{j,i} \bullet \mathbf{K}|^{\mathbb{I}}_{\mathbb{I}}
\end{aligned} \tag{11}$$

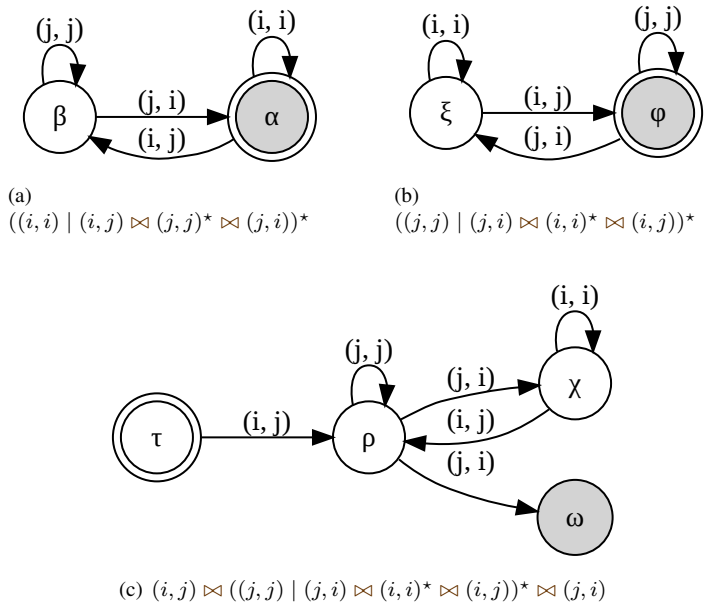
for $\mathbf{K}|^{\mathbb{I}}_{\mathbb{I}}$, $\mathbf{K}|^{\mathbb{I}}_{\mathbb{J}}$, $\mathbf{K}|^{\mathbb{J}}_{\mathbb{I}}$ *and* $\mathbf{K}|^{\mathbb{J}}_{\mathbb{J}}$ *then* $\mathbf{K}|^{\mathbb{I}}_{\mathbb{I}} \sqcup \mathbf{K}|^{\mathbb{I}}_{\mathbb{J}} \sqcup \mathbf{K}|^{\mathbb{J}}_{\mathbb{I}} \sqcup \mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})^{\star}$.

Proof. The expressions for $\mathbf{K}|^{\mathbb{I}}_{\mathbb{J}}$ and $\mathbf{K}|^{\mathbb{J}}_{\mathbb{I}}$ satisfy the constraints (8) and (9) with $\mathbf{Y}^{\mathbb{I}}_{\mathbb{J}} = \mathbf{P}_{i,j}$ and $\mathbf{Y}^{\mathbb{J}}_{\mathbb{I}} = \mathbf{P}_{j,i}$.

To show that the expression for $\mathbf{K}|^{\mathbb{I}}_{\mathbb{I}}$ satisfies equation (10) it suffices to show that $\mathbf{Y}^{\mathbb{I}}_{\mathbb{J}} \bullet \mathbf{K}|^{\mathbb{J}}_{\mathbb{J}} \bullet \mathbf{Y}^{\mathbb{J}}_{\mathbb{I}} \subseteq \mathbf{K}|^{\mathbb{I}}_{\mathbb{I}}$, or equivalently, that $\mathbf{P}_{i,j} \bullet (\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star} \bullet \mathbf{P}_{j,i} \subseteq (\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}^{\star}_{j,j} \bullet \mathbf{P}_{j,i})^{\star}$. Notice that by expanding the definition of the transitive closure operation and by distributing the join operation over the union operations, we can write each side of the expression as a union of expressions of the form $\mathbf{P}_{a_1,b_1} \bullet \mathbf{P}_{a_2,b_2} \bullet \ldots \bullet \mathbf{P}_{a_n,b_n}$ where the sequence of subscripts $(a_s, b_s) \in \{i, j\}$ with $s \in 1..n$ must obey specific rules. We can consider the possible pairs (i, i), (i, j), (j, i) and (j, j) as the four symbols of an alphabet and the sequences of symbols that correspond to expressions that are included in the union as the words of a language. To derive the language rules from the expression, note that the relation join operation $\bullet$ corresponds to the concatenation operation $\bowtie$ for words, the relation union operation $\cup$ corresponds to the alternation operation $\mid$ for words, while the transitive closure operation results in the *Kleene star* operation $\star$ that designates a word that is replicated a number of times. We can thus represent all the words for the language for the expression $(\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}^{\star}_{j,j} \bullet \mathbf{P}_{j,i})^{\star}$ as the regular expression $((i, i) \mid (i, j) \bowtie (j, j)^{\star} \bowtie (j, i))^{\star}$. We can then construct a finite deterministic automaton that recognizes the words from this language with well-known techniques (see figure 1(a)). We can similarly construct a regular expression $((j, j) \mid (j, i) \bowtie (i, i)^{\star} \bowtie (i, j))^{\star}$ and automaton (figure 1(b)) for the expression $(\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star}$. This automaton is symmetric to the automaton for $(\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}^{\star}_{j,j} \bullet \mathbf{P}_{j,i})^{\star}$: the only difference is that the initial and final (accepting) states are swapped.

The regular expression for $\mathbf{P}_{i,j} \bullet (\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star} \bullet \mathbf{P}_{j,i}$ can be constructed from the regular expression for $(\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star}$ by adding the prefix (i, j) and suffix (j, i): $(i, j) \bowtie ((j, j) \mid (j, i) \bowtie (i, i)^{\star} \bowtie (i, j))^{\star} \bowtie (j, i)$. This results in an automaton (figure 1(c)) that includes the main states of the automaton for $(\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star}$ as a sub-automaton. It can then be seen that any path from the initial to the final state through the automaton for $\mathbf{P}_{i,j} \bullet (\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}^{\star}_{i,i} \bullet \mathbf{P}_{i,j})^{\star} \bullet \mathbf{P}_{j,i}$ corresponds to a word for which a path in the automaton for $(\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}^{\star}_{j,j} \bullet \mathbf{P}_{j,i})^{\star}$ exists that corresponds to the same word: it suffices to require that the first edge is an edge that emits (i, j) and the last edge is an edge that emits (j, i). All expressions that occur in the

union expansion of $\mathbf{P}_{i,j} \bullet (\mathbf{P}_{j,j} \cup \mathbf{P}_{j,i} \bullet \mathbf{P}_{i,i}^{\star} \bullet \mathbf{P}_{i,j})^{\star} \bullet \mathbf{P}_{j,i}$ therefore also occur in the union expansion of $(\mathbf{P}_{i,i} \cup \mathbf{P}_{i,j} \bullet \mathbf{P}_{j,j}^{\star} \bullet \mathbf{P}_{j,i})^{\star}$. We can thus conclude that the expression for $\mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ satisfies equation (10). We can similarly show that $\mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$ satisfies equation (10).



(a) $((i,i) \mid (i,j) \bowtie (j,j)^{\star} \bowtie (j,i))^{\star}$

(b) $((j,j) \mid (j,i) \bowtie (i,i)^{\star} \bowtie (i,j))^{\star}$

(c) $(i,j) \bowtie ((j,j) \mid (j,i) \bowtie (i,i)^{\star} \bowtie (i,j))^{\star} \bowtie (j,i)$

Figure 1. Finite state automata for the recognition of the specified regular expressions. The initial state is indicated by a double border, while the final (accepting) state is indicated by a greyed background.

Since $\mathbf{1}_{\mathbb{I}} \subseteq \mathbf{K}|_{\mathbb{I}}^{\mathbb{I}}$ and $\mathbf{1}_{\mathbb{J}} \subseteq \mathbf{K}|_{\mathbb{J}}^{\mathbb{J}}$ we also know that $\mathbf{1}_{\mathbb{I} \cup \mathbb{J}} \subseteq \mathbf{K}$, thereby satisfying the second requirement for lemma 1.3.

Since the resulting $\mathbf{K}$ satisfies the required constraints that ensure it is transitively closed by lemma 1.3 and $\mathbf{C}(L, \mathbb{S}, \mathbf{P}) \subseteq \mathbf{K}$, we have $\mathbf{C}(L, \mathbb{S}, \mathbf{P})^{\star} \subseteq \mathbf{K}$. Since for every edge $k \in \mathbf{K}$ a path with edges in $\mathbf{P}_{i,i} \sqcup \mathbf{P}_{i,j} \sqcup \mathbf{P}_{j,i} \sqcup \mathbf{P}_{j,j}$ with the same endpoints must exist by construction, we have $\mathbf{K} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})^{\star}$. □ □

1.3.2. More than Two Indices

We can now apply the derived expressions recursively using algorithm 1. In this algorithm,

- SplitEvenly is a function that splits a set of n indices into two sets with $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n+1}{2} \rfloor$ indices,
- a combined relation $\mathbf{R}_{I,J}$ with a pair (I, J) of index sets as subscripts is a combined relation where only domains corresponding to the indices in I and ranges corresponding to the indices in J are considered,
- the union operation applied to a pair of combined relations is evaluated by applying the union operation to the constituent relations for corresponding indices and

- the join operation applied to a combined relation $\mathbf{R}_{I,J}$ from indices I to J and $\mathbf{S}_{J,K}$ from indices J to K is defined by setting

$$\bigvee_{(i,k)}^{I\times K}(\mathbf{R}_{I,J}\bullet\mathbf{S}_{J,K})_{i,k}=\bigcup_{j}^{J}\mathbf{R}_{i,j}\bullet\mathbf{S}_{j,k} \tag{12}$$

Algorithm 1 $\Psi(\mathbf{P}$: CombinedRelation; F: IndexSet): CombinedRelation

if $|F|>1$ **then**
  $(L,R)\triangleq$ SplitEvenly(F)
  $\mathbf{S}\triangleq\Psi(\mathbf{P}_{L,L},L)$
  $\mathbf{T}\triangleq\Psi(\mathbf{P}_{R,R},R)$
  $\mathbf{K}_{L,L}\triangleq\Psi((\mathbf{S}\cup\mathbf{P}_{L,R}\bullet\mathbf{T}\bullet\mathbf{P}_{R,L}),L)$
  $\mathbf{K}_{R,R}\triangleq\Psi((\mathbf{T}\cup\mathbf{P}_{R,L}\bullet\mathbf{S}\bullet\mathbf{P}_{L,R}),R)$
  $\mathbf{K}_{L,R}\triangleq\mathbf{K}_{L,L}\bullet\mathbf{P}_{L,R}\bullet\mathbf{K}_{R,R}$
  $\mathbf{K}_{R,L}\triangleq\mathbf{K}_{R,R}\bullet\mathbf{P}_{R,L}\bullet\mathbf{K}_{L,L}$
else
  $\mathbf{K}_{F,F}\triangleq\mathbf{P}_i^\star$ (with $\{i\}=F$)
end if
return $\mathbf{K}_{F,F}$

1.4. Analysis

Let us first assume that the number of indices is a power of two. A single call to the recursive closure algorithm for a set with $n=2^m$ indices then results in

- four recursive calls to the algorithm for index sets each with size $\frac{n}{2}$,
- 2 union operations on combined relations on $\frac{n}{2}$ indices resulting in a total of $\phi(n)\triangleq 2(\frac{n}{2})^2$ union operations on constituent relations and
- 8 join operations on combined relations from $\frac{n}{2}$ indices to $\frac{n}{2}$ indices resulting in $8(\frac{n}{2})^2$ evaluations of equations of the form (12), which in turn results in a total of $\zeta(n)\triangleq 8(\frac{n}{2})^3$ join operations and $\tau(n)\triangleq 8(\frac{n}{2})^2(\frac{n}{2}-1)$ union operations on constituent relations

if $n>1$ or a single transitive closure operation on a constituent relation if $n=1$. This allows to derive the total number of required operations:

Theorem 1.2. *Applying the recursive transitive closure algorithm to a combined relation with a set of $n=2^m$ indices results in the execution of*

- n^3-n^2 *join operations for constituent relations*
- $n^3-n^2(\frac{3}{2}\log_2 n+1)$ *union operations for constituent relations*
- n^2 *transitive closure operations for constituent relations*
- $\frac{1}{3}(n^2-1)$ *invocations of the recursive algorithm*

This leads to a time complexity of $\mathcal{O}(n^3)$. This also holds when the number of indices is no power of two (we can then consider the next higher power of two).

1.5. Application Example: Transitive Closure with a Polyhedral Abstraction

To apply the transitive closure operation while restricting the representable relations to a polyhedral abstraction, we must determine the abstractions of the required operations:

- The join operation for two polyhedral relations $\mathbf{S}$ (from s to t) and $\mathbf{T}$ (from t to u) can be obtained directly using the definition:

$$\begin{aligned}\mathbf{S} \bullet \mathbf{T} &= \{(i,k) \mid (i,j) \in \mathbf{S} \wedge (j,k) \in \mathbf{T}\} \\ &= \pi_{\mathbb{S}_s \times \mathbb{S}_u}((\mathbf{S} \times \mathbb{S}_u) \cap (\mathbb{S}_s \times \mathbf{T}))\end{aligned} \tag{13}$$

  where π denotes the projection operation.
- The abstract union operation is the lowest polyhedral upper bound (i.e., the convex hull) of the union of input relations.
- To obtain the result of the transitive closure of a constituent polyhedral relation $\mathbf{P}$ on a set $\mathbb{S}$, we consider the set of distance vectors of the relation:

$$\{z \mid z = y - x \wedge (x,y) \in \mathbf{P}\} \tag{14}$$

  This set is the projection of a polyheder on the coordinates of z. Using the Farkas lemma we can immediately derive a set of linear functions that are non-negative for all the vectors in this set. Since a function that is non-negative for a set of vectors will also be non-negative for linear combinations of these vectors, the linear functions will also be valid for the indirect distance vectors that result from the transitive closure of the relation. Since $\mathbf{1}_{\mathbb{S}_s} \subseteq \mathbf{P}^\star$ we can obtain a generating set of the non-negative functions for $\mathbf{P}^\star$ by duplicating the linear part obtained using the distance vector set and allowing an additional positive constant difference.

2. A Recursive Constraint System Closure Algorithm

2.1. Preliminaries

In [3] sets of tuples that are defined by a set of pairwise constraints in the form of *constraint matrices* are considered. In this section we consider an extension of this concept in its most general, relation form:

Definition 2.1 (Constraint system)**.** *A constraint system* $\mathbf{m}$ *on a (finite) set of indices* L *is a combined relation* $\mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *that represents the set*

$$\Gamma(\mathbf{m}) = \{x \in \overset{L}{\underset{k}{\times}} \mathbb{S}_k \mid \underset{i,j}{\overset{L^2}{\forall}} (x_i, x_j) \in \mathbf{P}_{i,j}\} \tag{15}$$

for a chosen total order[4] *of the indices in* L*.*

[4] the chosen order is only a formal requirement for choosing an order of occurence of the sets in the cardinal product $\overset{L}{\underset{k}{\times}} \mathbb{S}_k$ and thus the ordering of elements within the tuple

The constraints in a constraint system are not independent with respect to the set of tuples it represents. From two relations $\mathbf{P}_{i,j}$ and $\mathbf{P}_{j,k}$ one can deduce that for an $x \in \Gamma(\mathbf{m})$ the pair of elements (x_i, x_k) must not only obey the explicit constraints $(x_i, x_k) \in \mathbf{P}_{i,k}$, but also the *implicit constraints* $(x_i, x_k) \in \mathbf{P}_{i,j} \bullet \mathbf{P}_{j,k}$ for every $j \in L$ because a common x_j must exist in order to satisfy both constraints simultaneously. We can define an operation that allows to derive all these implicit constraints:

Definition 2.2 (Constraint system transition operation)**.** *Given index sets I, J and K and two combined relations $\mathbf{p} = \mathbf{C}((I, J), \mathbb{S}, \mathbf{P})$ and $\mathbf{q} = \mathbf{C}((J, K), \mathbb{S}, \mathbf{Q})$ on a number of mutually disjoint sets $\mathbb{S}_i$ with $i \in I \cup J \cup K$. Applying the* constraint transition operation $\blacktriangle$ *on the relations $\mathbf{p}$ and $\mathbf{q}$ results in a relation $\mathbf{p}\blacktriangle\mathbf{q}$ where for any pair $(i, k) \in I \times K$ we define that for a pair $(a, c) \in \mathbb{S}_i \times \mathbb{S}_k$ we have $(a, c) \in (\mathbf{p}\blacktriangle\mathbf{q})_{i,k}$ iff $(a, c) \in \mathbf{p}_{i,j} \bullet \mathbf{q}_{j,k}$ for each index j in J.*

Note that the transition operation is structurally similar to the join operation for combined relations.

The transition operation provides a new combined relation representing constraints that must be satisfied in addition to the explicit constraints that are given. These implicit constraints may lead to further implicit constraints. Since we also want to consider implicit constraints that occur through multiple steps of the transition operation we define a constraint system exponentiation operation:

Definition 2.3 (Constraint system exponentiation)**.** *Given a constraint system represented by a combined relation $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$ and a natural number $n > 0$, the constraint system $\mathbf{p}^{\blacktriangle(n)}$ is defined as*

$$\mathbf{p}^{\blacktriangle(n)} = \begin{cases} \mathbf{p} & n = 1 \\ \mathbf{p}^{\blacktriangle(n-1)}\blacktriangle\mathbf{p} & n > 1 \end{cases} \tag{16}$$

The combined relation that represents the explicit constraints as well as all implicit constraints obtained through any number of exponentiation steps explicitly can then be defined as:

Definition 2.4 (Transitive constraint system closure)**.** *For a constraint system represented by a combined relation $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$, the* transitive constraint system closure $\mathbf{p}^{\ddagger}$ *of $\mathbf{p}$ is defined as*

$$\mathbf{p}^{\ddagger} = \bigcap_{s=1}^{+\infty} \mathbf{p}^{\blacktriangle(s)} \tag{17}$$

Note that this operation is idempotent:

Lemma 2.1 (Idempotence of transitive constraint system closure)**.** *For every constraint system represented by a combined relation $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$, we have $\mathbf{p}^{\ddagger} = (\mathbf{p}^{\ddagger})^{\ddagger}$.*

The implicit constraints obtained through the repeated application of the transition operation are not the only implicit constraints that exist with respect to the set of tuples represented by the constraint system. For the case where $i = j$ in the definition of constraint system, the pair (x_i, x_j) must not only satisfy the explicit and implicit constraints

discussed above, but also the implicit constraint $x_i = x_j$, because both expressions represent the same component of a single tuple. We will define a constraint system where all such constraints are explicit as *coherent*:

Definition 2.5 (Coherent)**.** *A constraint system represented by a combined relation* $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *is* coherent *iff* $\overset{L}{\underset{k}{\forall}} \mathbf{p}|^{\mathbb{S}_k}_{\mathbb{S}_k} \subseteq \mathbf{1}_{\mathbb{S}_k}$.

A constraint system can easily be made coherent without changing the set of tuples it represents by intersecting the combined relation with the maximal coherent combined relation $\mathbf{m}_L = \mathbf{C}(L, \mathbb{S}, \mathbf{M})$, defined by

$$\mathbf{M}_{i,j} = \begin{bmatrix} \mathbf{1}_{\mathbb{S}_i} & i = j \\ \mathbb{S}_i \times \mathbb{S}_j & i \neq j \end{bmatrix} \quad \text{with } (i,j) \in L^2 \tag{18}$$

Since we are interested in finding a normal form of the representation of the constraint system, we define a closure that also takes these implicit constraints into account:

Definition 2.6 (Constraint system closure)**.** *For a constraint system represented by a combined relation* $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$*, the* constraint system closure $\mathbf{p}^\dagger$ *of* $\mathbf{p}$ *is defined as*

$$\mathbf{p}^\dagger = (\mathbf{p} \cap \mathbf{m}_L)^\ddagger \tag{19}$$

From this definition, it follows immediately that the constraint system closure results in a coherent relation.

Theorem 2.1 (Normal form)**.** *For two constraint systems represented by combined relations* $\mathbf{p}$ *and* $\mathbf{q}$ *we have* $\Gamma(\mathbf{p}) = \Gamma(\mathbf{q})$ *iff* $\mathbf{p}^\dagger = \mathbf{q}^\dagger$.

Definition 2.7 (Closed constraint system)**.** *A constraint system represented by a combined relation* $\mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *is closed iff* $\mathbf{C}(L, \mathbb{S}, \mathbf{P}) = \mathbf{C}(L, \mathbb{S}, \mathbf{P})^\dagger$.

A closed constraint system is interesting because all constraints are explicit in the constituent relations. Interesting operations on the represented sets can then be implemented using pointwise extensions of the operations on the constituent relations.

Due to the idempotency of the constraint system closure operation, the constraint system closure of any relation is a closed constraint system. We are therefore interested in an algorithm that computes the constraint system closure.

2.2. Problem Statement

Problem 2.1. *Given a constraint system represented by a combined relation* $\mathbf{C}(L, \mathbb{S}, \mathbf{P})$ *and an algorithm to compute the result of the operations (for every* $(i, j, k) \in L^3$*)*

- $\mathbf{R} \bullet \mathbf{Q}$ *for relations* $\mathbf{R}$ *from* $\mathbb{S}_i$ *to* $\mathbb{S}_j$ *and* $\mathbf{Q}$ *from* $\mathbb{S}_j$ *to* $\mathbb{S}_k$*and*
- $\mathbf{R} \cap \mathbf{Q}$ *for relations* $\mathbf{R}$ *and* $\mathbf{Q}$ *from* $\mathbb{S}_i$ *to* $\mathbb{S}_j$

we want to find an algorithm that computes a representation of the constraint system closure $\mathbf{C}(L, \mathbb{S}, \mathbf{P})^\dagger$ *of the constraint system.*

2.3. Solution

2.3.1. The Case of Two Subsets of Indices

Let us first consider the case where the index set L is partitioned into subsets I and J, i.e. $L = I \sqcup J$. Since the recursion must work at the level of indices rather than at the level of nodes, we will consider a restriction of combined relations to a subset of the index set:

Definition 2.8 (Index Set Restriction)**.** *Given a combined relation* $\mathbf{p} = \mathbf{C}((I, J), \mathbb{S}, \mathbf{P})$ *from index set* I *to* J*, we define the* index set restriction *of* $\mathbf{p}$ *to* (G, H) *with* $G \subseteq I$ *and* $H \subseteq J$ *as*

$$\mathbf{p}||_H^G = \mathbf{p}|^{\bigsqcup_{g}^{G} \mathbb{S}_g}_{\bigsqcup_{h}^{H} \mathbb{S}_h} \tag{20}$$

a combined relation from indices G *to* H*.*

We will use a lemma similar to lemma 1.2 to decompose the relation into a disjoint union of relations on a partition of its domain and range index sets:

Lemma 2.2. *For a combined relation* $\mathbf{p} = \mathbf{C}((I, J), \mathbb{S}, \mathbf{P})$ *from index set* I *to* J*, subsets* I_1 *and* I_2 *of* I *that partition* I *and subsets* J_1 *and* J_2 *of* J *that partition* J *we have* $\mathbf{p} = \mathbf{p}||_{J_1}^{I_1} \sqcup \mathbf{p}||_{J_2}^{I_1} \sqcup \mathbf{p}||_{J_1}^{I_2} \sqcup \mathbf{p}||_{J_2}^{I_2}$.

Let us use the more succinct notation $\mathbf{w}$ for $\mathbf{C}(L, \mathbb{S}, \mathbf{P})^\dagger$, the closure of the given constraint system. To find an expression for the relations $\mathbf{w}||_I^I$, $\mathbf{w}||_J^I$, $\mathbf{w}||_I^J$ and $\mathbf{w}||_J^J$ we use a lemma that serves the purpose that lemma 1.3 did for the case of transitive closure:

Lemma 2.3. *For a coherent constraint system represented by a combined relation* $\mathbf{p}$*, we have* $\mathbf{p}^\dagger = \mathbf{p}$ *iff* $\mathbf{p} = \mathbf{p} \blacktriangle \mathbf{p}$*.*

The expression for $\mathbf{w} \blacktriangle \mathbf{w}$ in terms of the decomposed combined relation can be written as

$$\begin{aligned} \mathbf{w} \blacktriangle \mathbf{w} &= \Big(\bigsqcup_{(a,b)}^{\{I,J\}^2} \mathbf{w}||_b^a \Big) \blacktriangle \Big(\bigsqcup_{(a,b)}^{\{I,J\}^2} \mathbf{w}||_b^a \Big) \\ &= \bigsqcup_{a,d}^{\{I,J\}^2} \bigcap_{t}^{\{I,J\}} \mathbf{w}||_t^a \overset{t}{\blacktriangle} \mathbf{w}||_d^t \end{aligned} \tag{21}$$

Just like for the transitive closure case we can now find the required constraints by identifying the corresponding combined relations:

$$\begin{aligned} \mathbf{w}||_I^I &= \mathbf{w}||_I^I \overset{I}{\blacktriangle} \mathbf{w}||_I^I \cap \mathbf{w}||_J^I \overset{J}{\blacktriangle} \mathbf{w}||_I^J \\ \mathbf{w}||_J^I &= \mathbf{w}||_I^I \overset{I}{\blacktriangle} \mathbf{w}||_J^I \cap \mathbf{w}||_J^I \overset{J}{\blacktriangle} \mathbf{w}||_J^J \\ \mathbf{w}||_I^J &= \mathbf{w}||_J^J \overset{J}{\blacktriangle} \mathbf{w}||_I^J \cap \mathbf{w}||_I^J \overset{I}{\blacktriangle} \mathbf{w}||_I^I \\ \mathbf{w}||_J^J &= \mathbf{w}||_J^J \overset{J}{\blacktriangle} \mathbf{w}||_J^J \cap \mathbf{w}||_I^J \overset{I}{\blacktriangle} \mathbf{w}||_J^I \end{aligned} \tag{22}$$

Note that we have a lemma analogous to lemma 1.5:

Lemma 2.4. *For any closed constraint system represented by a combined relation* $\mathbf{p}$ *on a set of indices* I*, every index set restriction* $\mathbf{p}||^J_J$ *with* $J \subseteq I$ *is a closed constraint system as well:* $\mathbf{p}||^J_J = (\mathbf{p}||^J_J)^\dagger$.

Since this implies that $\mathbf{w}||^I_I$ is coherent we know that

$$\begin{aligned}\mathbf{w}||^I_I \overset{I}{\blacktriangle} \mathbf{w}||^I_J &= \bigcup_{i,j}^{I\times J} \bigcap_{k}^{I} (\mathbf{w}||^I_I)_{i,k} \bullet (\mathbf{w}||^I_J)_{k,j} \\ &\subseteq \bigcup_{i,j}^{I^2} (\mathbf{w}||^I_I)_{i,i} \bullet (\mathbf{w}||^I_J)_{i,j} \\ &\subseteq \bigcup_{i,j}^{I^2} (\mathbf{w}||^I_J)_{i,j} = \mathbf{w}||^I_J\end{aligned} \tag{23}$$

and from the second expression in (22) we immediately have $\mathbf{w}||^I_J \subseteq \mathbf{w}||^I_I \overset{I}{\blacktriangle} \mathbf{w}||^I_J$ so that $\mathbf{w}||^I_J = \mathbf{w}||^I_I \overset{I}{\blacktriangle} \mathbf{w}||^I_J$. In the same way, it can be shown that $\mathbf{w}||^I_J = \mathbf{w}||^I_J \overset{J}{\blacktriangle} \mathbf{w}||^J_J$. A similar derivation can be made for $\mathbf{w}||^J_I$ so that we know that the expressions for $\mathbf{w}||^I_J$ and $\mathbf{w}||^J_I$ must have a form

$$\begin{aligned}\mathbf{w}||^I_J &= (\mathbf{w}||^I_I \overset{I}{\blacktriangle} \mathbf{z}^I_J) \overset{J}{\blacktriangle} \mathbf{w}||^J_J = \mathbf{w}||^I_I \overset{I}{\blacktriangle} (\mathbf{z}^I_J \overset{J}{\blacktriangle} \mathbf{w}||^J_J) = \mathbf{w}||^I_I \overset{I}{\blacktriangle} \mathbf{z}^I_J \overset{J}{\blacktriangle} \mathbf{w}||^J_J \\ \mathbf{w}||^J_I &= (\mathbf{w}||^J_J \overset{J}{\blacktriangle} \mathbf{z}^J_I) \overset{I}{\blacktriangle} \mathbf{w}||^I_I = \mathbf{w}||^J_J \overset{J}{\blacktriangle} (\mathbf{z}^J_I \overset{I}{\blacktriangle} \mathbf{w}||^I_I) = \mathbf{w}||^J_J \overset{J}{\blacktriangle} \mathbf{z}^J_I \overset{I}{\blacktriangle} \mathbf{w}||^I_I\end{aligned} \tag{24}$$

where $\mathbf{z}^I_J$ and $\mathbf{z}^J_I$ are relations that must yet be determined.

Note that we did not immediately remove the braces, because the operations $\overset{I}{\blacktriangle}$ and $\overset{J}{\blacktriangle}$ do not associate in general. This is caused by the fact that in order to prove associativity, we must be able to reorder the quantifiers in the definition of $\overset{I}{\blacktriangle}$ with the quantifiers in the definition of $\overset{J}{\blacktriangle}$. Since the quantifiers in the definition are of a different type this cannot be done in general. For the join operation this poses no problem because only one type of quantifier is involved in its definition. However, in order to proceed as we did with the transitive closure algorithm, we must be able to ensure that a string of transitions results in the same relation regardless of the order in which the transitions are applied. To this end, we propose that the associativity of the transition operation is taken as the mathematical property that is required for the normal form closure algorithm.

This naturally leads to the question *Is it possible to find required or sufficient conditions for an abstract domain of a basic pairwise relation such that the transition operation is associative for a constraint system that is built on relations described with this abstract domain?*

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[5] from 01/01/2008 to 31/08/2009
[6] from 01/07/2006 to 31/12/2007

A. Appendix: proofs

A.1. Lemma 1.6

The join operation $\bullet$ is associative.

Proof. Consider four sets, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$, and a relation $\mathbf{P}$ from $\mathbb{A}$ to $\mathbb{B}$, a relation $\mathbf{Q}$ from $\mathbb{B}$ to $\mathbb{C}$ and a relation $\mathbf{R}$ from $\mathbb{C}$ to $\mathbb{D}$, then

$$\begin{aligned}
&(s,r) \in ((\mathbf{P} \bullet \mathbf{Q}) \bullet \mathbf{R}) \\
&= \overset{\mathbb{C}}{\underset{c}{\exists}}((s,c) \in (\mathbf{P} \bullet \mathbf{Q})) \wedge ((c,r) \in \mathbf{R}) \\
&= \overset{\mathbb{C}}{\underset{c}{\exists}}(\overset{\mathbb{B}}{\underset{b}{\exists}}((s,b) \in \mathbf{P}) \wedge ((b,c) \in \mathbf{Q})) \wedge ((c,r) \in \mathbf{R}) \\
&= \overset{\mathbb{C}}{\underset{c}{\exists}}\overset{\mathbb{B}}{\underset{b}{\exists}}((s,b) \in \mathbf{P}) \wedge ((b,c) \in \mathbf{Q}) \wedge ((c,r) \in \mathbf{R}) \\
&= \overset{\mathbb{B}}{\underset{b}{\exists}}\overset{\mathbb{C}}{\underset{c}{\exists}}((s,b) \in \mathbf{P}) \wedge ((b,c) \in \mathbf{Q}) \wedge ((c,r) \in \mathbf{R}) \\
&= \overset{\mathbb{B}}{\underset{b}{\exists}}((s,b) \in \mathbf{P}) \wedge (\overset{\mathbb{C}}{\underset{c}{\exists}}((b,c) \in \mathbf{Q}) \wedge ((c,r) \in \mathbf{R})) \\
&= \overset{\mathbb{B}}{\underset{b}{\exists}}((s,b) \in \mathbf{P}) \wedge ((b,r) \in (\mathbf{Q} \bullet \mathbf{R})) \\
&= (s,r) \in (\mathbf{P} \bullet (\mathbf{Q} \bullet \mathbf{R}))
\end{aligned} \tag{25}$$

for any $(s,r) \in \mathbb{A} \times \mathbb{D}$. □ □

A.2. Lemma 1.4

The join operation $\bullet$ distributes over the union operation.

Proof. Consider two sets S and T (that must not necessarily be finite) and the relations $\mathbf{R}_s$ from $\mathbb{A}$ to $\mathbb{B}$ with $s \in S$ and $\mathbf{Q}_t$ from $\mathbb{B}$ to $\mathbb{C}$ with $t \in T$. Then

$$
\begin{aligned}
&(a,c) \in (\bigcup_{s}^{S} \mathbf{R}_s) \bullet (\bigcup_{t}^{T} \mathbf{R}_t) \\
&= \exists_{b}^{\mathbb{B}}((a,b) \in \bigcup_{s}^{S} \mathbf{R}_s) \wedge ((b,c) \in \bigcup_{t}^{T} \mathbf{R}_t) \\
&= \exists_{b}^{\mathbb{B}}(\exists_{s}^{S}(a,b) \in \mathbf{R}_s) \wedge (\exists_{t}^{T}(b,c) \in \bigcup_{t}^{T} \mathbf{R}_t) \\
&= \exists_{b}^{\mathbb{B}} \exists_{(s,t)}^{S \times T} (a,b) \in \mathbf{R}_s \wedge (b,c) \in \mathbf{R}_t \\
&= \exists_{(s,t)}^{S \times T} \exists_{b}^{\mathbb{B}} (a,b) \in \mathbf{R}_s \wedge (b,c) \in \mathbf{R}_t \\
&= \exists_{(s,t)}^{S \times T} (a,c) \in \mathbf{R}_s \bullet \mathbf{R}_t \\
&= (a,c) \in \bigcup_{(s,t)}^{S \times T} \mathbf{R}_s \bullet \mathbf{R}_t
\end{aligned}
\tag{26}
$$

for any $(a,c) \in \mathbb{A} \times \mathbb{C}$. □ □

A.3. Lemma 1.1

For every relation $\mathbf{R}$, we have $\mathbf{R}^\star = (\mathbf{R}^\star)^\star$.

Proof. It is clear that $\mathbf{R}^\star \subseteq (\mathbf{R}^\star)^\star$ since $\mathbf{R}^\star$ is the second term in $\bigcup_{s=0}^{+\infty} (\mathbf{R}^\star)^s = (\mathbf{R}^\star)^\star$. Furthermore, $(\mathbf{R}^\star)^\star \subseteq \mathbf{R}^\star$ since for every term $(\mathbf{R}^\star)^s$ in $\bigcup_{s=0}^{+\infty} (\mathbf{R}^\star)^s$ we have a join of s times $\mathbf{R}^\star$. By applying the definition of the transitive closure operation and using the distributivity of the join operation w.r.t. the union operation we see that $(\mathbf{R}^\star)^s$ consists of a union of terms of the form $\mathbf{R}^{t_1} \bullet \mathbf{R}^{t_2} \bullet \ldots \bullet \mathbf{R}^{t_n} = \mathbf{R}^{\sum_{q=1}^{n} t_n}$. Each such term is included in $\mathbf{R}^\star$ so that $(\mathbf{R}^\star)^\star \subseteq \mathbf{R}^\star$. □ □

A.4. Lemma 1.2

For a relation $\mathbf{R}$ from a set $\mathbb{A}$ to a set $\mathbb{B}$, subsets $\mathbb{A}_1$ and $\mathbb{A}_2$ of $\mathbb{A}$ that partition[7] $\mathbb{A}$ and subsets $\mathbb{B}_1$ and $\mathbb{B}_2$ of $\mathbb{B}$ that partition $\mathbb{B}$ we have $\mathbf{R} = \mathbf{R}|_{\mathbb{B}_1}^{\mathbb{A}_1} \sqcup \mathbf{R}|_{\mathbb{B}_2}^{\mathbb{A}_1} \sqcup \mathbf{R}|_{\mathbb{B}_1}^{\mathbb{A}_2} \sqcup \mathbf{R}|_{\mathbb{B}_2}^{\mathbb{A}_2}$.

Proof. By definition of the restriction operation, we have

$$
\begin{aligned}
&\mathbf{R}|_{\mathbb{B}_1}^{\mathbb{A}_1} \cup \mathbf{R}|_{\mathbb{B}_2}^{\mathbb{A}_1} \cup \mathbf{R}|_{\mathbb{B}_1}^{\mathbb{A}_2} \cup \mathbf{R}|_{\mathbb{B}_2}^{\mathbb{A}_2} \\
&= (\mathbf{R} \cap \mathbb{A}_1 \times \mathbb{B}_1) \cup (\mathbf{R} \cap \mathbb{A}_1 \times \mathbb{B}_2) \cup (\mathbf{R} \cap \mathbb{A}_2 \times \mathbb{B}_1) \cup (\mathbf{R} \cap \mathbb{A}_2 \times \mathbb{B}_2)
\end{aligned}
\tag{27}
$$

As a relation from $\mathbb{A}$ to $\mathbb{B}$, $\mathbf{R}$ must be a subset of $\mathbb{A} \times \mathbb{B}$. Since $\mathbb{A}_1$ and $\mathbb{A}_2$ partition $\mathbb{A}$ and $\mathbb{B}_1$ and $\mathbb{B}_2$ partition $\mathbb{B}$, we can derive that $\mathbb{A}_1 \times \mathbb{B}_1$, $\mathbb{A}_1 \times \mathbb{B}_2$, $\mathbb{A}_2 \times \mathbb{B}_1$ and $\mathbb{A}_2 \times \mathbb{B}_2$ partition $\mathbb{A} \times \mathbb{B}$. Each element $(a,b) \in \mathbf{R}$ will therefore be contained in one of the partitions, and will thus be contained in one of the four terms that is combined by the union operations.

[7] a pair of sets $(\mathbb{A}, \mathbb{B})$ partitions a set $\mathbb{C}$ iff $\mathbb{A} \sqcup \mathbb{B} = \mathbb{C}$

We thus know that $\mathbf{R} \subseteq \mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_1} \cup \mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_2} \cup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_1} \cup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_2}$. Since each term is intersected with $\mathbf{R}$ we also know that $\mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_1} \cup \mathbf{R}|^{\mathbb{A}_1}_{\mathbb{B}_2} \cup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_1} \cup \mathbf{R}|^{\mathbb{A}_2}_{\mathbb{B}_2} \subseteq \mathbf{R}$. □ □

A.5. Lemma 1.3

For a relation $\mathbf{R}$ on $\mathbb{A}$, we have $\mathbf{R}^\star = \mathbf{R}$ iff $\mathbf{R} \bullet \mathbf{R} = \mathbf{R}$ and $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}$.

Proof. We show the equivalence for a set $\mathbf{R}$ on any set $\mathbb{A}$ by showing either characterisation implies the other:

- $\Leftarrow$: Since $\mathbf{R} = \mathbf{R} \bullet \mathbf{R}$ implies $\mathbf{R}^n = \mathbf{R}$ for any natural number $n > 0$ we have $\mathbf{R}^\star = \mathbf{1}_\mathbb{A} \cup \mathbf{R}$. Since $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}$ we thus have $\mathbf{R}^\star = \mathbf{R}$.
- $\Rightarrow$: If $\mathbf{R}^\star = \mathbf{R}$ we have $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}$ since $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}^\star$. We now show that if $\mathbf{R}^\star = \mathbf{R}$ we also have $\mathbf{R} \bullet \mathbf{R} = \mathbf{R}$ by showing that $\mathbf{R} \bullet \mathbf{R} \supseteq \mathbf{R}$ and $\mathbf{R} \bullet \mathbf{R} \subseteq \mathbf{R}$:
  - $\mathbf{R} \bullet \mathbf{R} \supseteq \mathbf{R}$: Since $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}^\star$ we have $\mathbf{1}_\mathbb{A} \bullet \mathbf{R}^\star \subseteq \mathbf{R}^\star \bullet \mathbf{R}^\star$. Since $\mathbf{R}^\star = \mathbf{1}_\mathbb{A} \bullet \mathbf{R}^\star$ we thus have $\mathbf{R}^\star \bullet \mathbf{R}^\star \supseteq \mathbf{R}^\star$.
  - $\mathbf{R} \bullet \mathbf{R} \subseteq \mathbf{R}$: Since $\bullet$ distributes over $\cup$ we have

$$\mathbf{R}^\star \bullet \mathbf{R}^\star = (\bigcup_{n=0}^{+\infty} \mathbf{R}^n) \bullet (\bigcup_{m=0}^{+\infty} \mathbf{R}^m) = \bigcup_{(n,m)}^{\mathbb{N}^2} \mathbf{R}^n \bullet \mathbf{R}^m \tag{28}$$

    Any relation in this union is also contained in the union that we obtain by applying the definition of transitive closure to $\mathbf{R}^\star$ since for any pair (n, m) of natural numbers $\mathbf{R}^n \bullet \mathbf{R}^m = \mathbf{R}^{n+m}$ as a consequence of the associativity of $\bullet$. □ □

A.6. Lemma 1.5

For any transitively closed relation $\mathbf{R}$ on a set $\mathbb{A}$, every restriction $\mathbf{R}|^\mathbb{B}_\mathbb{B}$ with $\mathbb{B} \subseteq \mathbb{A}$ is transitively closed as well: $\mathbf{R}|^\mathbb{B}_\mathbb{B} = (\mathbf{R}|^\mathbb{B}_\mathbb{B})^\star$.

Proof. Let $\mathbb{C} = \mathbb{A} \setminus \mathbb{B}$, then $\mathbb{B}$ and $\mathbb{C}$ partition $\mathbb{A}$. If $\mathbf{R}$ is transitively closed we can use lemma 1.3 to show that $\mathbf{R}|^\mathbb{B}_\mathbb{B} = (\mathbf{R}|^\mathbb{B}_\mathbb{B})^\star$ by deriving that $\mathbf{R}|^\mathbb{B}_\mathbb{B} = \mathbf{R}|^\mathbb{B}_\mathbb{B} \bullet \mathbf{R}|^\mathbb{B}_\mathbb{B}$ and $\mathbf{1}_\mathbb{B} \subseteq \mathbf{R}|^\mathbb{B}_\mathbb{B}$. Since $\mathbf{R}$ is transitively closed we have $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}$ so that $\mathbf{1}_\mathbb{A}|^\mathbb{B}_\mathbb{B} = \mathbf{1}_\mathbb{A} \cap \mathbb{B} \times \mathbb{B} = \mathbf{1}_\mathbb{B}$ allows us to infer $\mathbf{1}_\mathbb{B} = \mathbf{1}_\mathbb{A}|^\mathbb{B}_\mathbb{B} \subseteq \mathbf{R}|^\mathbb{B}_\mathbb{B}$ by restricting $\mathbf{1}_\mathbb{A} \subseteq \mathbf{R}$ to $(\mathbb{B}, \mathbb{B})$. We now show that $\mathbf{R}|^\mathbb{B}_\mathbb{B} = \mathbf{R}|^\mathbb{B}_\mathbb{B} \bullet \mathbf{R}|^\mathbb{B}_\mathbb{B}$ by showing that $\mathbf{R}|^\mathbb{B}_\mathbb{B} \supseteq \mathbf{R}|^\mathbb{B}_\mathbb{B} \bullet \mathbf{R}|^\mathbb{B}_\mathbb{B}$ and $\mathbf{R}|^\mathbb{B}_\mathbb{B} \subseteq \mathbf{R}|^\mathbb{B}_\mathbb{B} \bullet \mathbf{R}|^\mathbb{B}_\mathbb{B}$.

- $\mathbf{R}|^\mathbb{B}_\mathbb{B} \supseteq \mathbf{R}|^\mathbb{B}_\mathbb{B} \bullet \mathbf{R}|^\mathbb{B}_\mathbb{B}$:

$$\begin{aligned}
(\bigcup_{(x,z)}^{\{\mathbb{B},\mathbb{C}\}^2} \mathbf{R}|^x_z)^2|^\mathbb{B}_\mathbb{B} &= (\bigcup_{(x,y,z)}^{\{\mathbb{B},\mathbb{C}\}^3} \mathbf{R}|^x_y \bullet \mathbf{R}|^y_z)|^\mathbb{B}_\mathbb{B} \\
&= \bigcup_{(x,y,z)}^{\{\mathbb{B},\mathbb{C}\}^3} \mathbf{R}|^x_y|^\mathbb{B}_\mathbb{A} \bullet \mathbf{R}|^y_z|^\mathbb{A}_\mathbb{B} \\
&= \bigcup_{(x,y,z)}^{\{\mathbb{B},\mathbb{C}\}^3} \mathbf{R}|^{x \cap \mathbb{B}}_{y \cap \mathbb{A}} \bullet \mathbf{R}|^{y \cap \mathbb{A}}_{z \cap \mathbb{B}} \\
&= \bigcup_{y}^{\{\mathbb{B},\mathbb{C}\}} \mathbf{R}|^\mathbb{B}_y \bullet \mathbf{R}|^y_\mathbb{B}
\end{aligned} \tag{29}$$

By lemma 1.3 we also have $\mathbf{R} \bullet \mathbf{R} = \mathbf{R}$ so that

$$\begin{aligned}(\bigcup_{(x,z)}^{\{\mathbb{B},\mathbb{C}\}^2} \mathbf{R}|_z^x)^2|_{\mathbb{B}}^{\mathbb{B}} =& (\bigcup_{(x,z)}^{\{\mathbb{B},\mathbb{C}\}^2} \mathbf{R}|_z^x)|_{\mathbb{B}}^{\mathbb{B}} \\ =& \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}\end{aligned} \tag{30}$$

such that

$$\bigcup_{y}^{\{\mathbb{B},\mathbb{C}\}} \mathbf{R}|_y^{\mathbb{B}} \bullet \mathbf{R}|_{\mathbb{B}}^y = \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \tag{31}$$

Therefore, $\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \bullet \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \subseteq \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}$.

- $\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \subseteq \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \bullet \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}$: Since $\mathbf{1}_{\mathbb{B}} \subseteq \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}$ we have $\mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} = \mathbf{1}_{\mathbb{B}} \bullet \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \subseteq \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}} \bullet \mathbf{R}|_{\mathbb{B}}^{\mathbb{B}}$. □ □

A.7. Theorem 1.2

Applying the recursive transitive closure algorithm to a combined relation with a set of $n = 2^m$ indices results in the execution of

- $n^3 - n^2$ join operations for constituent relations
- $n^3 - n^2(\frac{3}{2}\log_2 n + 1)$ union operations for constituent relations
- n^2 transitive closure operations for constituent relations
- $\frac{1}{3}(n^2 - 1)$ invocations of the recursive algorithm

Proof. If the first call is at level 0 of the recursion, then we have 4^l calls at level $l \leq m$. Each of these calls at level l is applied to index sets with size 2^{m-l}.

- The total number of join operations can then be written as

$$\begin{aligned}\sum_{l=0}^{m-1} 4^l \zeta(2^{m-l}) = \sum_{l=0}^{m-1} 4^l 8(\frac{2^{m-l}}{2})^3 = \sum_{l=0}^{m-1} 2^{2l+3+3(m-l-1)} = \sum_{l=0}^{m-1} 2^{3m-l} \\ = 2^{2m} \sum_{l=0}^{m-1} 2^{m-l} = 2^{2m}(2^m - 1) = n^3 - n^2\end{aligned} \tag{32}$$

- The total number of union operations can be written as

$$
\begin{aligned}
&\sum_{l=0}^{m-1} 4^l(\phi(2^{m-l}) + \tau(2^{m-l})) \\
&= \sum_{l=0}^{m-1} 4^l(2(\frac{2^{m-l}}{2})^2 + 8(\frac{2^{m-l}}{2})^2(\frac{2^{m-l}}{2} - 1)) \\
&= \sum_{l=0}^{m-1} 2^{2l+1+2(m-l-1)} + 2^{2l+3+2(m-l-1)}(2^{m-l-1} - 1) \\
&= \sum_{l=0}^{m-1} 2^{2m-1} + 2^{2m+1}(2^{m-l-1} - 1) \\
&= \sum_{l=0}^{m-1} 2^{2m-1} + 2^{3m-l} - 2^{2m+1} = 2^{2m}(\sum_{l=0}^{m-1} 2^{m-l} - \frac{3}{2}) \\
&= n^2(2^m - 1 - \frac{3}{2}\log_2 n) = n^3 - n^2(\frac{3}{2}\log_2 n + 1)
\end{aligned}
\tag{33}
$$

- The total number of transitive closure operations can be written as $4^m = n^2$.
- The total number of invocations of the algorithm can be written as

$$
\sum_{l=0}^{m-1} 4^l = \frac{4^m - 1}{4 - 1} = \frac{1}{3}(4^m - 1) = \frac{1}{3}(n^2 - 1) \tag{34}
$$

□

□

A.8. Lemma 2.1

For every constraint system represented by a combined relation $\mathbf{p} = \mathbf{C}(L, \mathbb{S}, \mathbf{P})$, we have $\mathbf{p}^\ddagger = (\mathbf{p}^\ddagger)^\ddagger$.

Proof. In the same way as in the proof for lemma 1.1 we can show that each term in the intersection resulting from expanding the definition of the transitive closure of either side of the equation $\mathbf{p}^\ddagger = (\mathbf{p}^\ddagger)^\ddagger$ also occurs in the other side. □

A.9. Lemma 2.2

For a combined relation $\mathbf{p} = \mathbf{C}((I, J), \mathbb{S}, \mathbf{P})$ from index set I to J, subsets I_1 and I_2 of I that partition I and subsets J_1 and J_2 of J that partition J we have $\mathbf{p} = \mathbf{p}||_{J_1}^{I_1} \sqcup \mathbf{p}||_{J_2}^{I_1} \sqcup \mathbf{p}||_{J_1}^{I_2} \sqcup \mathbf{p}||_{J_2}^{I_2}$.

Proof. Since

$$
\begin{aligned}
\mathbf{p} &= \mathbf{p}||_{J_1}^{I_1} \sqcup \mathbf{p}||_{J_2}^{I_1} \sqcup \mathbf{p}||_{J_1}^{I_2} \sqcup \mathbf{p}||_{J_2}^{I_2} \\
&= (\mathbf{p}|\substack{\bigsqcup_g^{I_1}\mathbb{S}_g \\ \bigsqcup_h^{J_1}\mathbb{S}_h}) \sqcup (\mathbf{p}|\substack{\bigsqcup_g^{I_1}\mathbb{S}_g \\ \bigsqcup_h^{J_2}\mathbb{S}_h}) \sqcup (\mathbf{p}|\substack{\bigsqcup_g^{I_2}\mathbb{S}_g \\ \bigsqcup_h^{J_1}\mathbb{S}_h}) \sqcup (\mathbf{p}|\substack{\bigsqcup_g^{I_2}\mathbb{S}_g \\ \bigsqcup_h^{J_2}\mathbb{S}_h}) \\
&= \mathbf{p}
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
(\bigsqcup_g^{I_1}\mathbb{S}_g) \sqcup (\bigsqcup_g^{I_2}\mathbb{S}_g) &= \bigsqcup_g^{I_1 \sqcup I_2}\mathbb{S}_g &&= \bigsqcup_g^{I}\mathbb{S}_g \\
(\bigsqcup_h^{J_1}\mathbb{S}_h) \sqcup (\bigsqcup_h^{J_2}\mathbb{S}_h) &= \bigsqcup_h^{J_1 \sqcup J_2}\mathbb{S}_h &&= \bigsqcup_h^{J}\mathbb{S}_h
\end{aligned}
\tag{36}
$$

the lemma follows from applying lemma 1.2. □ □

A.10. Lemma 2.3

For a coherent constraint system represented by a combined relation $\mathbf{p}$, we have $\mathbf{p}^\dagger = \mathbf{p}$ iff $\mathbf{p} = \mathbf{p}\blacktriangle\mathbf{p}$.

Proof. Let $\mathbf{p} = \mathbf{C}(I, \mathbb{S}, \mathbf{P})$. Since $\mathbf{p}$ is coherent we have $\mathbf{p} \subseteq \mathbf{m}$. Then

- $\mathbf{p} = \mathbf{p}\blacktriangle\mathbf{p} \Rightarrow \mathbf{p}^\dagger = \mathbf{p}$ since $\mathbf{p}^\dagger = (\mathbf{p} \cap \mathbf{m})^\ddagger = \mathbf{p}^\ddagger$ and $\mathbf{p} = \mathbf{p}\blacktriangle\mathbf{p}$ implies that $\mathbf{p}^\ddagger = \mathbf{p}$ because all terms in the intersection of the expansion of $\mathbf{p}^\ddagger$ are equal to $\mathbf{p}$ and
- $\mathbf{p} = \mathbf{p}\blacktriangle\mathbf{p} \Leftarrow \mathbf{p}^\dagger = \mathbf{p}$ because if $\mathbf{p}^\dagger = \mathbf{p}$ then
  - $\mathbf{p} \subseteq \mathbf{p}\blacktriangle\mathbf{p}$ since $\mathbf{p}\blacktriangle\mathbf{p}$ is the second term in the intersection of the expansion of $\mathbf{p}^\dagger = \mathbf{p}$ and
  - $\mathbf{p} \supseteq \mathbf{p}\blacktriangle\mathbf{p}$ since

$$
\begin{aligned}
\mathbf{p}\blacktriangle\mathbf{p} &= \bigcup_{i,j}^{I^2} \bigcap_k^I \mathbf{p}_{i,k} \bullet \mathbf{p}_{k,j} \\
&\subseteq \bigcup_{i,j}^{I^2} \mathbf{p}_{i,i} \bullet \mathbf{p}_{i,j} \\
&\subseteq \bigcup_{i,j}^{I^2} \mathbf{p}_{i,j} = \mathbf{p}
\end{aligned}
\tag{37}
$$

because for all $i \in I$ we have $\mathbf{p}_{i,i} \subseteq \mathbf{1}_{\mathbb{S}_i}$ due to coherence. □ □

A.11. Lemma 2.4

For any closed constraint system represented by a combined relation $\mathbf{p}$ on a set of indices I, every index set restriction $\mathbf{p}||_J^J$ with $J \subseteq I$ is a closed constraint system as well: $\mathbf{p}||_J^J = (\mathbf{p}||_J^J)^\dagger$.

Proof. Since a closed constraint system is coherent we have $\underset{k}{\overset{I}{\forall}}\mathbf{p}|_{\mathbb{S}_k}^{\mathbb{S}_k} \subseteq \mathbf{1}_{\mathbb{S}_k}$ so that $\underset{k}{\overset{J}{\forall}}\mathbf{p}|_{\mathbb{S}_k}^{\mathbb{S}_k} \subseteq \mathbf{1}_{\mathbb{S}_k}$ because $J \subseteq I$, and therefore $\mathbf{p}||_J^J$ is coherent too. We can thus use lemma 2.3 to show that $\mathbf{p}||_J^J = (\mathbf{p}||_J^J)^\dagger$ by showing that $\mathbf{p}||_J^J = \mathbf{p}||_J^J \blacktriangle \mathbf{p}||_J^J$.

Let $K = I \setminus J$ so that $I = J \sqcup K$.

- $\mathbf{p}||_J^J \subseteq \mathbf{p}||_J^J \overset{J}{\blacktriangle} \mathbf{p}||_J^J$: Using the definition of the transition operation we have

$$(\underset{a,b}{\overset{A^2}{\bigsqcup}}\mathbf{p}||_b^a)^{\blacktriangle(2)}||_J^J = \mathbf{p}||_J^J \overset{J}{\blacktriangle} \mathbf{p}||_J^J \cap \mathbf{p}||_K^J \overset{K}{\blacktriangle} \mathbf{p}||_J^K \tag{38}$$

and since $\mathbf{p}$ is closed we also have

$$(\underset{a,b}{\overset{A^2}{\bigsqcup}}\mathbf{p}||_b^a)^{\blacktriangle(2)}||_J^J = \underset{a,b}{\overset{A^2}{\bigsqcup}}\mathbf{p}||_b^a||_J^J = \mathbf{p}||_J^J \tag{39}$$

so that $\mathbf{p}||_J^J \subseteq \mathbf{p}||_J^J \overset{J}{\blacktriangle} \mathbf{p}||_J^J$.

- $\mathbf{p}||_J^J \supseteq \mathbf{p}||_J^J \overset{J}{\blacktriangle} \mathbf{p}||_J^J$:

$$\begin{aligned}\mathbf{p}||_J^J \overset{J}{\blacktriangle} \mathbf{p}||_J^J &= \underset{i,j}{\overset{J^2}{\bigcup}} \underset{k}{\overset{J}{\bigcap}} (\mathbf{p}||_J^J)_{i,k} \bullet (\mathbf{p}||_J^J)_{k,j} \\ &\subseteq \underset{i,j}{\overset{J^2}{\bigcup}} (\mathbf{p}||_J^J)_{i,i} \bullet (\mathbf{p}||_J^J)_{i,j} \\ &\subseteq \underset{i,j}{\overset{J^2}{\bigcup}} (\mathbf{p}||_J^J)_{i,j} = \mathbf{p}||_J^J\end{aligned} \tag{40}$$

since $\mathbf{p}||_J^J$ is coherent.

□

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