Author name

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Title

Clifford algebra, 3- and 4-dimensional analytic functions with applications. Manuscripts of the last century.

Abstract

The greatest revolution in the number from the days of Pythagoras.
The similarity between quantum mechanics and electromagnetism.
Are we a three-dimensional television show?
These and other fascinating topics are addressed by the author in this paper at once popular and mathematical, which leads us to a world still largely unexplored.
Are we facing with what is (up to now) the true language of Physics?
“Clifford's algebra — he called it 'geometric algebra' — is now well recognized as the natural algebra for describing physics in 3-space, but it hasn't yet caught on in engineering, or even in standard treatments of electricity and magnetism or fluid dynamics, where vector analysis with its ugly cross product still holds sway” (Mark Buchanan, Nature Physics 7, 442, 2011).
But can physics laws be derived from Clifford algebra and analytic functions? And why?
From simple postulates of geometrical nature (or, I mean, which simply precisely define our language) it seems that we arrive at equations of relativistic dynamics, electromagnetism, fluid dynamics and quantum mechanics.
Issues covered more or less in depth in this paper are: numbers and algebra, the analysis and the ∂^* operator, analytic functions in 3 and 4 dimensions, Maxwell's and Dirac equations, analytic functions in circular waveguides, analytic functions in four dimensions, i.e. spherical cavity, Physical Optics and heuristic derivation of the Hydrogen spectral lines.
Many disciplines are then influenced by this approach in a way that the paper often only suggests, so as it suggests several areas of future development.
I mulled over these topics for more than 40 years, and I then summarized in an unpublished manuscript dated March 2000, which is almost entirely reported here in his complete even if naïve form.

MANUSCRIPTS AT THE END OF THE CENTURY

The greatest revolution in the number starting from the days of Pythagoras.

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MANUSCRIPTS AT THE END OF THE CENTURY

This paper offers some of my work, results of studies conducted since the early '70s - '80, and summarized in a manuscript dated March 2000.
The manuscript of March 2000 is reproduced here in full, including the underlining, the footnotes and so on.
I preferred this solution, thus leaving even errors, inaccuracies etc. contained in the manuscript, in order to maintain the spontaneity of the original writing.
I just allowed some "minor changes":
- do not copy some heavy math, but confine them in special appendices, to avoid encumbering the text;
- add some current note, however, bringing it to the bottom of page that says n.d.r.

August 2011

Preface

The greatest revolution in the number starting from the days of Pythagoras.
The similarity between quantum mechanics and electromagnetism.
Are we a three-dimensional television show?
These and other fascinating topics are addressed by the author in this work at once mathematical and popular and, which leads us to a world still largely unexplored [1].

March 2000

1 - Introduction

These ideas are not my final ideas, but my ideas to start. And since it worked, I think it's fair to expose the reasoning that I followed.
"Everything is what we call electromagnetic field".
Moving on to more concrete statements, of what I'm speaking?
Essentially, from the end, I believe that many, or all, of the very fundamental equations of physics are reduced to the condition of analyticity: the one we know, in 2 dimensions, extended to 4 dimensions. But not all. Perhaps most striking is the following: I believe that the aforesaid condition of analyticity has a general validity because it says ….. absolutely nothing .... or, rather, it affirms a "true in itself", and therefore it is always valid. Defines our language.
What is the thing, "true in itself", which the condition of analyticity states, which is always true, and therefore becomes a fundamental law?
The content from the epistemological point of view (*) of the condition of analyticity, is:
"Now I'm going to discuss things as a function of space and time, and I'm not kidding."
One can see that you can derive many equations .... , or many "laws" .... with this statement. Why?
The phrase "and I'm not kidding" is the phrase: if I start with this single logical premise, that I will use quantities that are functions of space and time, but seriously, not fake ... the results are the Maxwell equations, equations of fluid, ... etcetera etcetera etcetera.
To be clear, is a single equation results, which has gradually various subcases, down, down to the ordinary analytical equations on the plane.
This general equation, which is nothing but the condition of analyticity I said before, concerns eight component functions. When it comes to the plane the components are reduced to two, as well as two are the coordinates on the plane. But, just to give you an example, the coordinates can also be one "space" and one "time" or two of "space" and one "time", and so on.
Came on to other things.
"To put ideas in a mathematical form" as Hestenes says, leads to work, therefore, with the coordinates of space and time.
Today we are accustomed to consider the constancy of the speed of light as a established physical fact. That is to say, now accepting a geometry in which, among other, shows the form $x^2 + y^2 + z^2 - c^2t^2 = 0$ is not scare. It is part of common experience. In this square, as you know, you see the square of the time with the minus sign.
This minus sign is run with the introduction of an imaginary.
Now, in few words, if we identify the space and time axes with unit vectors, the end result is the following: a mathematics borns between "imaginary" numbers to eight components, which are those certain types of numbers fully equipped to handle the eight components that I mentioned earlier Some, not commutative.
But even here, there's more.
From Hamilton, then, to switch to Dirac, until you get to Cambridge, "these things have been rediscovered over and over again" in the words of HestenesI believe that there is an underlying problem: in a so complicated mathematics, it is important to find the appropriate language. I mean that there are different formulations of language, but only the "right one" will keep things simple.

(*)
In any good physics paper you must use at least one time the word "epistemological" and say at least once "heuristic point of view."

The problem is obvious because <u>we</u> are, men, with our language, we <u>describe,</u> we <u>tell.</u> If language is not only coherent, but also good, things are simple.
I will describe the good language that I was able to find.
I see clearly that is similar, or perhaps equal to other formulations that I see around in the Internet, but I am in my own language for several reasons.
There are different types of reasons, which I think is worthwhile briefly to exhibit.
One is that I'm used to.
Another reason is that .... is a formulation suitable for technicians and engineers. I prefer to stay in a formulation (where possible) familiar even for technicians and engineers , and not convert it to spinors, quaternions, strange spaces, "octonions" and so on.
So far, what I described is equivalent of a mathematical thing, which we call a <u>“algebra”.</u>
But, in addition to algebra, with these numbers you handle, even better, the mathematical <u>analysis</u> (ie …. the condition of analyticity).
There's more.
The geometric view of a "thing" with its eight components and its intuitive understanding (I assume that it must exist, otherwise the language is not that good) are not immediate.
I did not know, and I don’t know, if this view could exist. But long ago I told myself: if there is a range of applications in which the meaning of these bodies can be understood, must be a situation where they already exist geometric entities of this type having physical meaning.
This field is electromagnetism.
Let me be clear on this point. I repeat. If we want to take over the meaning of symbols strange, unusual, <u>simple but</u> potentially can become complicated, if we want to see them, if we want to be sure of getting the right symbols, if .... etc.,.. we must find a known application. <u>What we see.</u> The only field of application which is already there ….. is the electromagnetism.
What do I mean?
Electromagnetism we already have six components, three electric and three magnetic. Applying the eight components numbers to the six components of electromagnetism, the result is the following: are the right things, things clear. In addition, we find that the other two components can justify the charge and I say, elementary particles.
We come then, finally, to another issue that I describe, which is related, among other things, to the Dirac equation. The statement that "everything is electromagnetic field" implies among other things, justify and describe in terms of electromagnetic field the charge, mass, ... and last but not least, the electron.
I must say that I did not succeed (*). However I have some ideas which can not, among other things, that go through a re-interpretation of quantum mechanics.
To be very brief and concise Dirac equations, I say, should actually be reduced to Maxwell's equations, and conversely the electromagnetic field should prove suitable to describe particles with charge and mass, and quantum mechanics should be reduced to that something called "theory of random signals", ie signals for which the phase is not known, and they only study the spectrum $\Psi\Psi^*$. Well? Well, in the text I think I have some results in this direction. It presents these results. This will be done especially in the last chapters gradually highlighting the key points to which I arrived. They are as follows. The Dirac equation can be made to coincide with Maxwell's equations. The eight components (two four-component spinors) of the Dirac wave function can be made to coincide with the components of the Maxwell field, giving them a meaning.

(*)
Old dream of many physicists, then <u>excluded.</u>

You can also do a dual operation and ask: what would happen if instead I wanted to represent the radar signals in a waveguide with a "Dirac equation"?
The result is that wavepackets of radar waves can actually be described by a Dirac equation or the Klein Gordon equation , having mass, and obedient to the equations of mechanics. This is one of the results that I consider most significant, however, even if obvious.
There are a whole series of gradually accessories results, I do not think happen by accident, but because we are managing and describing the same fundamental reality.
A critic could certainly define some simple analogies, I can not resist mentioning the statement, I think Chinese or Indian. "When the wise points at the moon, the fool sees the finger".
It is certainly important that the work should be thorough. Strangely, all the electromagnetic world seems disinterested. But it is important, and it is important for electromagnetism and perhaps even more for the elementary particles. If the particles really have an electromagnetic background, I think only good expert of electromagnetics can handle. We need a good base knowledge on waveguides, cavities, waveguide modes, the polarization, the complex signals, the signal theory, and also, I would say, the circuit theory (because the Dirac equation can also be represented with an introduction, in Maxwell's equations, of local capacity and inductance. This is another interesting result, which I intend to present).
In conclusion let me summarize the basic ideas, which are two, even three, the first I do not care of itself.
The first is the reformulation of the study of electromagnetism and fluids etc…. with the Clifford algebra, which could also lead to new ideas or new understanding.
This in itself does not interest me, even though studies on these in the U.S., if they existed and were carriers of results, such as beams for military use, would be secret.
The other two are the following ideas instead.
The first concerns the possibility that indeed "this is so, because we are not saying anything". I repeat that a good language, truly universal, and truly universal laws, would be such a good chance if they were "true in itself".
It seems to me that so are things, here.
The second idea is more philosophical.
I have made the conviction, starting from the Upanishads, and Schroedinger to get to, and Hestenes perhaps, which many have thought of a universal vibration [2]. Even today, in the Internet, legions of people talk about, more or less explicit. Among these, some probably think of the electromagnetic field.
Of course the problem is, if it is true, prove it. The problem is to find formulas. And find people who work there, who are scientists, not mystics. It should also be carefully considered the impact resulting, for which many people I think, also if convinced, they fear of losing face (*).
If you succeed in proving the electromagnetic basis of elementary particles, humanity would be almost at a turning point All statements, from the Upanishads on, that "all is One", and the like, become a fact. One might reasonably assume of signals modulation to get on .... matter. All of us would become a giant three-dimensional television show, with someone who holds the remote control. I can not even think about the impact on religions. All the philosophical basis of the so-called "quantum logic" should be belly up. The philosophy start from scratch.
For these reasons, I believe, Schroedinger <u>has made</u> statements about it, but very shy, and those few were enough to have him put aside and mocked. However, again as a similar occasion Schroedinger said "it is necessary however that someone has the courage to work, at the cost of being laughed at". I believe that in this direction, on these hypotheses, work <u>should be</u> done.

(*)
Also because there are, as is known, various "proofs" and why this is <u>not</u> possible.

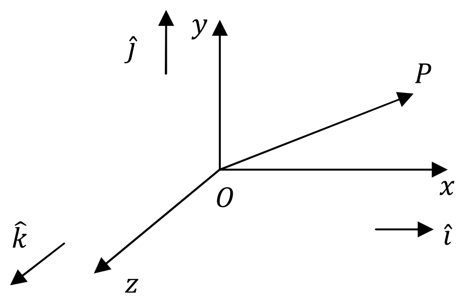
In any case, surely, will deepen the use of a new mathematical tool that is proving, in the world (*), extremely suitable to reformulate the physics and mathematics, and I believe, should be taken in hand by the electromagneticians to be fully understood in cahoots with the mathematicians. Otherwise, they go on their own.
(There are also potential applications, eg. in the study of fluids, but I'm not fully appreciating).
One of the things that I consider essential and then I tried to highlight the text, is that you should not start with the basics of Clifford algebra and analytic functions, already complicated as new, to go to "upward" (Lie algebra , SU (4) symmetry, "spin groups", etc..) but come to "downward", that is to understand the meanings in simple cases.
What I know for sure is that not many years, all the mathematical physics, and electromagnetics, and elementary mathematics, it will do so.
(Do not know what symbolism is in fashion, as happened with Gibbs, you will see).
The topics of paragraphs or chapters are as follows:
2 - Number and algebra
3 - The analysis and ∂^* operator
4 - Analytic functions in 3 and 4 dimensions
5 - Maxwell's equations
6 - The Dirac equation
7 - The equations of Maxwell and Dirac compared
8 - The waveguides
9 - The analytic functions in the circular waveguide
10 – Plot of "charged" solutions in the circular waveguide
11 - Analytic solutions in 4 dimensions, or spherical cavity
12 - Physical Optics and heuristic derivation of the atom spectral lines
13 - Considerations on the fine structure constant
14 - Conclusion
15 - Appendices (**)

(*)
In a few places actually. And I sometimes get the impression that those who do are treated as heretics. In addition, they are not experts in electromagnetics.

(**)
I added a paragraph Appendices in the current edition (n.d.r.).

2 – Number and algebra

We start from the usual three-dimensional symbolism:



where $\hat{\imath}\ \hat{\jmath}\ \hat{k}$ are the axes unit vectors. A point P is represented by a vector

$$\overrightarrow{OP} = \vec{x}$$

with

$$\vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

The modulus, which is the measure of OP, is given by

$$\overline{OP}^2 = \vec{x} \cdot \vec{x} = x^2 + y^2 + z^2$$

We simplify a little, dealing with x, $\hat{\imath}$, etc., as if they were any numbers, so omitting the symbol (·) of inner product and (×), outer product, and writing as if they were numbers:

$$\vec{x}\vec{x} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = x^2\hat{\imath}\hat{\imath} + xy\hat{\imath}\hat{\jmath} + etc\ + yx\hat{\jmath}\hat{\imath} + y^2\hat{\jmath}\hat{\jmath} + etc$$

and with the rules $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = 1$ and other rules $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$ and so we get precisely:

$$\vec{x}\vec{x} = x^2 + y^2 + z^2$$

Note, in respect of the rule $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$, its obvious meaning and its peculiarities.

On the one hand we are led to think intuitively the product between $\hat{\imath}$ and $\hat{\jmath}$ as .... an outer product (the only possible between $\hat{\imath}$ and $\hat{\jmath}$, 90° among them). So we are led to think that, being as we have always said:

$$\hat{\imath}\hat{\jmath} = \hat{k}$$

even:

$$\hat{\jmath}\hat{\imath} = -\hat{k}$$

i.e. as assumed:

$$\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$$

On the other hand, most <u>definitely not</u> write $\hat{\imath}\hat{\jmath} = \hat{k}$, but write that $\hat{\imath}\hat{\jmath}$ .... is equal to $\hat{\imath}\hat{\jmath}$ and we will let written so (* see final note).

In summary we take as justified by intuition and with a precise geometric meaning the rules:

$$\hat{\imath}^2 + \hat{\jmath}^2 + \hat{k}^2 = 1$$
$$\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$$
$$\hat{\imath}\hat{k} = -\hat{k}\hat{\imath}$$
$$\hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath}$$

that we will consider rules between numbers.

It's important to note this simplification, that we have no need to treat them as unit vectors or to distinguish the inner product (·) or the outer product (×), but we treat them as if they were arbitrary numbers. (The rule that, however, some anticommute is intuitively justified).

Since we are in the year 2000 and is now common knowledge that

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

(relativity) we can introduce the time in the form

$$\tau = ct$$

and a 4 th axis "time" with its unit vector $\hat{T}$. If we assume $\hat{T}^2 = -1$, and also we continue to maintain the anticommutativity within all the symbols i.e.

$$\hat{\imath}\hat{T} = -\hat{T}\hat{\imath}$$

etc., we get in 4 dimensions $\vec{x}$ which is generalized as follows: $\vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} + \tau\hat{T}$ and:

$$\overline{OP}^2 = \vec{x}\vec{x} = x^2 + y^2 + z^2 - \tau^2$$

as they wished.

Since are numbers, we can multiply them in every possible way and we have:

1 / $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ $\hat{T}$/ $\hat{\imath}\hat{\jmath}$ $\hat{\imath}\hat{k}$ $\hat{\imath}\hat{T}$ $\hat{\jmath}\hat{k}$ $\hat{\jmath}\hat{T}$ $\hat{k}\hat{T}$ / $\hat{\imath}\hat{\jmath}\hat{k}$ $\hat{\imath}\hat{\jmath}\hat{T}$ $\hat{\jmath}\hat{k}\hat{T}$ $\hat{\imath}\hat{k}\hat{T}$ / $\hat{\imath}\hat{\jmath}\hat{k}\hat{T}$

1+4+6 +4+1= 16 different numbers that we could say is an “algebra” (consisting of scalars and vectors, bivectors, trivectors, four-vectors). The algebra is closed in the sense that for various products is in itself. However there is an important peculiarity:

- the even sub-algebra is closed;
- the odd sub-algebra is not.

That is if we take only the numbers of even sub-algebra (scalar 1, the bivectors and the four-vector) we get 8 numbers that are a sub-algebra that is closed in itself (even x even = even).

Note well that, yet for some time, we're not talking about rules introduced ad hoc, but only the consequences of the initial intuitive hypotheses.

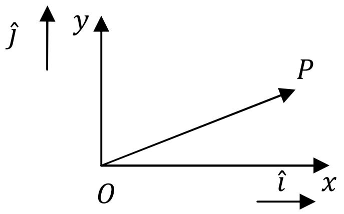
For example, we implicitly know how to make the divisions, because no matter what, ex. $\hat{\imath}$, as divided by any other, ex. $\hat{k}$, (ie, multiplied by the inverse of $\hat{k}$) is:

$$\hat{\imath}\frac{1}{\hat{k}} = \hat{\imath}\frac{1}{\hat{k}}\frac{\hat{k}}{\hat{k}} = \hat{\imath}\frac{\hat{k}}{\hat{k}^2} = \hat{\imath}\hat{k}$$

In general, there is no need to remember any particular methodology. Just write. And respect the anticommutativity.

We digress at this point on the plane before introducing the complex numbers in 4D.

On plane we define $z = \hat{\imath}\vec{x}$.



$$\vec{x} = x\hat{\imath} + y\hat{\jmath}$$

$$z = \hat{\imath}\vec{x} = \hat{\imath}(x\hat{\imath} + y\hat{\jmath}) = x + \hat{\imath}\hat{\jmath}y$$

i.e. also $\vec{x} = \hat{\imath}z$

It is as if we had tried that number z which, multiplied by $\hat{\imath}$, provides $\vec{x}$. Or, if you will, it is as if we had "measured $\vec{x}$ against $\hat{\imath}$, taken as the unit of measure" (ie making the ratio between $\vec{x}$ and $\hat{\imath}$).

There is no need for any additional rules or symbols as calling i (the usual imaginary) the number or bivector $\hat{\imath}\hat{\jmath}$, automatically find that:

$z = x + iy$ $\qquad$ $i^2 = -1$

Your correspondence $z = \hat{\imath}\vec{x}$ or $\vec{x} = \hat{\imath}z$ is a one to one correspondence between vectors and the "complex numbers" of the plane.

However, it clarified the meaning of $i = \hat{\imath}\hat{\jmath}$ which is the "bivector orthogonal to the xy plane."

Similarly for the other planes.

Ok. Without intermediate steps we can directly go to 4 dimensions and define even the same type of interdependence between

$$\vec{x} = \overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} + \tau\hat{T}$$

and $\vec{x}$ "referred to the real axis 1", ie:

$$z = \hat{\imath}\vec{x}$$

$$z = 1x + \hat{\imath}\hat{\jmath}y + \hat{\imath}\hat{k}z + \hat{\imath}\hat{T}\tau$$

$$z = x + iy + jz + T\tau$$

with:

$$i^2 = j^2 = -1$$

$$T^2 = +1$$

$$ij = -ji$$

$$iT = -Ti$$

etc. As you can see the rules are: anticommutativity among all indexes, square -1 for the imaginary i, j and square +1 for the imaginary T. Again, these are not new rules, but tedious application of the initial intuitive rules.

(I recognize that at this point there is some ambiguity in my symbolism, because the use of $\hat{\imath}, i$ $\hat{\jmath}, j$ and $\hat{T}, T$.

However, I preferred doing so to keep two well established uses:

- one, the use of $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ as unit vectors of the axes;
- the other, the use of the symbol i as imaginary (in $z = x + iy$), so I must necessarily extend to other imaginary ..... calling them j e T.

A further ambiguity is that sometimes I will use z as a coordinate and sometimes as $z = x + iy$ or $z = x + iy + jz$. However, all these ambiguities are clarified by the context).

Wanting to keep even the rule that gives, on the xy plane, the modulus $|a|$ and the rule $|ab| = |a||b|$

$$z^*z = zz^* = x^2 + y^2$$

it is found that the modulus is defined as follows:

$(ab)^* = b^*a^*$ and so $i^* = -i$ $j^* = -j$ $T^* = -T$

That's all. (Note: obviously and intuitively $\hat{\imath}^* \equiv \hat{\imath}$ etc.).

The symbols 1 $\hat{\imath}\hat{\jmath}$ $\hat{\imath}\hat{k}$ $\hat{\jmath}\hat{k}$ (ie 1 i j ji) we observe that they have the properties corresponding to the 4 symbols of Hamilton quaternions.

Likewise, you can find additional correlations that do not interest me. What is important is what follows.

With the symbols real (1) and imaginary (i j T) comes an algebra with 8 symbols

$$1 \;\; i \;\; j \;\; T \; ij \;\; iT \;\; jT \;\; ijT$$

which we observe that consists of 1 scalar, 6 bivectors and and 1 four-vector, but also "numbers", ie:

- is a closed algebra of numbers ("complex numbers")

(complex, but very real, because their meaning is crystal clear and geometrically defined: such is $i = \hat{\imath}\hat{\jmath}$, the bivector perpendicular to the xy plane);

- are numbers in the ordinary sense of the term that is "dimension free" or "numbers well suited for treating physical formulas" as any physical quantity whatever the original:
- if you already had a number, that stands;
- if it was a number with a physical dimension ex. $[lm^{-1}]$ compared to the sample unit becomes a number;
- if it was a vector, compared to a unit vector $\hat{\imath}$ becomes .... a number (complex);
- etc.

In short, the 8 quantities with indices 1 i j T..ij iT jT ijT are numbers.
The number so defined acts both on numbers or vectors, by altering, by turning, just as it does ... the number $e^{i\varphi}$ on vectors (or numbers ....) of the x, y plane.
(I should note that I disagree with Hestenes when it interprets $\hat{\imath}$ as a <u>line,</u> $\hat{\imath}\hat{\jmath}$ $\hat{\imath}\hat{k}$ etc. so as an <u>area,</u> $\hat{\imath}\hat{\jmath}\hat{k}$ so as <u>a volume</u> and so on. <u>Right.</u> But better left to 1 $\hat{\imath}\hat{\jmath}$ $\hat{\imath}\hat{k}$ etc.the distinction of number, which was born as such, or that it has become as a ratio between physical quantities. If not, should, according to Hestenes, saying that in the formula:

$$e^{i\varphi} = cos\varphi + isin\varphi = cos\varphi + \hat{\imath}\hat{\jmath}sin\varphi$$

$cos\varphi$ is a scalar and $isin\varphi$ is an area. I seem correct to recall that i is the " $\hat{\imath}\hat{\jmath}$ bivector", but in a context of mathematical physics, is not <u>an area</u> for me, is a number).
I would like to summarize. It is possible to achieve a 2D extension of the complex number

$$z = x + iy \qquad \text{with}$$
$$zz^* = x^2 + y^2$$

coming to realize that "strange" imaginary are needed, to have the same in 4 dimensions:

$$\boxed{z = x + iy + jz + T\tau}$$
$$z^* = x - iy - jz - T\tau$$
$$\boxed{zz^* = x^2 + y^2 + z^2 - \tau^2}$$

with the "strange" rules, $i^* = -i$ etc., $i^2 = -1$….. $T^2 = +1$, $ij = -ji$ etc., …. and so on.
Whichever way you turn around, these are the only rules that provide this generalization of the complex number (**).
Only after a restart by vectors $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ $\hat{T}$ we can understand that these rules match ..., are <u>congruent,</u> or <u>contemporary,</u> with those intuitive on the unit vectors $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ (and $\hat{T}$).
And, as will be seen in a while, these are the same rules that are used for the <u>derivative</u> operator.

(*) final note
You say, "but we know that $\hat{\imath}\hat{\jmath} = \hat{k}$". <u>NO!</u> We were smart but clumsy.
$\hat{k}$ <u>is $\hat{k}$</u>, $\hat{\imath}\hat{\jmath}$ <u>is $\hat{\imath}\hat{\jmath}$</u>.
$\hat{k}$ is <u>oriented on z as a vector</u> (if the axes $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ change sign, <u>he changes),</u> $\hat{\imath}\hat{\jmath}$ is <u>oriented on z as bivector: a</u> $\vec{\Omega}$ (if the axes $\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ change the sign, he <u>does not</u> change).

(**)
I mean, irrespective from details, choice of names ($\hat{\imath}$ $\hat{\jmath}$ $\hat{k}$ $\hat{T}$ or e_1, e_2, e_3, e_4), choice of spacetime signature ($+ + + -$ or $- - -$ $+$) and so on (n.d.r.).

Exercises.
1- Prove that the index Tji commutes with all other 7 possible indices.
2- Check that $(Tji)^* = Tji$.
3- Prove that $(ji)^2 = -1$.
4- Prove $(Tji)^2 = -1$ and $(Ti)^2 = +1$.
5- Demonstrate that are <u>commutative to each other</u> 1 i Tji Tj.
6- Demonstrate that are <u>commutative to each other</u> 1 j Tji Ti; idem 1 T Tji ij.

3 - The analysis and ∂^* operator

I could start from the end.
The operator

$$\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + T\frac{\partial}{\partial \tau}$$

has the property that is:

$$\partial\partial^* = \partial^*\partial = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}$$

This second operator is the operator of the "wave equation" $\partial\partial^* F = 0$ (or even the Klein-Gordon relativistic equation that obeys any particle, $\partial\partial^* F = m_0^2 F$).
Who is the first operator? Who is ∂^*? I'd say that before Dirac they do not know (Dirac doesn't noticed it, despite having had written, because it had written with spinors and 4x4 matrices). It is at once "the square root of the Laplacian" so to speak

$$\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}$$

and is also at the same time ... the operator of analyticity (or monogenicity), ie, what defines in a blow Cauchy Riemann conditions for there to be an analytic function (*)

$$\partial^* F = 0$$

As I say I started from the end except to note again that, however, ∂ is the derivative (or a derivative):

$$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z} - T\frac{\partial}{\partial \tau}$$

It should, after this brief preview of the final results, review some meanings, which I think are important, in terms of analytic functions.
Proceed with order. A point P on the plane is characterized (if it is deemed appropriate) by the complex number $z = x + iy = \rho e^{i\varphi}$ that identifies the point P.
Suppose a physical phenomenon that takes place on a plane, eg the flow of a fluid. (Things would be worth similar for a plane electric field).
Suppose I say: bah! try to see if there is a quantity (eg two-component, 1 and i) which would describe this phenomenon on the physical plane. Then I will describe with a two component function f, 1 and i:

$$f(x,y) = f = u(x,y) + iv(x,y)$$

I can write:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

or with a linear change of variables from x, y to z, z^*:

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial z^*}dz^*$$

(*)
Cambridge does not use these symbols, or Hestenes, but I prefer to keep it for this simple reason that this symbology generalizes in the obvious way what happens on the plane where

$$\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

$$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

give, respectively, the Cauchy Riemann conditions and the derivative ($\frac{\partial}{\partial z} = \frac{1}{2}\partial$) (see next Note)

Now here is one of the most significant passages, I believe, of this discourse. Be careful. Until now, no assumptions are made, if not the attempt to see if I can describe this phenomenon, as the flow of a fluid, with f having two components $u(x,y), v(x,y)$. Truly a hypothesis was made, that the phenomenon in question is described as a function of the point P, but seriously, I add (as I said in the introduction).
That is: "not a joke".
Ie: the phenomenon in question, and its two components function f, I, a human being that I am going with my "mathematical language" to describe the phenomenon, I say that as far as I know my description will a function of the point P.
Now: the point P is uniquely characterized by $z = x + iy$. It therefore has no sense that the function is function of z and even z^*. So must be $\frac{\partial f}{\partial z^*} = 0$.
So, whatever it is that law on the plane, must obey (*):

$$\boxed{\partial^* f = 0}$$

These are, with this single logical premise:
- the equations of fluids on the plane [3];
- the equations of electric fields on the plane;
- the equations of magnetic fields on the plane
and so on and so forth.
But they are also the analyticity conditions of ….... to be a unique function of a point (it is impressive I must say, from a logical point of view, which in 4 dimensions by imposing the same requirements ... to use an unique function of a point …. we obtain Maxwell's equations. See later. These minimal assumptions seem to me, the only logical content we state in writing Maxwell's equations).
Remain only at this point to define some rules and a reminder of $grad\ div\ rot$ (which at this point, however, are by-passable).
The first is:
the analyticity of $u + iv$ means that to the conjugate $u - iv$ equations $rot\vec{v} = 0$,
$div\ \vec{v} = 0$ apply,
(see, for example, Tricomi, "Analysis") (then: found the solutions you need to pass the conjugate).
The second observation is:
the ∂^* operator provides in one go all possible operations $grad\ div\ rot$ and provides through $\partial\partial^*$ the $\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

(*) Note
You can check all the steps, which already exist eg. albeit with different symbols, which give:

$$\frac{\partial}{\partial z} = \frac{1}{2}\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\partial^* = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

4 - Analytic functions in 3 and 4 dimensions

The dream of extending the analytic functions to 3-dimensions has been the dream of all, by Maxwell Hamilton et cetera.
Maxwell said, see [4]: "… the method … is much more powerful than any known method applicable to three dimensions. The method depends on the properties of conjugate functions of two variables".
Hilbert said ("in order to characterize the futility of all attempts in this direction"): "time is one-dimensional, space is three- dimensional, however the number, that is, the perfect complex number, has two dimensions" (Sommerfeld).
Some of this research led to aesthetic reasons, but certainly for all the aim was to extend the convenient properties of analytic functions to more than 2 dimensions. Of course, now we can say that these functions exist, are manageable, and you can solve problems. By itself it would be worthwhile to open a study entitled "Applications of the ∂^* operator to engineering".
I will give, for convenience, directly the expression to the analytic functions in 4 dimensions, from which then the case 3 is obtainable as a subcase.
I say that a mathematical quantity structured (*) as:

$$F = (U_1 + iU_2 + jU_3) + Tji(U_4 + iU_5 + jU_6)$$

is such that it its analyticity implies for the complex conjugate

$$(U_1 - iU_2 - jU_3) + Tji(U_4 - iU_5 - jU_6)$$

the validity of Maxwell's equations. In fact, introducing the names relevant to a physical quantity "electromagnetic field":

$$F = \left(E_x + iE_y + jE_z\right) + Tji(H_x + iH_y + jH_z)$$

the analyticity of F means:

$$\partial^* F = 0$$

ie:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + T\frac{\partial}{\partial \tau}\right)\left[\left(E_x + iE_y + jE_z\right) + Tji\left(H_x + iH_y + jH_z\right)\right] = 0$$

Developing and then noting that for the satisfaction of the equation must be simultaneously zero "parts" 1, i, j, etc., we have:

1 $$\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = 0$$

i $$\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} + \frac{\partial H_z}{\partial \tau} = 0$$

j $$\frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x} - \frac{\partial H_y}{\partial \tau} = 0$$

T $$\frac{\partial E_x}{\partial \tau} + \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0$$

ji $$\frac{\partial H_x}{\partial \tau} + \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = 0$$

Tj $$-\frac{\partial H_x}{\partial y} + \frac{\partial E_z}{\partial \tau} - \frac{\partial H_y}{\partial x} = 0$$

Ti $$\frac{\partial H_x}{\partial z} + \frac{\partial E_y}{\partial \tau} + \frac{\partial H_z}{\partial x} = 0$$

Tji $$\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} - \frac{\partial H_z}{\partial z} = 0$$

(*)
For the moment I will limit myself to 6 components.

These are Maxwell's equations, as seen for example from the first equation which, for the conjugate with components $\left(E_x - iE_y - jE_z\right)$, is the equation $div\,\vec{E} = 0$. And so on (*).
But there's more:
each specialization of these conditions of analyticity to a smaller number of dimensions (or components) leads to a physical law.
I can do a variety of examples.
Take for example a physical quantity (**):

$$U = (U_1 + iU_2 + jU_3 + TU_4)$$

The condition of analyticity for it, made the necessary steps, leads to

$$
\begin{array}{ll}
1 & \frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial y} - \frac{\partial U_3}{\partial z} + \frac{\partial U_4}{\partial \tau} = 0 \\
\mathrm{i} & \frac{\partial U_2}{\partial x} + \frac{\partial U_1}{\partial y} = 0 \\
\mathrm{j} & \frac{\partial U_3}{\partial x} + \frac{\partial U_1}{\partial z} = 0 \\
\mathrm{T} & \frac{\partial U_4}{\partial x} + \frac{\partial U_1}{\partial \tau} = 0 \\
\mathrm{ij} & \frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} = 0 \\
\mathrm{iT} & \frac{\partial U_4}{\partial y} - \frac{\partial U_2}{\partial \tau} = 0 \\
\mathrm{jT} & \frac{\partial U_4}{\partial z} - \frac{\partial U_3}{\partial \tau} = 0
\end{array}
$$

We can see for example that if $U_4 = 0$, these conditions <u>require</u> that (U_1, U_2, U_3) are independent of time. For the remaining components (U_1, U_2, U_3) we can see that ... they mean the cancellation of the rotor and divergence for the conjugate $(U_1, -U_2, -U_3)$.
This immediately generalizes what happens for 2D analytic functions.
In the general case, the equations mean, for the conjugate $(U_1, -U_2, -U_3, -U_4)$, putting $\vec{v} = (U_1, -U_2, -U_3)$, the equations:

$$rot\ \ \vec{v} = 0$$

$$div\,\vec{v} + \frac{\partial v_4}{\partial \tau} = 0$$

$$\frac{\partial \vec{v}}{\partial \tau} + grad\ v_4 = 0$$

equations <u>of motion of a irrotational, compressible fluid,</u> in 3 dimensions (if $v_4 = 0$, even stationary).
Staying for the moment in 3D (x, y, z) it is easy to build up analytic functions with the following property, however general:

If A is harmonic, $v = \partial A$ is analytic.

(*)
The modulus FF^* is equal to $EE^* - HH^* + Tji(EH^* + HE^*)$ that contains the two famous field invariants $\left|\vec{E}\right|^2 - \left|\vec{H}\right|^2$ and $\vec{E} \cdot \vec{H}$.

(**)
that, in fact, has the added component U_4 .

In fact, A harmonic means:

$$\partial\partial^* A = 0$$

However $\partial\partial^* = \partial^*\partial$, so

$$\partial^*\partial A = 0$$

ie, putting $v = \partial A$, follows $\partial^* v = 0$, so v it's analytical CVD.
This extends the concept of "potential" of a field, where the field is the ∂ derivative. But we can also, again, see this as a empirical way to make analytic functions.
Just as an example (which you can work hard) computing the analytic function that provides the flow of a fluid around a sphere of radius a.
Skipping the way to derive it from a harmonic potential.
The analytic function is:

$$U = 1 + \frac{1}{2}a^3 \frac{1 - \frac{3xz^*}{r^2}}{r^3}$$

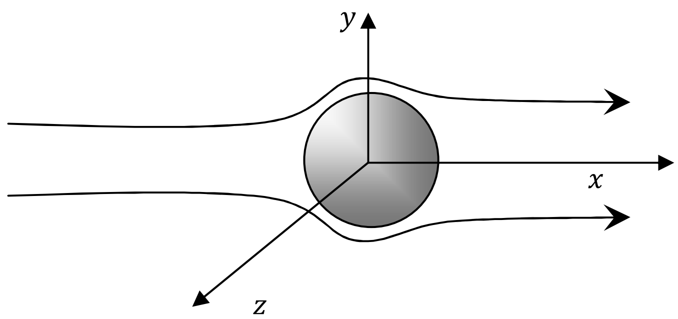
The condition $\partial^* U = 0$ is verified.
For the conjugate $\vec{v} = (v_x, v_y, v_z) = (U_1, -U_2, -U_3)$ apply the formulas:

$$v_x = -\frac{3}{2}\frac{a^3x^2}{r^5} + \frac{1}{2}\frac{a^3}{r^3} + 1$$

$$v_y = -\frac{3}{2}\frac{a^3xy}{r^5}$$

$$v_z = -\frac{3}{2}\frac{a^3xz}{r^5}$$

Note a term 1 of speed parallel to the x axis The fluid moves in the direction of the positive x



Other interesting analytic functions:

$$U = \frac{1 + \frac{z^*}{r}}{x + r} = \frac{1}{r} - i\frac{\frac{y}{r}}{x + r} - j\frac{\frac{z}{r}}{x + r}$$

$$U = \frac{z^*}{r^3}$$

(I think the first around I don’t remember what, the second on a sink, but I do not remember)
Go on to other things.
(In this final section I report only the main concepts. This part contains indeed a number of interesting and picturesque aspects that would be explored. I do not have reported too much math, which would have greatly burdened the text.).
Show particular interest those I would call "powers (analytical) of the number".
Who are they?
Are the same of powers z^n in 2D.
In 2D, taking $z = x + iy = \rho e^{i\varphi}$ the variable, or "point P", its (analytical) powers are z^2, z^3, …., $z^n = \rho^n e^{in\varphi}$. In 2D there are two ways to get:

- or by the product of z and z and z …. n times;
- or defining them as <u>eigenfunctions</u> (analytical) <u>of angular momentum:</u> in fact

$$-i\frac{\partial}{\partial\varphi}z^n = nz^n$$

The latter picturesquely calls the operator of angular momentum in quantum mechanics that is

$$L_z = -i\frac{\partial}{\partial\varphi}$$

but as we shall see (and to prevent people from thinking we're doing quantum mechanics) it appears separating the variables in the equation $\partial^* f = 0$.

<u>The 2-th way, and not the first, it generalizes to 3 dimensions.</u>

In fact do not exist in 3D powers (meaning: analytical, ie such that $\partial^* f = 0$) <u>made by product of various terms $z = x + iy + jz$ (*).</u>

Indeed neither $z = x + iy + jz$ it is analytical.

How does the operator of angular momentum appears in the equation $\partial^* f = 0$?

The way it should be noted in writing ∂^* (which is then equated to zero) in the form:

$$\partial^* = \frac{z}{r}\frac{1}{r}(z^*\partial^*)$$

So when one writes $\partial^* = 0$ brings up $z^*\partial^* = 0$. Let the 2D polar coordinates r, φ and seek solutions that are analytic in the form $f = R(r)\Phi(\varphi)$ to separate variables. Immediately the operator of angular momentum appears. In fact

$$z^*\partial^* = (x - iy)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\varphi}$$

$-i\frac{\partial}{\partial\varphi}$ is the operator of angular momentum.

At this point it is clear that invoking "an eigenvalue for the operator $-i\frac{\partial}{\partial\varphi}$" is equivalent to being able to separate the variables r, φ . The fact that this eigenvalue is an integer arises from which the function covers as it spins. With the eigenvalue n we measure the "turns" that makes the function (in this case, on the x, y plane).

In 3 dimensions is:

$$z^*\partial^* = (x - iy - jz)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + etc.\right) + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + etc.\right)$$
$$= r\frac{\partial}{\partial r} + \Gamma^*$$

$-\Gamma^*$ is the operator of angular momentum The explicit expression of the $-\Gamma^*$ operator is:

$$-\Gamma^* = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - j\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) - ji\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

Following the notations of almost all the books of quantum mechanics is (unless the current physical meaning of i etc.) we have:

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\varphi}$$

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

(*)Note

Ambiguity between notations z and z solved in context.

If $-\Gamma^*$ has an eigenvalue l , this means:

$$-\Gamma^*\Psi = l\Psi$$

ie:

$$[L_z + j(L_x + iL_y)]\Psi = l\Psi$$

and if, perhaps simultaneously, L_z has an eigenvalue m , is:

$$L_z\Psi = m\Psi$$

We will see the exceptions of the latter.
I would insist and repeat that all these concepts that seem to be peculiar to quantum mechanics are simply the mathematics of analytic functions (and waveguides and spherical cavities, is it by chance? We will see).
Well. After this long introduction let 3D analytical solutions that are in the form $f = R(r)\Phi(\vartheta,\varphi)$ with separate variables.
Let, as in 2D, in the same form of powers $z^n = \rho^n e^{in\varphi}$

$$\Psi_l = r^l\psi_l(\theta,\varphi)$$

with the condition, as in 2D, to be analytical

$$\partial^*\Psi_l = 0$$

and with the condition, as in 2D, to be eigenfunctions of angular momentum

$$-\Gamma^*\Psi_l = l\Psi_l$$

Incidentally this last thing, if you think about it, is equal to put them in shape $\Psi_l = r^l\psi_l(\theta,\varphi)$.
They …. are dependent on another index m .
This second index is written at the top of the form Ψ_l^m (*).
The powers Ψ_l^l are trivial. They are:

$$\Psi_l^l = (x + iy)^l$$

Then, for each l , there are additional powers of lower grade m :

$$\Psi_l^m \rightarrow \Psi_l^{l-1}, \quad \Psi_l^{l-2}, \ldots.. \quad \text{until} \quad \Psi_l^{-l}$$

Total: $(2l + 1)$.
I repeat that I can not write too much because we would like a text ad - hoc. No book or article attempts to investigate the physical meaning, or I would say even better geometric, or arithmetic. It turns out that such banal .... m who is it?
Every Ψ_l^m (ie: any ψ_l^m) turned around l times, and this makes m rounds on the x, y plane (or when it is negative m turns in the opposite direction on the x, y plane).
The statement is not strict, but he interprets the facts in a picturesque way.
At this point we must proceed with an observation concerning the operator $L_z = -i\frac{\partial}{\partial\varphi}$ which should measure the m turns on the x, y plane.
Does not always L_z give an eigenvalue m. Sometimes, in fact, applied to Ψ_l^m does not give an eigenvalue precisely nothing (which, formally, we make it impossible to separate the variable φ into equation of analyticity $\partial^* = 0$).

(*)
and correspondingly the angular functions $\psi_l(\theta,\varphi)$, they become ψ_l^m It is quite interesting, given the shape of the powers $z^n = \rho^n e^{in\varphi}$ and $\Psi_l^m = r^l\psi_l^m(\theta,\varphi)$, to note that the functions $\psi_l^m(\theta,\varphi)$ containing the angular dependence are the 3D generalization of $e^{in\varphi}$.

The only operator that gives an eigenvalue on Ψ_l^m is a L_z "modified" whose name is J_z. in quantum mechanics. It gives on Ψ_l^m the eigenvalue of ... $\left(m+\frac{1}{2}\right)$:

$$J_z\Psi_l^m = (m+\frac{1}{2})\Psi_l^m$$

and is thus formed:

$$J_z = L_z + S$$

Its action on Ψ_l^m is of the following:

$$J_z\Psi_l^m = -\frac{\partial}{\partial\varphi}\Psi_l^m i - \frac{1}{2} i\Psi_l^m i$$

and in quantum mechanics the three terms respectively give the total angular momentum, orbital angular momentum and spin.
As you can see the onset of these eigenvalues and operators (or if you want these geometric features), however, does not have anything to do with quantum mechanics. It is characteristic of this new mathematics of analytic functions, all to be investigated.
Is this by chance? Are there coincidences? In my opinion not, and indeed opens before us a huge matter of study, both theory and application. Apply these "spin" properties in paragraphs 9 and 11.
Needless to say these Ψ_l^m are orthogonal, form a base, they can develop any analytic function (... is the analogue of zeros and poles, are the analogues of ... z^n on the plane .... and there are also the states $1/_{z^n}$ that are ... to negative n). The ψ_l^m generate a single stroke a lot of mathematical special functions such as Legendre polynomials P_l^m and / or "spherical armonics" $Y_l^m(\theta,\varphi)$. The explicit expression (Doran) of ψ_l^m with the associated Legendre polynomials (see Gradshteyn & Ryzhik for the definition of P_l^m) is:

$$\psi_l^m = (l+m+1)P_l^m e^{im\varphi} + jP_l^{m+1}e^{i(m+1)\varphi}$$

I end here, but I anticipate some kind of picturesque account of paragraphs 9 and 11.
Turns out that the enumeration of the analytical solutions Ψ_l^m and their type of angular distribution ψ_l^m match to the states of the electrons in the hydrogen atom.
"Strangely," it also gets information about the spatial distribution of the orbits, which are described by ψ_l^m and are listed by the ψ_l^m.
However instead will be entirely clear and unambiguous the problem of spherical cavity, where the oscillation modes are also described by ψ_l^m, and enumerated by ψ_l^m. In this second case there is no ambiguity about the fact that the field components are described in their spatial distribution by the solutions of Maxwell equations $\partial^* F = 0$.
Conclusion: there are strong suspicions of relationship between the two cases: the possible oscillation modes in the cavity and the orbits of the electrons in the atom. There are elementary geometric entities that are the Ψ_l^m or the ψ_l^m that dominate the scene. The ψ_l^m are three-dimensional analogous of the angular distributions $e^{in\varphi}$. The Ψ_l^m.... are the analogous of the elementary geometric (or numerical) entities z^n and $1/_{z^n}$ of the x, y plane.
Why this happens?
We are discovering, as the Cambridge group says, the properties of space and time?
None of this. The question, in my opinion, is as follows:
what chance is there that the basic entities of our language are different (so to speak) of elementary particles?
The answer is in my opinion (if the language is "centered"): none.

And so we should not be surprised that this happens. At the end is our problem. In physics an elementary reality too often escapes, which is obvious, but then is forgotten with phrases like "I found the law ...", "I discovered". In physics, we can only tell you what happens. Or try to tell.
Now if you think about it (this is probably incorrect but suggestive diction) we say that bodies are made of atoms, elementary particles, etc., and functions are made up of zeros, poles and so on. Compose the bodies with the atoms and compose functions with the elementary functions.
It would be wise, even if not required, that the elementary entities of the mathematical language coincided with the basic realities of the surrounding reality that we intend to tell. That is why this fact can reasonably happen: elementary physical entities are elementary functions. They are, not for aesthetic magic, or coincidence, but because we, in turn and reversal of secular thought and mathematical and physical laws, we can not merge, more or less unconsciously, in a right mathematical language, in which elementary functions, such as analytic functions Ψ_l^m , necessarily correspond in some way to elementary physical entities.
You might say: but it is not. It is still not exactly so. Well: Schroedinger say: "In my day it was said that the research is not over yet."
Finally (and I stress this, which in my opinion is the most satisfying aspect) we would pointed out and stated our fundamental "ignorance". We only pointed out the class of functions that will work in the description of the phenomena of space and time. We, in fact, first pointed out the space and time, with their unit vectors, and the formal rules required, consequential, and then the class of functions, analytic function of the point P of spacetime. The general law valid in all of physics is that we use those functions there. The law $\partial^* F = 0$ adds nothing to the substance of what we know of the electromagnetic field (and thus eventually would be for elementary particles). End.

5 - Maxwell's equations

I have already spoken of Maxwell's equations but now propose again in a complete form with 8 components.
Introduce a quantity F electromagnetic field (*):

$$F = \left(E_x + iE_y + jE_z - TH_\tau\right) + Tji(H_x + iH_y + jH_z + TE_\tau)$$

The analyticity of F means

$$\partial^* F = 0$$

ie:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + T\frac{\partial}{\partial \tau}\right)\left[\left(E_x + iE_y + jE_z - TH_\tau\right) + Tji\left(H_x + iH_y + jH_z + TE_\tau\right)\right] = 0$$

Developing and then separating the "parts" 1, i, j, etc, we have 8 scalar equations:

$$1 \qquad \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} - \frac{\partial H_\tau}{\partial \tau} = 0$$

$$\mathrm{i} \qquad \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} + \frac{\partial H_z}{\partial \tau} - \frac{\partial E_\tau}{\partial z} = 0$$

$$\mathrm{j} \qquad \dots\dots\dots\dots\dots\dots\dots\dots \text{etcetera}$$

With 6 components we have Maxwell's equations in empty space. With non-zero H_τ and E_τ terms appear, related to density of electric and magnetic charges and currents.
You can see that even the term H_τ it is enough to give Maxwell's equations with charges and currents. Consider for example the first equation written. Moving on as usual to the conjugate $\left(E_x - iE_y - jE_z + TH_\tau\right)$ and then placing $\rho = -\frac{\partial H_\tau}{\partial \tau}$ we see that it says:

$$div\,\vec{E} = \rho$$

Even on this you can write a book, but I want to move quickly to write down some peculiarities.
In these conditions of maximum generality the analyticity condition $\partial^* F = 0$ applied to a 8 components F results in terms which are <u>all</u> interpretable in electromagnetic sense.
The resolution of these equations may seem a rather complicated problem, it is already difficult to solve the Maxwell equations without currents or fields derived from charges and currents that generate them. In fact, paradoxically, the opposite is true: that is easy to produce solutions in abundance in fully automatic mode.
How?
The premise holds, of which I have already mentioned, that if any "thing" A is harmonic, ∂A is analytic.
So the <u>thing</u> A , if harmonic, can be anything:
a scalar $\varphi(z,t)$, or $\varphi(x,y,z,t)$, a thing <u>with indices,</u> such that $e^{i(\omega t - kz)}$ which, if $\omega = k$, is harmonic, and so on. We can say that A playing the role of potential (generalized) of the field, but no longer obeying the Lorenz gauge condition but the more "simple" condition:

$$\Box A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}\right)A = \partial^*\partial A = 0$$

Differentiating, we get a field $F = \partial A$ that provides a structure of electric fields, magnetic fields, charges and currents, interpreted every time, and who satisfies the Maxwell equations in the sense that these currents generate these fields according to Maxwell's equations.
(At least this formally. It should of course be met physical condition, which make precisely

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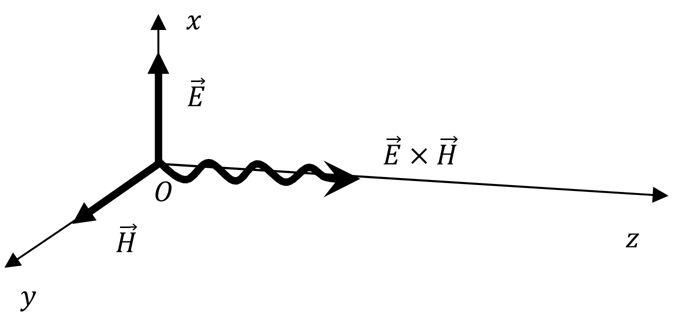
(*) Note
The decision to introduce the other two components, and writing them as $-TH_\tau$ and TE_τ is done in order to be able to write F in a form that will be used later for comparison with the Dirac equation (n.d.r.)

physically acceptable those current and those fields).
We can say that these currents generate these fields according to Maxwell's equations (*), but at this point would be equally justified in saying that these fields generate the current ... and then in general we tend to say that those fields and currents are self-sustaining .
Not only. Reached a solution F with some indices (indices which we do not like, or if we wanted to reach the interpretation of a solution with different indices) is the observation that while F it is analytical, yet analytical Fi, FT, Fji , or whatever, for any index multiplied by the right . This exchange and reinterprets all original indexes of F.
Example: multiplying F by Tji exchanging electric fields to the magnetic switching from a TE to TM (or, if you started from TM , you get a TE).

(*)Note
I think it is important to note that these "generalized" equations to 8 components are not different from the usual Maxwell equations. Are the usual Maxwell's equations and more precisely are the usual Maxwell equations in the presence of charges and currents. The only (so to speak) difference is that the charges and currents that appear in the equations are no longer assigned outside (and then they will derive these fields), but appear automatically as part of the solution.
In other words, the electromagnetic fields $\vec{E},\vec{H}$ that the equations provide are the fields that were (are) drawn from the usual Maxwell equations in the presence of charges and currents if you were able to generate this distributions of charges and currents.
So this is to examine from time to time if charges and currents correspond to physically realizable situations (n.d.r.).

Exercises.
Exercise 1 - The potential $A = Tie^{i(\omega t-kz)}$ with different signs for ω and k is harmonic, provided that $\omega^2 = k^2$. Differentiating with the ∂ operator the four analytical solutions are obtained
$F_1 \quad (1+Tj)e^{i(-kz+\omega t)}$
$F_2 \quad (1+Tj)e^{i(+kz-\omega t)}$
$F_3 \quad (1-Tj)e^{i(+kz+\omega t)}$
$F_4 \quad (1-Tj)e^{i(-kz-\omega t)}$
The first two are propagated in the z positive direction, with spin $e^{\pm i\omega t}$ opposite. Ditto the second, to negative z .
Exercise 2 To interpret in terms of Poynting vector.

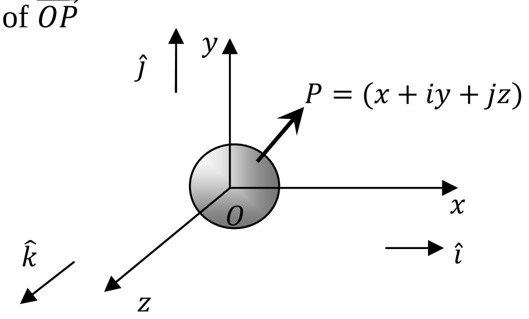


Exercise 3 - Discuss the sums $\frac{F_1+F_2}{2}$ and $\frac{F_1-F_2}{2}$.
Solution:
$\frac{F_1+F_2}{2} = (1+Tj)\cos(\omega t-kz)$ Linear $E_x\ H_y$ Horizontal;
$\frac{F_1-F_2}{2} = (1+Tj)i\sin(\omega t-kz)$ Linear $E_y\ H_x$ Vertical.

Exercise 4 - Interpret the analytical solution $F = \frac{z^*}{r^3}$.
Solution. Turning to the conjugate, we have:

$$\frac{z^*}{r^3} \rightarrow \frac{z}{r^3} = \frac{1}{r^2}\frac{z}{r}$$

$\frac{z}{r}$ is the unit vector of $\overrightarrow{OP}$

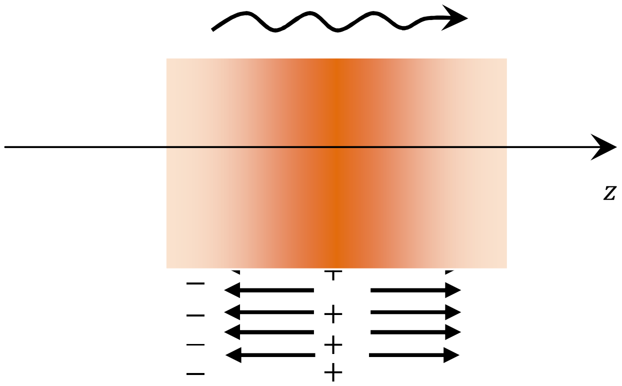


The field is $E_r = \frac{1}{r^2}$ radial, static.
Exercise 5 - The potential $A = \cos{(\omega t - kz)}$ is harmonic if $\omega = k$. The function

$$F = \partial A \propto (j - T) sin\,(\omega t - kz)$$

is analytic. Interpret in terms of electromagnetic field.

Solution:



It has a E_z wave (not polarized in the transverse plane), which propagates accompanied by oscillating charge $\frac{\partial H_\tau}{\partial \tau}$. There are no electromagnetic waves in the ordinary sense (*).

(*)Note
This is an example of "scalar waves" which is widely discussed on the Internet. They arise naturally in these so-called "generalized" electromagnetic field equations , also accompanied by other "strange" solutions. Actually, that is, without making assumptions, and without posing the problem of arbitrary "gauge conditions", the most natural analytic solutions of the equation $\partial^* F = 0$ are solutions to 8 components. It remains to be seen if these solutions can have a physical sense. These solutions give automatically charges and fields which are self supporting in the empty space. What's better suited for an electromagnetic interpretation of elementary particles?

6 - The Dirac equation

The fluid equations, special functions P_l^m, the powers of the point in 2D and 3D, Maxwell's equations, the waveguide equations , all could lead us to say that the combination of many things: (1) has a mysterious meaning or (2) is a matter of elegance, or (3) means nothing and is only an analogy completely irrelevant, meaning only that in all cases is formally present the same ∂^* or (4) might suggest that we are facing fundamental "laws" even beyond our intuition, or that (5) whenever there is at stake $div\ rot$ there are similarities of which is useless to ask the meaning, analogous to the observation (3), or (6) analogy follows from mathematics, which formally put into play the analytic functions (2D, 3D, 4D) and so for this common properties born, common functions etc. or ..... and so on.

In fact I believe that what we are facing is an appropriate definition of number, and a deep statement (*) and a general physical law, which is very general because it says .... nothing and says nothing, more, in the form of tautology, and therefore presents itself as a thing in itself true, and that is no more a law of physics but a law of our language ..... and is therefore also, as a herald of information, information-rich, because it responds to what I think is the fundamental question of physics, that is, "but we, what the hell we wanted to say?" (ie, a review of language).

This law can be expressed in various ways, but roughly it is:

"I will describe the phenomenon (which I'm starting to describe and what happens in space and time) … with functions of space and time", followed (or accompanied by, or equivalent to) also tautological statements such as:" the functions of space and time I will use will obey the general laws .... which must obey the functions of space and time (∂^*=0)".

It is clear that when faced with statements so powerful no one can attack.

It is equally clear that this is not physics, but the study of ourselves. We might even get to say, grandly, and ambitiously, to "have discovered a new law of physics", or have discovered the law of physics if you completed the previous statements with this other: "I will use the condition ∂^*=0 in all cases where this is appropriate to use it".

At this point (I think) we are incontestable.

So?

So when in 1928 he invented Dirac the operator "square root of $E^2 = p^2 + m_0^2$" is found, without realizing it, to discover (**) the ∂^* operator.

The problem of Dirac was, ultimately, to bring to 1-st order the relativistic 2-th order Klein Gordon equation. Solved it with a 4x4 matrix, and "spinors". The Dirac equation is thus look quite unfriendly. Developed in full, in one of its various possible equivalent "forms", it is equivalent to a system of 4 equations in 4 complex quantities ψ_1, ψ_2 , ψ_3, ψ_4, with indices 1, i (and therefore a system of 8 equations in 8 real quantities).

If we now take a quantity F (or Ψ) to 8 components and write

$$\partial^*\Psi = \hat{\imath} m_0 \Psi i \hat{T}$$

we get, by developing, a system of equations formally identical to the Dirac equation.

(*)

Which denied, as Niels Bohr said, is still a deep statement.

(**)

Being $\partial\partial^* = \partial^*\partial = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}$ and $\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + T\frac{\partial}{\partial \tau}$ and being $\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{\partial^2\Psi}{\partial \tau^2} = m_0^2\Psi$ the Klein Gordon equation, ie between operators $-p^2 + E^2 = m_0^2$.

Moreover, taking the ∂ of both members and considering that (as is easy to verify)

$$\partial\hat{\imath} = \hat{\imath}\partial^*$$
$$i\hat{T}i\hat{T} = 1$$

we get immediately (with $m_0 > 0$ or with $m_0 < 0$):

$$\partial\partial^*\Psi = \partial\hat{\imath}m_0\Psi i\hat{T} = \hat{\imath}\partial^* m_0\Psi i\hat{T} = \hat{\imath}m_0\left(\hat{\imath}m_0\Psi i\hat{T}\right)i\hat{T} = m_0^2\Psi$$

and then the desired Klein Gordon equation.
But what lies beneath?
Why these equations so reminiscent of Maxwell's equations $\partial^* F = 0$? (Just put formally $m_0 = 0$).
It is said in a paper of Cambridge group:
“when in 1928 Dirac “square rooted” the quantum operator $E^2 = p^2 + m_0^2$ not only … (etc.) but he also uncovered the geometric rules governing Minkowsky spacetime!”. (The exclamation point is in the writing of Cambridge).
But why?
And Dirac encountered in the properties of spacetime, or in the properties of our language? (In the sense of the properties we have assigned to our descriptive language, with the introduction of the ∂^* operator). And why Dirac's equations are so? Would not it be more logical if they were $\partial^*\Psi = 0$?
I will try in the following paragraphs to illustrate the concept, or if you want to call it the hypothesis that the Dirac equations are approximate in the sense of a "ray approximation" or physical optics, and that the true equations are actually ... $\partial^*\Psi = 0$.
I remember what they are (and how they relate) the Schroedinger equation, Pauli, Dirac and Klein Gordon equation.
They are the differential equations that govern the spacetime evolution of the "wave function" Ψ, which, however, we consider only the meaning of $\Psi\Psi^* = |\Psi|^2$ or so.
The Schroedinger equation is to second order in space and first order in time.
The Klein Gordon equation is the relativistic Schroedinger (to second derivatives) and includes, as non-relativistic approximation, the Schroedinger equation. All are scalar equations. The Pauli equation is no longer scalar, but with components, so that it can take account of the polarization (say the spin). The Dirac equation is to the 1-st order, 4 complex components 1, i, relativistic, includes and generates all the others and is the analogue of Maxwell's equations with respect to the wave equation (which here becomes the Klein-Gordon equation, ie with mass).
Practically the Klein Gordon equation is equivalent to the formula of relativistic mechanics

$$E^2 = p^2 + m_0^2$$

while the Schroedinger equation is equivalent to the approximate mechanical relationship

$$\frac{1}{2}m_0 v^2 \cong E_{(kinetic)} = \frac{1}{2}\frac{p^2}{m_0}$$

This is the nonrelativistic approximation of $E^2 = p^2 + m_0^2$.
Note that in this context, the Maxwell equations are the equivalent of the mechanical relationship

$$E = pc$$

valid for the light.
Each scalar component of Dirac equation still satisfies the Klein Gordon equation (as well as each scalar component of the Maxwell's equations satisfy the wave equation).

7 - The equations of Maxwell and Dirac compared

The equations of Maxwell and Dirac are, for a 8-components F:

$$\partial^* F = 0$$
$$\partial^* F = \hat{\imath} m_0 F i \hat{T}$$

Developing the equations, for example in a system of 4 complex equations with components 1, i, or 8 scalar equations between scalar quantities, the two equations coincide, one being a particularization of the other with $m_0 = 0$. Put in Maxwell's equations:

$$F = (E_t + jE_l) + Tji(H_t + jH_l)$$
$$E_t = E_x + iE_y$$
$$E_l = E_z + iE_\tau$$
$$H_t = H_x + iH_y$$
$$H_l = H_z + iH_\tau$$

This is just a different way of grouping the field quantities, being F the usual (and already defined in paragraph 5):

$$F = \left(E_x + iE_y + jE_z - TH_\tau\right) + Tji(H_x + iH_y + jH_z + TE_\tau)$$

The grouping, as we see, highlights the transverse quantities (x, y plane) with respect to the longitudinal ones ("perpendicular" to the x, y plane). With a little of steps, by separating the indices as indicated, we reach to:

$$j, ij \qquad \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) E_l + \frac{\partial}{\partial z} E_t + \frac{\partial}{\partial \tau} iH_t = 0$$
$$1, i \qquad \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) E_t - \frac{\partial}{\partial z} E_l + \frac{\partial}{\partial \tau} iH_l = 0$$
$$Ti, T \qquad \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) iH_l + \frac{\partial}{\partial z} iH_t + \frac{\partial}{\partial \tau} E_t = 0$$
$$Tji, Tj \qquad \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) iH_t - \frac{\partial}{\partial z} iH_l + \frac{\partial}{\partial \tau} E_l = 0$$

These are exactly the Dirac equation with $m_0 = 0$ for the complex quantities (*):

$$\psi_4 = E_l$$
$$\psi_3 = E_t$$
$$\psi_2 = iH_l$$
$$\psi_1 = iH_t$$

as written in books and developing the formula $\partial^* F = \hat{\imath} m_0 F i \hat{T}$, namely:

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} - im_0\right)\psi_1 = 0$$
$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im_0\right)\psi_2 = 0$$
$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_2 + \frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} + im_0\right)\psi_3 = 0$$
$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_1 - \frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im_0\right)\psi_4 = 0$$

(*)
In my later paper of 2009, "Clifford Algebra and Dirac equation for TE, TM waveguide", viXra: 0910.0059 and later I used an opposite sign in $\partial^* F = \hat{\imath} m_0 F i \hat{T}$, ie $\partial^* F = -\hat{\imath} m_0 F i \hat{T}$. This results, in the Dirac equations written in extended form, to a change of sign of m_0 that is reflected in an exchange between the pair ψ_1, ψ_2 and the pair ψ_3, ψ_4, with the result that ψ_1, ψ_2 are identified as electric fields and ψ_3, ψ_4 magnetic (n.d.r.).

Why the comparison? What interests? What about the comparison? What is?
Before continuing, I would like to make some considerations at least in words, to explain the meaning (one of the meanings) of these ruminations.
For a long time, the meaning of the Dirac equation has been hidden. Indeed, it is still hidden, since there is certainly no statement about my knowledge of the kind that I'm doing. For me, both equations must lead necessarily to a formulation even more common that we will discuss in a little, and that is, for both, the assertion of being electromagnetic fields (or, if you will, of being both the expression of analyticity).
We come to us.
The meaning, the hidden meaning of the Dirac Ψ components, expressed as field quantities, it was not clear for a long time, first of all .... because it was forbidden to ask. And it is still prohibited. The only meaning is given to $\Psi\Psi^*$ (*). I have said that to me this means that a priori they give up knowledge of phase and describe the phenomenon without the phase of Ψ (which, moreover, in the absence of two channels I and Q "phase detector", we would be forced to do already in the infrared, or already at 300 GHz, or already in W-band, if we had not the technology. Thus, the lack of knowledge and technology has led to a prohibition) [5].
Let's get back to us.
The correspondence I wrote between ψ_1, ψ_2, ψ_3, ψ_4 and the E_t, E_l, H_t, H_l components of the electromagnetic field is not the correspondence. It is a possible correspondence. In fact, the Dirac equations have several possible "forms of the Dirac equation" (evidently through all the different possible groupings that can be created between the components, while leaving intact certain quantities which are the only ones to which we give meaning, ex. $\Psi\Psi^*$).

However, the correspondence we found at least creates a clue, or a way to try to assign a meaning to the Dirac components. I could give many other examples but I do not want to dwell too much: only note that the plane wave solutions of the Dirac equation lead to the following 4 very simple solutions at rest:

$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4) = \left(e^{i\omega_0 t}, 0,0,0\right)$$
$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4) = \left(0, e^{i\omega_0 t}, 0,0\right)$$
$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4) = \left(0,0, e^{-i\omega_0 t}, 0\right)$$
$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4) = \left(0,0,0, e^{-i\omega_0 t}\right)$$

in quantum mechanics interpreted as matter and antimatter, spin here and there. These, transported in terms of E_t, E_l, H_t, H_l, have an electromagnetic suggestive meaning, but some clearly ambiguous and incomplete.
On the other hand, it does not have to worry about even the slightest, because if it is viewed in electromagnetic terms, the solution of the Dirac equation has a clear field components too light. For example, ψ_3=$e^{-i\omega_0 t}$ means $E_x + iE_y = e^{-i\omega_0 t}$: the field would be a transverse electric field that turns alone in circular polarization. Suggestive, but somewhat incomplete.
So, in summary, we may have groped a way for an identification. No more, no less. Go on in the next section.
But before I do another suggestive example. The ability to write Maxwell's equations in terms of ∂^* instead of div, rot,.... now opens the possibility to write the Dirac equation in terms of div, rot instead of ∂^*. They sound so for $\rho = 0$:

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{H} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} + im_0 \vec{H}, \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + im_0 \vec{E}$$

(*)
and some other “observable” quantities.

Taking $\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})+im_0(\vec{\nabla}\times\vec{H})$ and substituting $\vec{\nabla}\times\vec{H}$ it gets in a little steps:

$$\nabla^2\vec{E}-\frac{\partial^2}{\partial t^2}\vec{E}=m_0^2\vec{E}$$

that is, the wanted Klein-Gordon equation for the field components (the same for $\vec{H}$).
So those are "the 1-st order equations corresponding to the 2-th order Klein-Gordon equation", that is the Dirac equations.
But if you look at them, and think the imaginary i that appears in electrical terms, we see that the equation

$$\vec{\nabla}\times\vec{H}=\frac{\partial\vec{E}}{\partial t}+im_0\vec{E}$$

has a term that contains a current

$$\vec{J}=im_0\vec{E}$$

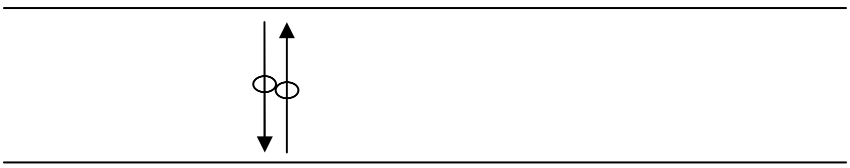
instead of the usual

$$\vec{J}=\sigma\vec{E}$$

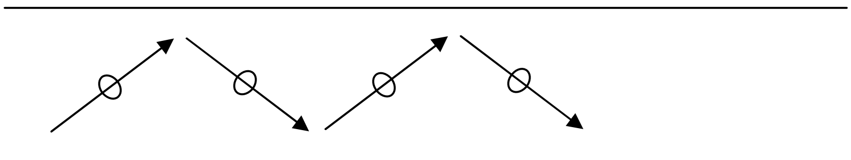
we usually consider.
$\vec{J}=\sigma\vec{E}$ (for a medium with $\sigma\neq 0$) means energy dissipation by Joule effect. A term $\vec{J}=im_0\vec{E}$ (such as $I=i\omega V$) means "quadrature", ie, reactive energy, ie energy that is there but does not dissipate.
Hence the obvious speculations (for example, "equivalent circuits") that I skip, of energy storage in space inductance or capacitance.

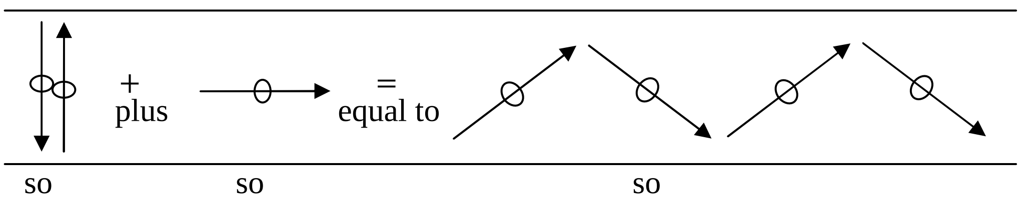
8 - The waveguides

The waveguides are the "place" where we can best give an account of how the equations for a particle with mass born.
I will detail how in waveguides is worth exactly the Schroedinger equation (or the corresponding "exact" relativistic Klein Gordon equation) for a particle of mass m_0 (ω_0).
Use "savagely" the units, in the sense that I write either, considering them equivalent, ω_0 or m_0 or the energy E_0 (and so I will for momentum p and the wave vector k). Suitably choosing the unit of measure for c (or the value for Planck's constant ħ) ω is identified with the energy ($E = ħω$) and the energy at rest is identified with mass ($E_0 = m_0c^2$).).
The fundamental characteristic of a particle with mass is to give an eigenvalue for the energy operator at rest that you identify with $m_0 = \omega_0$ (in other words this means that the wave function or the field have, at rest, the appearance $e^{\pm i\omega_0 t}$, as indeed we have seen is in the Dirac equation).
The structure of the field in a waveguide, in particular to a specific "mode" in waveguide with cutoff ω_0, is first of all it comes into play only if the frequency or energy reach at least the value ω_0 (otherwise not born a damn thing).
If the frequency, that is the available energy, is at least ω_0 for the first time a field born and appears in the waveguide.
<u>It</u> is <u>standing.</u>
(In fact, its energy is barely enough to keep it alive there, standing).
The "descriptive" structure of the field in these conditions, as is well known (see for example Ramo Whinnery) is that of a wave that dribbles between the wall and stays there, like this:



When, if and when the frequency or the energy grows, the field travels, like this:





There are three characteristic quantities related to each other, describing the field in waveguide in one of these generic situations, and they are:

- the frequency (ie ω , ie the energy E);
- the wavevector k_z (or the momentum p along the z axis of propagation);
- the k_c (or k_{cutoff}, or ω_0, or the rest energy or mass m_0;

and these quantities are related by:

$$\omega^2 = k_z^2 + k_c^2$$

ie:

$$E^2 = p^2 + m_0^2$$

At rest $p = 0$, $E^2 = m_0^2$.
In particular, therefore, at rest (at the cutoff frequency) where $k_z = 0$, it is $\omega^2 = \omega_0^2 = k_c^2$.
The condition on k_c is imposed on the boundary conditions.
So, conclusion:
the mode in question operates in obedience to mechanical relationships (relativistic) which obeys a particle of mass m_0, momentum p and energy E (E_0=ω_0). At rest, the energy is there, dribbling.
You can have fun digging in all the formulas and / or analogies that come to mind, but it all leads to a strict correspondence between the mode of "mass" $\omega_0 \equiv m_0$ and a particle of mass $m_0 \equiv \omega_0$ (eg in the waveguide the λ exhibited by the mode is exactly equal to the λ of a particle of mass $m_0 \equiv \omega_0$ moving with momentum p, exhibited in diffraction experiments performed with particle beams).
On the other hand, it is difficult to argue that it is a simple formal analogy (or rather: a stupid would say that it is a formal analogy. In reality we have …. light standing, or travelling, in both cases).
It is also however instructive to study all possible comparisons and analogies that I said (*).
But we go further, because interesting things are those that follow.
Each field component in waveguide fulfills the equation (see Ramo Whinnery)

$$(1) \qquad \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = k_c^2 F$$

that is .... the Klein Gordon equation for a particle moving with momentum $p = p_z$ in the direction z and mass $m_0^2 = k_c^2$.

$$E^2 - p_z^2 = m_0^2$$

$$(1bis) \qquad \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial \tau^2} = m_0^2 F$$

(Therefore, in non-relativistic approximation for small velocities, or for ω that differ little from ω_0, the Schroedinger equation).
But then ..... someone is stealing, what's the catch?
How is it that every component of the field satisfies the Klein Gordon equation .... as this is an electromagnetic field and then satisfies the Maxwell equations .... and then to the 2-th order satisfies the wave equation:

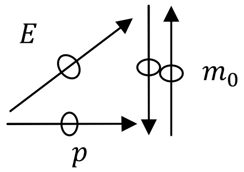
$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial \tau^2} = 0$$

and not to Klein Gordon equation:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial \tau^2} = m_0^2 F$$

It's simple.
Each field component meets (it seems that meets) in the waveguide the Klein-Gordon equation because each field component also meets the eigenvalue equation:

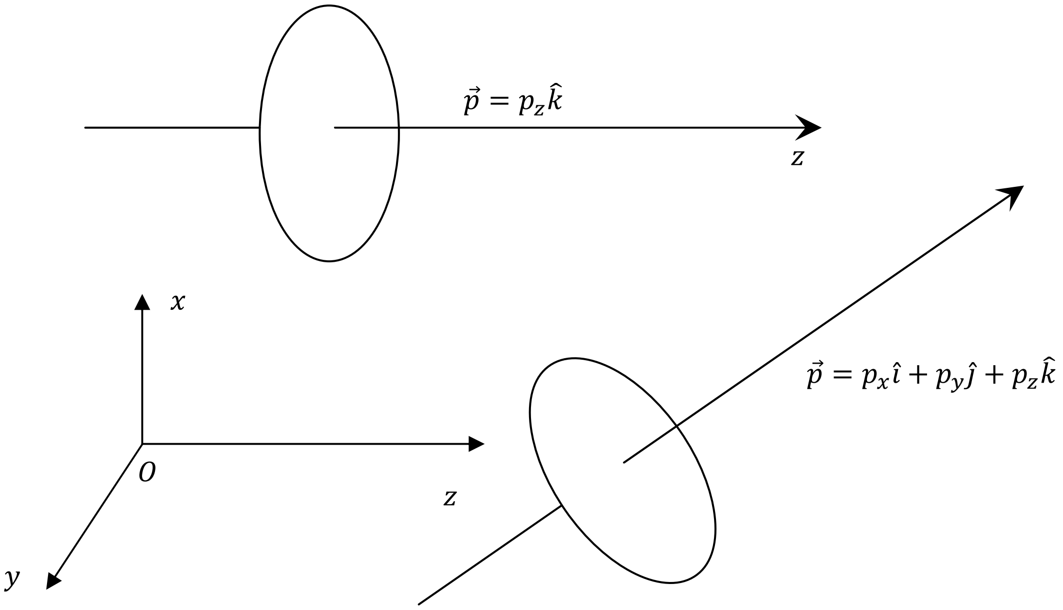
(*)
You can make a fun, picturesque and non casual observation by noting that the graphic composition of movements, with the notations.



according to the Pythagorean theorem corresponds to the relativistic formula $E^2 = p^2 + m_0^2$.

$$(2) \qquad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -k_c^2 F$$

Something (here the walls) results for the eigenvalue at rest k_c^2, ie the mass.
Let now the last step (which is obvious but I want to highlight). If we neglect the variation of the field on the transverse plane, or plane x, y, or assume a plane wave approximation of the field, the Klein Gordon equation (1bis) or (1) with respect to z becomes strictly true for any direction of propagation $p \neq p_z$:



that is true in its complete form:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial \tau^2} = m_0^2 F$$

(So this is not strictly for the field F, but for the plane wave approximation of the field F. We say, "physical optics").
(I just want to say that this would be a more than adequate for a wave packet or photon in the infrared (for example) that were propagated in a suitable infrared waveguide. Probably, not knowing anything about measuring the transverse plane, we would be more than happy to describe it for plane waves).
So we summarize things.
If Schroedinger in the ‘20s was more familiar with the fields in the waveguide he would have probably immediately written ... "the wave equation for a particle of mass m_0" that is
“ a scalar wave function $\psi(z, t)$ that properly represent all the mechanical characteristics of the motion (the relationship between mass, momentum, energy) and at the same time as to reproduce correctly the diffraction experiments with particle beams".
This would mean writing immediately (as indeed he did) while recognizing that he had only to assume, for the need, a $\psi(z, t)$ scalar component of a field in waveguide. With this he also highlighted the significance of rest mass: the dribbling of field energy, which does not travel, but it is there.
He also would have clearly understood what he was doing, namely, that it was a satisfying plane wave approximation, scalar (what, however, largely satisfactory). He acknowledged that he was doing a ray approximation, or as we say "physical optics".

Wanting to satisfy the wish for deflect these rays moving in within a field (eg electric) would have been enough, more or less (as indeed he did ....) to put in the phase (ie adding to the energy) even the energy imparted from the field (that is $eV = \frac{Ze^2}{r}$). Having put that term in the phase (ie the action) automatically to track the ray where the phase (the action) is stationary would have described "curved beams", which bend within the electric fields (or similar within magnetic fields). This is what he did, more or less.
The same thing he did Dirac when in the 1-st order equations adds to the "Hamiltonian" (total energy) the effect of electric and magnetic fields.
But, back to Schrödinger, I would say that would automatically have done much more, because:

- would write an equation that would naturally ... relativistic (ie directly the Klein Gordon equation, which would have shown "Schroedinger equation" for low speed or in non-relativistic approximation);
- would have been obvious that a scalar wave function (ie representing the plane wave with a single scalar component) could only represent a particle without spin, and to represent a particle with spin (the Pauli equation later) would should hold at least two field components.

And so on and so forth. There is a whole series of further matches:

- $\Psi\Psi^*$ is the energy of the field (*), and normalizing to 1 we obtain the "probability of finding the particle where" (of course: it is its energy distribution);
- the wave packet is lost: $\psi(z,t)$ after a while disappears (translation: the waveguide is dispersive, the group velocity is a function of frequency for which the various components of the Fourier spectrum of a wave packet after a while discarding).

Et cetera. Because of these clues we have, I believe, to examine possible relationships between the equations of 1 st order, that of Maxwell and Dirac. Up to now we would say clearly understood meanings and relationships between the 2-th order equations. From examination of the equations of order 1, we can expect all that huge enrichment of meanings that bring ..... Maxwell's equations rather than the most trivial wave equation. From what we saw on the role of "trick" performed by k_c^2, which is carried by 1-st to 2-th member so apparently changing the equation from the wave equation to the Klein Gordon equation, we can expect a game (or role) that is similar on the 1-st order equations ie the Maxwell equations.
In short, the suspicion is that the separation constant k_c (or ω_0) in Maxwell's equations is carried by the 1 st to 2 nd member, so generating a plane wave approximationor or "physical optics" for which the Dirac equation is true.
Also would love to write a book but step over (**).
Let's look at aspects of mathematical physics, and probably not random combinations with quantum mechanics.
I think there is in mathematics (or mathematical physics) a similar enrichment of meaning resulting from the transition to the study of ∂^* with respect to the study of $\partial\partial^*$.
The ∂^* Hestenes says has never been treated in mathematical physics ... simply because was an operator unknown in mathematical physics. Entire classes of polynomials and special functions,

----

(*)
More precisely $\Psi\hat{T}\Psi^*$, or $F\hat{T}F^*$ for the electromagnetic field, gives energy and momentum four-vector, see my later writings (n.d.r.)

(**)
I have faced and solved this in my later writings (n.d.r.)

which were all known from 2 nd order equations, are returned to the solution of equations of the 1 st order (as we shall see, Bessel functions J_n, spherical Bessel functions j_n, Legendre polynomials P_l^m, etc.) with types of links that I assimilate to those of $e^{i\varphi}$ and $cos\,\varphi$, $sin\,\varphi$ not knew as long as we define the parts $cos\,\varphi$, $sin\,\varphi$ through 2 nd order differential equations, without realizing that the real function (and Euler's formula) are defined by a differential equation with first derivatives. Etcetera. This, at least, is what it seems to me.
The better "place" to ask ourselves these kinds of questions is once again a problem of waveguides in circular symmetry (not rectangular) in which, presumably, formulas of circular polarization or "spin" assumed a more spontaneous form.
We must therefore examine in some detail the solutions in the circular waveguide, with the separation of variables not made with the 2 nd order k_c^2 but with a (presumably) 1^{st} order k_c ..... that is, in the Maxwell's equations $\partial^* F = 0$.
The problem is that ..... nobody has ever raised the issue of separating the variables in the Maxwell's equations, 1st order. Usually it is done in the wave equation, if only because no one in mathematical physics has never used the operator ∂^* if only because ... there was not (or maybe someone in the United States did. If someone did it and came across charged rays, I think someone else has asked the question "could be useful these charged rays to shoot him anyone?").
I will do in the next section.

9 - The analytic functions in the circular waveguide

The problem of separation of variables in the first order equation in circular waveguides

$$\partial^* F = 0$$

is of great simplicity and elegance (after it is done ….) although I will stay longtime in mathematical passages, and is preparatory to the more x, y, z, τ general problem in that they are then .... spherical cavities.

The above equation, ie the Maxwell equation in a circular waveguide, is from a purely physical mathematical point of view such as "find the functions that are analytic functions of the coordinates x, y, τ".

(Solve the problem for the electromagnetic field "at rest" for not introducing a boring additional dependence on z, so much is obvious).

The international research groups on Clifford algebra, in my opinion, discovered a "play that works" and gives a lot of mathematical results, but just not understanding why it works (that is, it works because it does not mean anything if not the fact, important, to clarify our language, and the functions that we propose to use in our language) they depart from the already complicated issues, like the Dirac equation, the "monogenic functions" Ψ_l^m, to go to 'upward as a complication (Lie algebra, n-dimensional spaces, symmetries SU (3) and SU (4), families of particles) because obviously being mathematicians, they generalize

That is, what is the starting point, in itself new, and difficult to appreciate and understand, to understand they generalize

Instead I think it's important to come to downward, that is to review all the most basic cases that we have taught as basic structures of mathematical physics or electromagnetic fields, to be able to handle and appreciate these new tools, to answer the famous fundamental question of physics, "but we, what the hell were we saying?".

So, returning to the separation of variables in the equation $\partial^* F = 0$ in the circular waveguide, we do not even have the faintest idea how or what it means to separate the variables.

There is, for example, an anticommutativity problem among indices, so the usual writing a solution, in order to separate, for example in the form:

$$F(\rho, \varphi, t) = R(\rho)\Phi(\varphi)T(t)$$

as we write here? (Because it does matter which order you write the terms).

Proceed by trial and I suppose, for various reasons, to put

$$F = F_1(x, y)e^{-i\omega_0\tau}$$

I make the assumption that $F_1(x, y)$ it contains indexes ex. $1, i, Ti, T$ and I suppose a term $e^{-i\omega_0\tau}$ with the index i which (as a bivector $i = \hat{\imath}\hat{\jmath}$) produces rotations on the x, y plane (*).

To say that the solution has that form means that there is k_c, an eigenvalue for the rest energy ω_0 or for the operator $\frac{\partial}{\partial \tau}(F)i$ which provides $\frac{\partial F}{\partial \tau} i = \omega_0 F$.

Then the correct operator which gives an eigenvalue ω_0 on $e^{-i\omega_0\tau}$ or better on the function $F = F_1 e^{-i\omega_0\tau}$ (where F_1 has various indexes) is more or less the same as in quantum mechanics

$$i\frac{\partial F}{\partial \tau} = \omega_0 F$$

but it really is $\frac{\partial F}{\partial \tau} i = \omega_0$ (it is a fundamental position of i on the right).

(*)

I absolutely do not remember quite I decided to arbitrarily choose the sign $e^{-i\omega_0\tau}$ instead of $e^{+i\omega_0\tau}$.
I probably chose the sign $e^{-i\omega_0\tau}$ instead of $e^{+i\omega_0\tau}$ only to better propose an analogy with quantum mechanics, where the energy operator is $i\frac{\partial F}{\partial \tau}$, and produces on $e^{-i\omega_0\tau}$ the result $i\frac{\partial F}{\partial \tau} = \omega_0 F$ that is a positive eigenvalue $+\omega_0$ for the energy (n.d.r.).

But in short, to sum up, apart from "energy eigenvalues" and so on, I have to solve the equation:

$$\partial^* F = 0$$

$$\partial^*_{xy} F + T\frac{\partial}{\partial \tau} F = 0 \qquad \rightarrow \qquad \partial^* = \partial^*_{xy} + T\frac{\partial}{\partial \tau}$$

Put, with $F_1 = F_1(x, y)$ having many indexes:

$$F = F_1(x, y)e^{-i\omega_0 \tau}$$

and get:

$$\partial^*_{xy} F_1 e^{-i\omega_0 \tau} + T\frac{\partial}{\partial \tau}\left(F_1 e^{-i\omega_0 \tau}\right) = \partial^*_{xy} F_1 e^{-i\omega_0 \tau} - T\omega_0 F_1 i e^{-i\omega_0 \tau}=0$$

It can therefore (in spite of such anticommutative indices) simplify $e^{-i\omega_0 \tau}$ from <u>right</u> and you get the new equation, having separate x, y or ρ, φ from τ:

$$\boxed{\partial^*_{xy} F_1 - T\omega_0 F_1 i = 0}$$

Writing explicitly $\partial^*_{xy} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ and allowing the possibility of a possible double sign for $k_c = \omega_0$ you finally reach equation (*):

$$\boxed{\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = \pm T k_c F i}$$

which is the first order equivalent to eq. (2) of the preceding paragraph

$$\boxed{\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -k_c^2 F}$$

<u>We then found the first order equivalent to separation of variables in the wave equation</u> (what immediately appears, among other things, is that <u>two</u> constants are possible that correspond to the k_c^2, ie $+k_c$ and $-k_c$).

I put now, looking for a TE, F_1 in this form:

$$\boxed{F_1 = E + TH}$$

where hypothesize E with indices $1, i$ and as well H with indices $1, i$, as I expect for a TE the components at rest E_x, E_y, H_z, or E_r, E_φ, H_z (see Ramo Whinnery) <u>and</u> therefore I expect that the term $e^{-i\omega_0 \tau}$, by rotation, produces on H the terms from TH_τ to TiH_z and from TiH_z to TH_τ.
Briefly, I would expect components E_x, E_y, H_z, H_τ and indexes $1, i, Ti, T$.
Substituting and separating the indices we have:

$$\partial^*_{xy} F_1 - T\omega_0 F_1 i = \partial^*_{xy}(E + TH) - T\omega_0 (E + TH) i = 0$$

$$\partial^*_{xy} E - \omega_0 H i = 0$$

$$\partial^*_{xy} TH - T\omega_0 E i = 0$$

I note that $\partial^*_{xy} T = T\partial_{xy}$ and then replacing and simplifying T from the left finally arrive to the following system of two equations in two complex quantities $1, i$:

(*)
<u>So</u> the key is: if F_1 is analytic on the transverse plane, $\partial^*_{xy} F_1 = 0$, you have a static field or, in motion, a TEM, <u>the mass is zero</u>. If it falls the analyticity on x, y, <u>we have mass.</u> The mass thus presents itself as the eigenvalue of the operator ∂^*_{xy} (in space will be ∂^*_{xyz}).

$$\partial_{xy}^* E - \omega_0 H i = 0$$
$$\partial_{xy} H - \omega_0 E i = 0$$

Change coordinates to r, φ where (as I seem to have already written elsewhere):

$$\partial^* = \frac{r^2}{r^2}\partial^* = \frac{zz^*}{r^2}\partial^* = \frac{z}{r}\frac{1}{r}z^*\partial^* = e^{i\varphi}\frac{1}{r}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\varphi}\right) = e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi}\right)$$

and then $\partial = e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right)$ so I have the system:

$$\boxed{\begin{aligned} e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi}\right)E - \omega_0 H i = 0 \\ e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right)H - \omega_0 E i = 0 \end{aligned}}$$

I try now a separation between the two remaining variables r and φ. I ask for this:

$$E = R_E \Phi_E$$
$$H = R_H \Phi_H$$

and replace

$$e^{i\varphi}\left(\frac{\partial R_E}{\partial r}\Phi_E + \frac{i}{r}\frac{\partial\Phi_E}{\partial\varphi}R_E\right) - \omega_0 R_H \Phi_H i = 0$$

$$e^{-i\varphi}\left(\frac{\partial R_H}{\partial r}\Phi_H - \frac{i}{r}\frac{\partial\Phi_H}{\partial\varphi}R_H\right) - \omega_0 R_E \Phi_E i = 0$$

One option (looking at the first equation) in order to simplify the angular functions Φ, thus leaving an equation for the only variable r, is that it is $e^{i\varphi}\Phi_E = \Phi_H$. In this way, between the first and last term, $e^{i\varphi}\Phi_E$ goes away with Φ_H.
If $e^{i\varphi}\Phi_E = \Phi_H$ it is also $\Phi_E = e^{-i\varphi}\Phi_H$, which is what you need to simplify Φ_E and $e^{-i\varphi}\Phi_H$ in the second equation.
On the other hand, this double condition

$$e^{i\varphi}\Phi_E = \Phi_H$$
$$e^{-i\varphi}\Phi_H = \Phi_E$$

is simply interpreted as:
"Φ_E turns, and Φ_H turns with a turn $e^{i\varphi}$ on more".
That is:

$$\Phi_H = e^{i(n+1)\varphi}$$
$$\Phi_E = e^{in\varphi}$$

with this, differentiating, we get (*):

$$\frac{\partial\Phi_H}{\partial\varphi} = i(n+1)\Phi_H$$
$$\frac{\partial\Phi_E}{\partial\varphi} = in\Phi_E$$

so, replacing everything, you get a huge simplification and remains:

(*)
Note: this is equivalent to the eigenvalues $(n+1)$ and n for the orbital angular momentum operator L_z. In fact $-i\frac{\partial\Phi_H}{\partial\varphi} = (n+1)\Phi_H$ and $-i\frac{\partial\Phi_E}{\partial\varphi} = n\Phi_E$.

$$\frac{\partial R_H}{\partial r} + \frac{1}{r}(n+1)R_H - iR_E = 0$$
$$\frac{\partial R_E}{\partial r} - \frac{1}{r}nR_E - iR_H = 0$$

These two coupled equations have a structure that recurs over and over again in quantum mechanics (*) (eg Fock, Schiff, Bjorken and Drell, Landau. They have the structure of the equations for the radial part of the spin components of the Dirac equation for Hydrogen atom).
They have a structure that always recurs ..... because it is the 1 st order equivalent to other "usual" second order equations. Here these equations are the 1 st order equation equivalent to the Bessel equation (which is usually presented ... as an equation of the 2 nd order).
In fact if I get R_H from the second one and replace in the first (and vice versa for R_E) I get either:

$$\frac{\partial^2 R_E}{\partial r^2} + \frac{1}{r}\frac{\partial R_E}{\partial r} + (1 - \frac{n^2}{r^2})R_E = 0$$
$$\frac{\partial^2 R_H}{\partial r^2} + \frac{1}{r}\frac{\partial R_H}{\partial r} + (1 - \frac{(n+1)^2}{r^2})R_H = 0$$

resolved by J_n, J_{n+1} (or N_n, N_{n+1} or $H_n^{(1)}, H_{n+1}^{(1)}$ and so on and so forth).
The equations of 1 st order (which, as we shall see in a little are also a single complex equation to the first partial derivatives) put in evidence, however, that there is "a particular cluster" of Bessel functions that makes sense, a precise "couple" (which is the equivalent of not having to take $\cos n\varphi$, $\sin n\varphi$ but the function $e^{in\varphi} = \cos n\varphi + i \sin n\varphi$).
Take for example $R_H = iJ_{n+1}$

$$H = iJ_{n+1}(r)e^{i(n+1)\varphi}$$

Substituting in the equation of 1 st order which gives R_E you have:

$$R_E = \frac{\partial J_{n+1}}{\partial r} + \frac{1}{r}(n+1)J_{n+1} = J_n$$

In fact, are valid for Bessel functions the following recurrence equations (which are nothing but .... that system of 1 st order differential equations … valid for J_n, N_n,, $H_n^{(1)}, H_n^{(2)}$)

$$\frac{\partial J_{n-1}}{\partial r} - \frac{n-1}{r}J_{n-1} + J_n = 0$$
$$\frac{\partial J_n}{\partial r} + \frac{n}{r}J_n - J_{n-1} = 0$$

Summing up and recombining all we come to the end result: the solution F is like

$$F = (J_n e^{in\varphi} + TiJ_{n+1}e^{i(n+1)\varphi})e^{-i\omega_0\tau}$$

I say "like" because: n it can be anything in particular we have states to $n > 0$ and states to $n < 0$, then we have the corresponding solutions $e^{+i\omega_0\tau}$, and the Bessel functions may be the Hankel functions or something, and in any event would occur in pairs, et cetera.
Summing up solutions $e^{-i\omega_0\tau}$ with positive n and negative n, and then combining them with the corresponding pair but having $e^{+i\omega_0\tau}$ we obtain the TE fields in the circular waveguide.
The individual solutions rather have a charge ($\frac{\partial H_\tau}{\partial \tau}$, positive or negative) and mass m_0 (positive or negative).

(*)
In fact it would probably be best to put H as $H = R_H \Phi_H i$, but I do not want to redo the calculation. If I place $H = R_H \Phi_H i$ in the equations would not have appeared the ugly i in iR_E and iR_H. I also put $\omega_0 = 1$.

By measuring the spin of the solution F or better by measuring the orbital angular momentum L_z (in short, doing $-i\frac{\partial}{\partial\varphi}$) it is observed that the 1 st term of electric field gives an eigenvalue n.
The 2 nd gives an eigenvalue $n+1$.
<u>F as a whole does not give an eigenvalue</u> for the operator $L_z = -i\frac{\partial}{\partial\varphi}$.
However you can see that <u>the other</u> operator that I have already written, J_z, that of the total angular momentum (orbital + spin)

$$J_z = L_z + S$$

gives an eigenvalue ..... that is intermediate between n and $n+1$, and is $n+\frac{1}{2}$.
“The angular momentum of the analytical solutions F has a value $n+\frac{1}{2}$ composed of an orbital angular momentum n and spin ½”. <u>(Other</u> solutions have rather spin -1/2).
It's just a way to tell? I do not know, what I know is that <u>the exact solutions of electromagnetic fields in a circular waveguide are obtained by combinations (*) of solutions, with charge and current, and rest mass ω_0, moving with mass, energy and momentum obedient to the mechanical relationships provided by relativity for particles of mass $m_0 = \omega_0$, and show values $n+\frac{1}{2}$ as eigenvalues of the total angular momentum operator J_z. Fictitious?</u>
<u>Boh</u>!
Reviewing the final solution there is a curious peculiarity that is that in the expression

$$F = (J_n e^{in\varphi} + TiJ_{n+1}e^{i(n+1)\varphi})e^{-i\omega_0\tau}$$

the variables r, φ <u>are not</u> separated (as they are, however, in the individual parts E and H). In fact, the brackets do <u>not</u> appear as the product of a function of φ and a function of r.
However, (using $e^{-i^{\varphi}/_2}T = Te^{+i^{\varphi}/_2}$)) it has the new expression

$$F = e^{-i^{\varphi}/_2}(J_n + TiJ_{n+1})e^{i(n+{}^1/_2)\varphi}e^{-i\omega_0\tau}$$

where r, φ and τ <u>are separated.</u> It is the special appearance of a "hypercomplex Bessel function" formed from the collection $(J_n + TiJ_{n+1})$, with indices $1, i, Ti, T$ and at the same time appeared the exponential $e^{i(n+{}^1/_2)\varphi}$ with the exponent $\left(n+\frac{1}{2}\right)$.
There is thus a suspicion that everything is related to the particular operator which <u>has</u> eigenvalues on F

$$J_z F = (n+\frac{1}{2})F$$

Restart from the beginning looking for F as eigenfunction of the operator J_z with eigenvalues $(n+\frac{1}{2})$. We arrive with a little steps to the equation for the radial part

$$\boxed{\frac{\partial\mathfrak{F}}{\partial r}i - \frac{i}{r}\left(n+\frac{1}{2}\right)\mathfrak{F} + \frac{1}{2r}\mathfrak{F}i - T\omega_0\mathfrak{F} = 0}$$

<u>This differential equation of order 1, provides in one fell swoop the Bessel functions J_n of the form "hypercomplex" $\mathfrak{F}(r) = J_n + TiJ_{n+1}$ and the recursive equations between them.</u>
(Not only J_n, but also N_n,, $H_n^{(1)}, H_n^{(2)}$).

(*)
See Appendix A1.

If I ask $\mathfrak{F} = R_E + TR_H$ I get

$$\frac{\partial R_E}{\partial r} - \frac{n}{r} R_E + i\omega_0 R_H = 0$$

$$\frac{\partial R_H}{\partial r} + \frac{n+1}{r} R_H + i\omega_0 R_E = 0$$

that is the usual system of equations with the usual solutions (it is written differently from the previous one because this time (*) I have tried solutions $e^{+i\omega_0\tau}$).

This exercise serves not least to understand this: the emergence of concepts such as "eigenvalues of total angular momentum" or "orbital angular momentum" or "half spin" does <u>not</u> appear necessary in connection with quantum mechanics. The exercise we have just done, to resolve $\partial^* F = 0$, may in fact be thought of <u>not</u> as an exercise in quantum mechanics but:
exercise on a circular waveguide;
but also
exercise of physics about circular membranes;
but also
a <u>mathematical</u> exercise <u>on the analytic functions.</u>
<u>These are the facts.</u>

(*)
For detailed steps see Appendix A2

10 – Plot of "charged" solutions in the circular waveguide

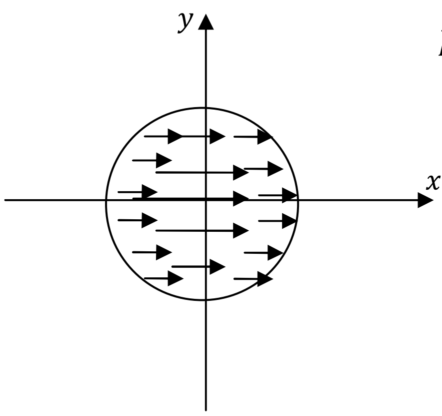
I provide only some indication for "plotting" the fields because it is too boring to consider the various solutions.
Take some states with n as low as possible:

$$F = (J_0 + TiJ_1e^{i\varphi})e^{-i\omega_0\tau}$$
$$F = (J_0 - TiJ_1e^{i\varphi})e^{+i\omega_0\tau}$$

or even ($J_{-1} = -J_1$)

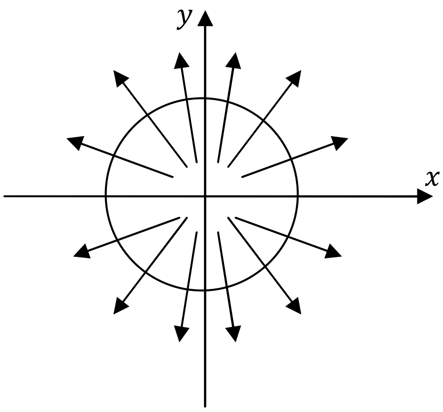
$$F = (J_{-1}e^{-i\varphi} + TiJ_0)e^{-i\omega_0\tau}$$
$$F = (J_{-1}e^{-i\varphi} - TiJ_0)e^{+i\omega_0\tau}$$

or combinations. For the plotting of fields should be undertaken as well.
1) by first we fix time. You draw a situation of electric field "frozen" on the plane, which then rotates rigidly with the operator $e^{\pm i\omega_0\tau}$ that induces rotations in the plane.
2) we pass to the conjugate .
3) we consider separately the electric field, which has components E_x, E_y or E_r, E_φ the x, y plane.
For example:



$\vec{E}$ electric field

J_0



$\vec{E}$ electric field

$J_{-1}e^{i\varphi}$

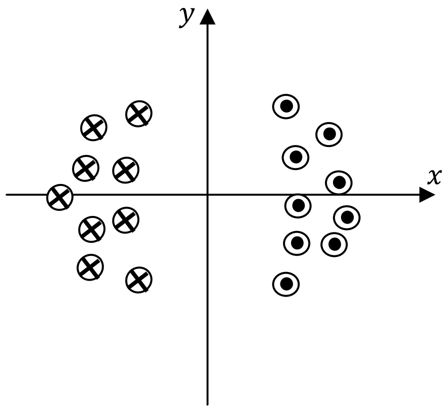
4) the magnetic field (*) has components T, Ti then H_τ, H_z (it should be split $e^{i\varphi} = \cos\varphi + i\sin\varphi$ and then separate H_z from H_τ).

(*)
Note: The operator $e^{\pm i\omega_0\tau}$ <u>does not</u> induce rotations on the x, y plane and on the H_z H_τ components. They oscillate with law $\cos\omega_0 t$ and $\sin\omega_0 t$. Can be drawn at a fixed t, then <u>oscillate.</u>

5) drawing H_z (example $TiJ_1 \, cos \, \varphi$ or J_0)

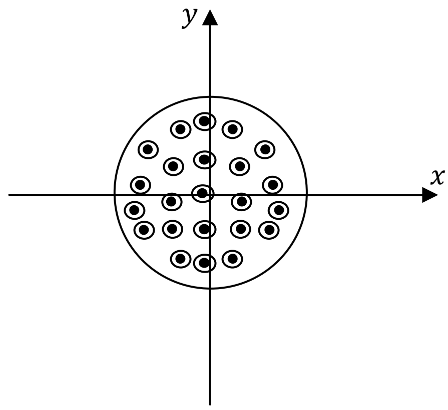
H_z magnetic field

$J_1 \, cos \, \varphi$



H_z magnetic field

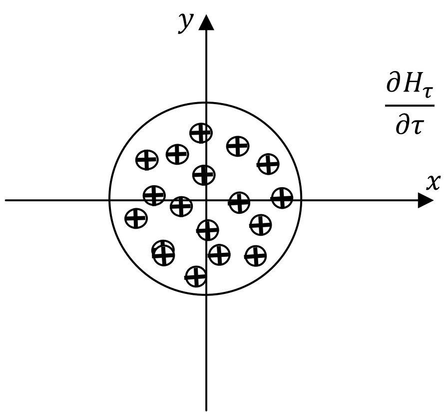
J_0



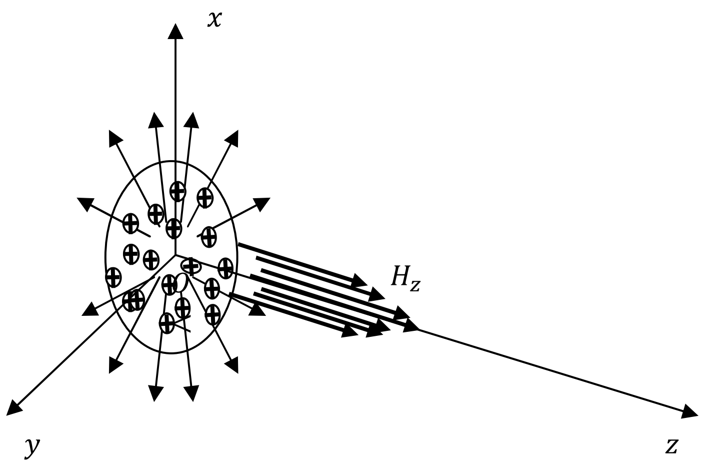
6) drawing the electric charge $\frac{\partial H_\tau}{\partial \tau}$ (for example with $H_\tau = J_0 \, sin \, \omega_0 \, t$ you made the derivative and then you draw)

charge

$$\frac{\partial H_\tau}{\partial \tau}$$



Overall, for example, we have solutions in which a charge oscillates exchanging with a pulsating component H_z.



7) if you want you can draw the currents.

8) you can try to interpret the various electric and magnetic fields that arise, relating to charges and currents that generate them.
The meanings are obscure.

Please note that from a physical point of view the electromagnetic field is self-sustaining, with terms of the Poynting vector (energy and momentum of the field) that are related to the Lorentz force at each point.
At each point the Lorentz force induced on each element of charge at that point a variation of energy and momentum of the field which, among them, are self-sustaining.

11 - Analytic solutions in 4 dimensions, or spherical cavity

As simple "electricians", having interpreted the Dirac equation as an approximation of physical optics of "heavy and charged beams", we can afford to find the orbital configurations of the electron in the hydrogen atom (*).
We can treat this problem without any reference to quantum mechanics.
In fact, since we have clear the kind of approximation, we are induced (at least I am induced) to look carefully the kind of spatial configuration of the field components resulting (**).
What we can have more trust .... are the values or eigenvalues of energy (we call them: the natural frequencies of oscillation) that result from the solution.
Yet strangely (and this must be said in honor of the tremendous capacity of the physical optics approximation) one gets also information on the spatial distribution of "orbits", which are described from ψ_l^m, and are listed by ψ_l^m .
However instead is entirely clear and unambiguous the problem of spherical cavity, where the oscillation modes are also described by ψ_l^m, and enumerated by ψ_l^m. In this second case (cavities) there is no ambiguity about the fact that the field components are described, in their spatial distribution, by the solutions of Maxwell's equation $\partial^* F = 0$.
Of course, at least I assume because of laziness I did not, even here, as in the case of waveguides, suitable sums and differences of the solutions provide the electromagnetic field (Stratton).
Conclusion: there are strong suspicions of kinship between the two cases: the possible modes in the cavity and the orbits of electrons in atoms. The correspondence between the two cases is one by one. The states are the same. Angular configurations described by ψ_l^m are the same.
The one present themselves as field configurations, the other (the orbits) as configurations of charge; however (in cavities) combinations of charge configurations of opposite charge (***) give the fields. Everyone can fantasize about what it follows, according to his own personal fantasy (for example, even if things are not in such simple terms, you could use this picturesque image: the fields are made of e^+ and e^-. If someone removes one e^+ from a electromagnetic field, what's left is the spatial configuration of an electronic orbit (a e^-).
But if you add a field e^+ to a field e^-, you get a "neutral" electromagnetic field, ie the oscillation in a cavity.
Moreover, the addition of a "neutral" mode of oscillation to a "charged" orbit, is passed to another charged orbit, "more excited" than the last. And vice versa. We move from one charged orbit to a second, the lower energy level, through the emission of a "neutral" electromagnetic field).
I will now make a long digression.

I'll repeat in part what I anticipated in the paragraph 4.

(*)
Or generic atoms of nuclear charge $+Ze$ ($Z = 1,2,3,\cdots$). See paragraph 12.

(**)
Distinction here between a detailed description (of more or less hypothetical "field components") and a more overall description on the spatial distribution of the orbits (n.d.r.).

(***)
Note that (see Bjorken Drell) the Dirac equation for the atom provides both. The solutions e^+ are discarded or interpreted as positrons and antimatter.

In all this talk there are elementary geometric entities that are the Ψ_l^m or the ψ_l^m that dominate the scene. The ψ_l^m are three-dimensional analogous of the angular distributions $e^{in\varphi}$. The Ψ_l^m.... are the analogous of the elementary geometric (or numerical) entities z^n and $1/_{z^n}$ of the x, y plane.
Why this happens?
We are discovering, as the Cambridge group says, the properties of space and time?
None of this. The question, in my opinion, is as follows:
what chance is there that the basic entities of our language are different (so to speak) of elementary particles?
The answer is in my opinion (if the language is "centered"): none.
And so we should not be surprised that this happens. At the end is our problem. In physics an elementary reality too often escapes, which is obvious, but then is forgotten with phrases like "I found the law ...", "I discovered".
In physics, we can only tell you what happens.
Or try to tell.
Hestenes wrote to me (in response to my strange ramblings) "no one will give you credit, though not explaining the interactions of the electron". Right. However, if we want to split hairs, we can not explain the interaction of the electron.
We can not explain the electron: we can tell the electron (which is there).
Now if you think about it (this is probably incorrect but suggestive diction) we say that bodies are made of atoms, elementary particles, etc., and functions are made up of zeros, poles and so on.
Compose the bodies with the atoms and compose functions with the elementary functions.
It would be wise, even if not required, that the elementary entities of the mathematical language coincided with the basic realities of the surrounding reality that we intend to tell. That is why this fact can reasonably happen: elementary physical entities are elementary functions. They are, not for aesthetic magic, or coincidence, but because we, in turn and reversal of secular thought and mathematical and physical laws, we can not merge, more or less unconsciously, in a right mathematical language, in which elementary functions, such as analytic functions Ψ_l^m , necessarily correspond in some way to elementary physical entities.
You might say: but it is not. It is still not exactly so. Well: Schroedinger say: "In my day it was said that the research is not over yet."
I think that is certain (*) that it may have a solution.
Well (and this is the last point at which end this long digression) here's more.
There's much more. In fact there are two more things.
First, here are a lot of things that match, it seems there are a lot of things that match. One might think that we have discovered the electron. We would rather (if it works) found the appropriate language. Here, then, what is magical or mysterious it may seem at times, sometimes mystical, sometimes its hard to understand why, or .... “interesting”, aesthetic or otherwise, or incomprehensible, or even irrelevant. This according to the imagination of the beholder.
That's why the "powers of the number" are .... the orbits of the electrons in the hydrogen atom, and states s, p, d ... (see below) correspond to the enumeration of ψ_l^m (see below).

(*)
Align the elementary entities of our language with elementary particles, such as the electron, is not a problem of the electron, but it is our problem. Moreover, it is clear that this is possible, just change the language (as long as they do not succeed, as long as he has not found a suitable language).

Second (and I stress this in my opinion is the most satisfying aspect) would, as I say, pointed out at last and said, our fundamental "ignorance". We only pointed out the class of functions that will work in the description of the phenomena of space and time. We, in fact, first pointed out the space and time, with their unit vectors, and the formal rules required, consequential, and then the class of functions, analytic function of the point P of spacetime. The general law valid in all of physics is that we use those functions there. The law $\partial^* F = 0$ adds nothing to the substance of what we know of the electromagnetic field (and thus eventually would be for elementary particles).
End of digression.
Returning to the subject, before solving the equation $\partial^* F = 0$ we establish a preliminary formula that is used in the calculations and concerns $\frac{z^*}{r}$. It is in 3 dimensions exactly the same .... of $\frac{z^*}{r}$ in two dimensions ... viz $\frac{z^*}{r} = \frac{re^{-i\varphi}}{r} = e^{-i\varphi}$. This establishes a similar relationship analogous to $e^{i(l-1)\varphi} = e^{-i\varphi}e^{il\varphi}$, the relation that connects the states l and $(l-1)$. Here is a similar relationship, but a little different, which connects the states l (with l positive or zero, $l \geq 0$) with states $-(l+2)$.
(Why? Boh!).
The formula in question is (*):

$$\boxed{-\Gamma^* T\frac{z}{r}\psi_l^m = -(l+2)T\frac{z}{r}\psi_l^m}$$

It says that if we start from a state ψ_l^m to $l \geq 0$ functions $T\frac{z}{r}\psi_l^m$ have a negative eigenvalue of angular momentum and precisely $-(l+2)$.
(A good example of this is the analytic function ..... $\Psi_0 = 1$ (ie, $l = 0$, $\psi_l = 1$) which corresponds to the function $\psi_{-2} = T\frac{z}{r} = \frac{z^*}{r}T$ that is part of the analytic function $\Psi_{-2} = r^{-2}\frac{z^*}{r} = \frac{z^*}{r^3}$ already met).
The solution of $\partial^* F = 0$ now proceeds in close analogy ("mutatis mutandis") with what happens for the waveguides. Is:

$$(1) \qquad \partial^* F = 0$$

or

$$(2) \qquad \partial^*_{xyz} F + T\frac{\partial}{\partial \tau} F = 0$$

I take:

$$(3) \qquad F = F_1(r, \vartheta, \varphi)e^{i\omega_0 t}$$

where $F_1(r, \vartheta, \varphi)$ has various indexes. I write (see paragraph 4):

$$z^*\partial^* = r\frac{\partial}{\partial r} + \Gamma^*$$

$$(4) \qquad \partial^*_{xyz} = \frac{z}{r}(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma^*)$$

and substituting

$$(5) \qquad \frac{z}{r}\left(\frac{\partial F_1}{\partial r} + \frac{1}{r}\Gamma^* F_1\right) + TF_1\omega_0 i = 0$$

where the exponential has been simplified from the right. Now I put:

$$(6) \qquad F_1 = E_1 + TH_1$$

(*)
For detailed steps see Appendix A3.

from which substituting in (5) and separating the parts with and without the index T is obtained

$$(7)\qquad \begin{cases} \frac{z}{r}\frac{\partial E_1}{\partial r}+\frac{z}{r}\frac{1}{r}\Gamma^* E_1 + H_1\omega_0 i = 0 \\ \frac{z}{r}\frac{\partial TH_1}{\partial r}+\frac{z}{r}\frac{1}{r}\Gamma^* TH_1 + TE_1\omega_0 i = 0 \end{cases}$$

To try to separate r from ϑ, φ I try to put

$$(8)\qquad \begin{cases} E_1 = R_E\psi_l^m \\ H_1 = \frac{z}{r}\psi_l^m R_H i \end{cases}$$

in order to write:

$$(9)\qquad \begin{cases} -\Gamma^* E_1 = -\Gamma^* R_E\psi_l^m = R_E l\psi_l^m \\ -\Gamma^* TH_1 = -\Gamma^* T\frac{z}{r}\psi_l^m R_H i = -(l+2)T\frac{z}{r}\psi_l^m R_H i \end{cases}$$

Substituting in (7), <u>simplify</u> all the terms from the left as $\frac{z}{r}\psi_l^m$ or $T\frac{z}{r}\psi_l^m$ and you get, with $\omega_0 = 1$ for convenience:

$$(10)\qquad \begin{cases} \frac{\partial R_E}{\partial r} - \frac{1}{r}lR_E - R_H = 0 \\ \frac{\partial R_H}{\partial r} + \frac{1}{r}(l+2)R_H + R_E = 0 \end{cases}$$

Getting R_H from the first and substituting in the second (and ditto for R_E) is obtained

$$(11)\qquad \begin{cases} \frac{\partial^2 R_H}{\partial r^2} + \frac{2}{r}\frac{\partial R_H}{\partial r} + (1 - \frac{(l+1)(l+2)}{r^2})R_H = 0 \\ \frac{\partial^2 R_E}{\partial r^2} + \frac{2}{r}\frac{\partial R_E}{\partial r} + (1 - \frac{l(l+1)}{r^2})R_E = 0 \end{cases}$$

These are the 2 nd order equations for the "spherical Bessel functions," already solved (Schiff, Stratton). If you take $R_E = j_l$ the (10), because of the recursive formulas (*) between j_l, we are obliged to take $R_H = -j_{l+1}$ for which

$$(12)\qquad F = (j_l\psi_l^m - Tj_{l+1}\frac{z}{r}\psi_l^m i)e^{i\omega_0 t}$$

This is the solution for the spherical cavity (**), very similar to that for the cylindrical waveguides (**). The enumeration of states follows from the ψ_l^m (for each l, $(2l+1)$ values for m, from $m = l, l-1, l-2, \cdots$ to $m = -l$).

(*)

Worth it for the "spherical Bessel functions" the following <u>recursive equations</u> (which are nothing but .... that system (10) of 1 st order differential equations)

$$\frac{\partial j_l}{\partial r} - \frac{l}{r}j_l + j_{l+1} = 0$$
$$\frac{\partial j_{l+1}}{\partial r} + \frac{(l+2)}{r}j_{l+1} - j_l = 0$$

(**)

We can verify, by replacing, if they really satisfy (1) or (5) or the system (7), see Appendix 4. <u>Only later</u> I've written the 1 st order equations as a <u>single</u> complex equation, see Appendix A5 (n.d.r.)

There are negative l states. To calculate such states I restart from (7).
This time to separate r from ϑ, φ I try to ask

$$(13) \qquad \begin{cases} E_1 = T\frac{z}{r}\psi_l^m R_E T \\ H_1 = T\psi_l^m T R_H i \end{cases}$$

which I need to write:

$$(14) \qquad \begin{cases} -\Gamma^* E_1 = -\Gamma^*\left(T\frac{z}{r}\psi_l^m R_E T\right) = -(l+2)(T\frac{z}{r}\psi_l^m R_E T) \\ -\Gamma^* T H_1 = -\Gamma^* \psi_l^m T R_H i = l\psi_l^m T R_H i \end{cases}$$

Substituting in (7), all the terms like $\frac{z}{r}\psi_l^m$ or $\frac{z}{r}T\frac{z}{r}\psi_l^m$ are simplified from the left and you get, always asking for convenience $\omega_0 = 1$:

$$(15) \qquad \begin{cases} \frac{\partial R_E}{\partial r} + \frac{(l+2)}{r}R_E - R_H = 0 \\ \frac{\partial R_H}{\partial r} - \frac{l}{r}R_H + R_E = 0 \end{cases}$$

These, taking into account the recursive equations for the "spherical Bessel functions"

$$\begin{cases} \frac{\partial j_l}{\partial r} - \frac{l}{r}j_l + j_{l+1} = 0 \\ \frac{\partial j_{l+1}}{\partial r} + \frac{(l+2)}{r}j_{l+1} - j_l = 0 \end{cases}$$

are satisfied by $R_H = j_l$ and $R_E = j_{l+1}$.
The end result is

$$F_1 = E_1 + TH_1 = T\frac{z}{r}\psi_l^m j_{l+1} T + \psi_l^m T j_l$$

$$(16) \qquad F = (T\frac{z}{r}\psi_l^m j_{l+1} T + \psi_l^m T j_l i)e^{i\omega_0 t}$$

(and so here $-\Gamma^* E = -(l+2)$ instead of (12) $-\Gamma^* E = lE$)
Combining "states to positive l and negative l " and states with $+\omega_0$ and $-\omega_0$ we obtain the electromagnetic fields in the cavity (I imagine: I did not but should work, because $\partial^* F = 0$ they are .... Maxwell's equations).
The enumeration of the states, here too, presents, for each l, $(2l+1)$ values for m (from $m = l, l-1, l-2, \cdots$ to $m = -l$).
The total states, including positive and negative (ω_0 always positive) are the same as available ψ_l^m:

for $l = 1$	$2l + 1 = \mathbf{3}$ states (+**3** negative)	**6** states
for $l = 2$	$2l + 1 = \mathbf{5}$ (+**5**)	**10** states
for $l = 3$	$2l + 1 = \mathbf{7}$ (+**7**)	**14** states
for $l = 0$	$2l + 1 = \mathbf{1}$ (+**1**)	**2** states

This corresponds to the enumeration **s**, **p**, **d**, **f**,···· of electrons (**2** electrons in **s** , **6** electrons in **p,** **10** electrons in **d, 14** electrons in **f, ...).**
The fact that I have probably made some mistakes in the calculation here and there, does not change the substance of the matter, all this is not quantum mechanics: it is mathematics, and comes from the technique of separation of variables in the equation $\partial^* F = 0$.
Or if you want, is electromagnetism, where appropriate combinations of "fictitious" solutions, with positive and negative states and solutions $+\omega_0$ and $-\omega_0$ (*), give the possible modes in the spherical cavity. The enumeration of solutions and their type of angular distribution ψ_l^m corresponds to the orbits of electrons in the atom. Could it be just a coincidence?
Well, the answer might be yes.
It may well be a coincidence but a coincidence in the fact that there are always us. We are always involved. A coincidence in the fact that we are describing similar basic phenomena. Which also we can not say what they are, as well as the electromagnetic field we do not know what it is. We create, basically, "models" representative of reality. But (this is the point) I think that what we call electromagnetic field, seen as $\partial^* F = 0$ "generalized", has in itself all the complications necessary for try to describe, with it, even the electron [6], [7], [8], [9]. Are to be solved, of course, several problems still unclear.

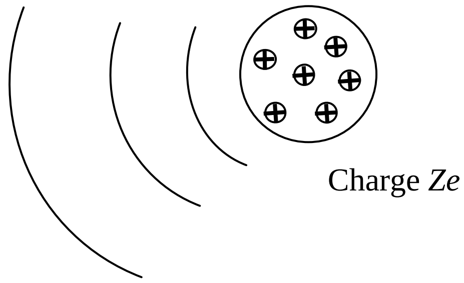
(*)
The solutions $-\omega_0$ are obtained by simply multiplying Ti from the right. This allows analytical the analytical solutions and we have:

$$F = (Tj_{l+1}\frac{z}{r}\psi_l^m T - Tij_l\psi_l^m)e^{-i\omega_0 t}$$

$$F = (+j_l\psi_l^m + T\frac{z}{r}\psi_l^m j_{l+1} i)e^{-i\omega_0 t}$$

12 - Physical Optics and heuristic derivation of the atom spectral lines

I mentioned a tentative interpretation of the Dirac equation in a few paragraphs earlier.
Now, however, for greater mental cleansing, I would leave aside the Dirac equation and forget it, think in terms of physical optics [10], "heuristic".
I mean here as physical optics a plane waves or rays approximation of Maxwell's equations, namely that we have discussed talking waveguides. Suppose that some solution of $\partial^* F = 0$, approximated by plane waves, represents a wave packet with mass (mean mass for energy "at rest" $E_0 = \omega_0 = m_0$), and suppose also that it is charged, ex. with a negative charge $-e$, ie it is affected by the presence of external electric fields.
I have already said that the movement of this "heavy and charged ray" can aptly described by "suitably" introducing, in addition to the value of free energy ω, the energy value, example $-\frac{Ze^2}{r}$, when it passes near a charge Ze.



introducing, I said, this value of energy in the wave phase, so that the trajectory of the wave bends according to the various positions imposed by the stationary phase (see for example Landau, "Th. du Champ").
Call "heuristics" throughout this procedure.
Thus I consider an F such as:

$$(17) \qquad F = F_1 e^{-i\left(E+\frac{Ze^2}{r}\right)t}$$

where, as usual, F_1 contains some number of indexes anticommutative.
As we saw in section waveguides this field, in a plane wave approximation, satisfies the 2 nd order Klein Gordon equation (I mean: for the free particle):

$$(18) \qquad \partial\partial^* F = \omega_0^2 F$$

However, the 2 nd order equation is completely forgetting the "components" structure of the plane wave, in particular does not take into account the polarization. One way "heuristic" again, to take account of the field components without throw them away is to consider the 1 st order equations corresponding to (18), for example in the form

$$(19) \qquad \partial^* F = -\hat{\imath}\omega_0 F i\hat{T}$$

I say that is an "heuristics" way (*), a little <u>because I can not do better,</u> and a little because I have no idea how on earth will be stored the plane wave components in (19) and what they <u>mean.</u>

(*)
The (Italian) dictionary, to the term "heuristic", says:
"Heuristic (mathematics): procedure by analogy, or intuitive, or approximate, which allows the deduction of empirical laws, before you can express with mathematical rigor".

One thing is certain: (19), instead of (18), keeps the information of the field components. As such it is a better physical optics approximation of the electromagnetic field.
Or at least hopefully.
Even if the assumptions we made are very rude, they correspond to a first approximation approach, let's see if there are solutions of (17) in (19).
Rewrite (19) in the form

$$(20) \qquad \partial^*_{xyz} F + T\frac{\partial F}{\partial t} = -\hat{\imath}\omega_0 Fi\hat{T}$$

It follows, using the expression (17):

$$T\frac{\partial F}{\partial t} = T\frac{\partial}{\partial t}F_1 e^{-i\left(E+\frac{Ze^2}{r}\right)t} = -TF_1 i\left(E+\frac{Ze^2}{r}\right)e^{-i\left(E+\frac{Ze^2}{r}\right)t}$$

and again:

$$-\hat{\imath}\omega_0 Fi\hat{T} == -\hat{\imath}\omega_0 F_1 i\hat{T}e^{-i\left(E+\frac{Ze^2}{r}\right)t}$$

$$\partial^*_{xyz} F = (\partial^*_{xyz} F_1)e^{-i\left(E+\frac{Ze^2}{r}\right)t}$$

so substituting (20) is simplified by the right the exponential (*) and you have an equation for $F_1(x,y,z)$, or $F_1(r,\vartheta,\varphi)$ if you prefer:

$$(21) \qquad \partial^*_{xyz} F_1 - T\left(E+\frac{Ze^2}{r}\right)F_1 i === -\hat{\imath}\omega_0 F_1 i\hat{T}$$

Suppose at this point F_1 in the form:

$$(22) \qquad F_1 = E_1 + TH_1$$

and introducing the usual expression of ∂^*_{xyz} in spherical coordinates we have:

$$\frac{z}{r}\left(\frac{\partial F_1}{\partial r} + \frac{1}{r}\Gamma^* F_1\right) - T\left(E+\frac{Ze^2}{r}\right)F_1 i = -T\omega_0(E_1 - TH_1)i$$

(I changed the 2 th member using $(E_1 + TH_1)\hat{T} = \hat{T}(E_1 - TH_1)$ and then using $\hat{\imath}\hat{T} = T$).
By separating the parts with and without the index T comes to:

$$(23) \qquad \begin{cases} \dfrac{z}{r}\dfrac{\partial E_1}{\partial r} + \dfrac{z}{r}\dfrac{1}{r}\Gamma^* E_1 - (E + \dfrac{Ze^2}{r} + \omega_0)H_1 i = 0 \\ \dfrac{z}{r}\dfrac{\partial TH_1}{\partial r} + \dfrac{z}{r}\dfrac{1}{r}\Gamma^* TH_1 - T(E + \dfrac{Ze^2}{r} - \omega_0)E_1 i = 0 \end{cases}$$

At this point the equations are exactly identical to (7), just different coefficient of the last term, so no need to redo that step, we separate the angular variables as we did, and you will come to the equations only with r:

(*)
In doing $\partial^*_{xyz}F$ I consider $\frac{Ze^2}{r}$ as a fixed term imposed from outside to F, as if it were constant and not subject to the operations of derivative.

$$(24)\qquad \begin{cases} \dfrac{\partial R_E}{\partial r} - \dfrac{l}{r} R_E + (E + \dfrac{Ze^2}{r} + \omega_0) R_H = 0 \\ \dfrac{\partial R_H}{\partial r} + \dfrac{(l+2)}{r} R_H + (E + \dfrac{Ze^2}{r} - \omega_0) R_E = 0 \end{cases}$$

These equations are known and already discussed, see eg.C. Doran et al., "Spacetime Algebra and Electron Physics" Adv. Imag. & Elect. Phys. (1996):

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \dfrac{k-1}{r} & -(E + \dfrac{Z\alpha}{r} + \omega_0) \\ (E + \dfrac{Z\alpha}{r} - \omega_0) & \dfrac{-K-1}{r} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Doran says, "K is a non zero positive or negative integer" $K = (l+1)$ or $K = -(l+1)$.
So with $K = (l+1)$,, and with $e^2 \equiv \alpha$ (*) we obtain exactly (24)

$$\begin{cases} u' - \dfrac{l}{r} u + (E + \dfrac{Ze^2}{r} + \omega_0) v = 0 \\ v' + \dfrac{(l+2)}{r} v - (E + \dfrac{Ze^2}{r} - \omega_0) u = 0 \end{cases}$$

He always says Doran: "The solution of these radial equations can be found in many textbooks (see, for example, JD Bjorken and SD Drell, "Relativistic Quantum Mechanics", vol 1. McGraw-Hill, 1964), and the energy spectrum is obtained from the equation

$$(25)\qquad E^2 = \omega_0^2 \left(1 - \frac{(Z\alpha)^2}{n^2 + 2n\nu + (l+1)^2} \right)$$

where n is a positive integer and $\nu = \sqrt{(l+1)^2 + (Z\alpha)^2}$". So Doran says.
What concerns us is that the system admits solutions only for certain eigenvalues of E that coincide with all the information given by the Dirac theory of the atom
(Spectral lines).

At the same time the solution $F = (R_E \psi_l^m + T\frac{z}{r}\psi_l^m R_H i) e^{-i\left(E + \frac{Ze^2}{r}\right)t}$, with $R_E(r), R_H$ known functions, contains all the information on the radial and angular distribution of electrons in atoms. Orbital states are once again listed and described by the ψ_l^m.
The solution (25) on the energy levels of spectral lines also contains in itself (obviously) all approximate solutions of Schroedinger and Bohr (but for these discussions we can see any good book on quantum mechanics).
Conclusion: while admitting that in this paragraph have been "heuristic," and perhaps even biased, it gives me to think that simply by (17) and (19) early, to be received (25) with all that follows. We used one of those fictitious solutions of Maxwell's equations that, we have seen, provide in a waveguide the waveguide modes. We have represented in the form of plane waves where someone or something has stored and set the value of $\omega_0 \equiv k_c$ which the wave carries. Such "heavy and charged rays" we made them travel in an external electric field ... and the result is all the Dirac theory on spectral lines, down to the fine structure of spectra.

(*)
Writing e^2 or α depends on the system of units adopted.

I would like to observe a thing. I insisted, for several paragraphs, knowing that this is not quantum mechanics, and neither is electromagnetism, but .... a pure question of mathematical language. Pure language, pure geometry, pure mathematics.
After then, jumping out (25), with inside atoms, spectral lines, experimental values of constants, such as c, e^2, m_0, h, α . In fact, is it possible that these last solutions also belong to the pure geometry, ie, <u>to the language?</u>

13 - Considerations on the fine structure constant

The solutions of the equation $\partial^* F = 0$

$$\begin{cases} 2D \text{ with } \partial^* = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} \\ 3D \text{ with } \partial^* = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} + j\dfrac{\partial}{\partial z} \\ 3D \text{ with } \partial^* = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} + T\dfrac{\partial}{\partial \tau} \\ 4D \text{ with } \partial^* = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} + j\dfrac{\partial}{\partial z} + T\dfrac{\partial}{\partial \tau} \end{cases}$$

give rise, respectively, if you require the separation of angular variables, (plus the condition ... that they repeat in angle) to analytic functions like this:

$$\begin{cases} z^n = \rho^n e^{in\varphi} \\ \Psi_l^m = r^l \psi_l^m(\theta,\varphi) \\ F_n = (J_n e^{in\varphi} + TiJ_{n+1}e^{i(n+1)\varphi})e^{-i\omega_0 t} \\ F_l^m = (j_l\psi_l^m + T\frac{z}{r}\psi_l^m j_{l+1} i)e^{-i\omega_0 t} \end{cases}$$

In addition to the conditions on the angles, giving rise to integer numbers, we have here eigenvalues regarding ω (or ω_0, rest mass, or the total energy E). These eigenvalues do not come more from conditions on space angles, but time-related conditions. The last two give special functions J_n and j_l, and correspond to the "normalized" eigenvalue $\omega_0 = 1$. The first two can be thought as, indeed are, the solutions corresponding to the eigenvalue $\omega_0 = 0$.

All these solutions appear as the basic solutions of the equation $\partial^* F = 0$ and correspond to eigenvalues $\omega_0 = 0$ e $\omega_0 = 1$.

Are there any others?

The solution to the atom relative in paragraph 12 does not correspond to any analytic solution of $\partial^* F = 0$ (we could obtain only with approximations ... not knowing what else to do, moreover, because we would not know how to write .... the presence of the body consisting of the nucleus Ze). It is possible that there is an analytic function, describing the whole system electron plus nucleus, or individually describing the electron and the nucleus, but in terms of course of exact analytical function $\partial^* F = 0$?

And yet, seeing things from another point of view, or skill, or fortunately, we were able to find a spectrum of energy eigenvalues, which proves very good (ie, Dirac was able to). Now, this is the point, overall this spectrum of energy eigenvalues is in fact a purely geometric (or numerical). Appropriately normalizing the units, next to the eigenvalue 0 and another eigenvalue 1, there is the additional eigenvalue $\left(1 - \frac{(Z\alpha)^2}{n^2+2n\nu+(l+1)^2}\right)^{\frac{1}{2}}$, where they appear only integers (n, l, Z) and $\alpha = {}^1/_{137{,}035999}$ ... Just to understand, the situation is a bit as if we had found, for certain basic functions of our geometric language, only 0, 1, and integers and π.

This is not a demonstration. It is a doubt.

May α be due to factors not yet understood geometry (*)? Suggestive clues of more or less evidence there are many.

(*)
Quantization conditions on spacetime angles? Skipping these reflections leaving the reader (n.d.r.).

14 - Conclusion

I do not know if Eddington had understood everything (*), but I was always struck by the final sentence of his book:
“We found that where science is more advanced, that the mind has recovered from nature what the same mind had placed in nature. We found a strange footprint on the beach of the unknown. We have devised profound theories, one after another, to explain its origin. At the end we managed to reconstruct the creature that had left the imprint. And lo! is our footprint". (Arthur Eddington, 1920)

(*)
Who, however, say the books, dedicated many years of his life to obsessed exploration with the number α.

15 - Appendices

A1

I point out, even if it is a bit boring, as to derive these fields of Ramo Whinnery. I take solutions $e^{-i\omega_0\tau}$ with positive n and negative n (and n even (*), $J_{-n} = +J_n$).

$$(J_{n-1}e^{i(n-1)\varphi} + TiJ_n e^{in\varphi})e^{-i\omega_0 t}$$
$$(J_{-n-1}e^{-i(n+1)\varphi} + TiJ_n e^{-in\varphi})e^{-i\omega_0 t}$$

The sum of magnetic fields gives:

$$TiJ_n \cos n\varphi\, e^{-i\omega_0 t}$$

With $e^{+i\omega_0\tau}$ I got:

$$(J_{n-1}e^{i(n-1)\varphi} - TiJ_n e^{in\varphi})e^{+i\omega_0 t}$$
$$(J_{-n-1}e^{-i(n+1)\varphi} - TiJ_n e^{-in\varphi})e^{+i\omega_0 t}$$

In this case the sum of magnetic fields gives:

$$-TiJ_n \cos n\varphi\, e^{+i\omega_0 t}$$

<u>Subtracting</u> the magnetic fields in the two cases (because I want a solution where the TH_τ field is removed) is only obtained a field H_z

$$+TiJ_n \cos n\varphi \cos \omega_0\, t$$

Electric fields can be rewritten first.
If $e^{-i\omega_0\tau}$ they are

$$(\frac{\partial J_n}{\partial r} + \frac{n}{r}J_n)e^{in\varphi}e^{-i\varphi}e^{-i\omega_0 t}$$
$$(\frac{\partial J_n}{\partial r} - \frac{n}{r}J_n)e^{-in\varphi}e^{-i\varphi}e^{-i\omega_0 t}$$

and the sum gives

$$(\frac{\partial J_n}{\partial r}\cos n\varphi + i\frac{n}{r}J_n \sin n\varphi)e^{i\varphi}e^{-i\omega_0 t}$$

If $e^{+i\omega_0\tau}$ they are

$$(\frac{\partial J_n}{\partial r} + \frac{n}{r}J_n)e^{in\varphi}e^{-i\varphi}e^{+i\omega_0 t}$$
$$(\frac{\partial J_n}{\partial r} - \frac{n}{r}J_n)e^{-in\varphi}e^{-i\varphi}e^{+i\omega_0 t}$$

and summing

$$(\frac{\partial J_n}{\partial r}\cos n\varphi + i\frac{n}{r}J_n \sin n\varphi)e^{-i\varphi}e^{+i\omega_0 t}$$

Now <u>subtracting</u> the electric fields in the two cases (like the magnetic field that gave H_z) it has

$$(\frac{n}{r}J_n \sin n\varphi\, e^{-i\varphi} - \frac{\partial J_n}{\partial r}\cos n\varphi\, ie^{-i\varphi}) \sin \omega_0 t$$

Combining everything together and passing to the conjugate, as is done for analytic functions on the plane (since we have seen that the equation $\partial^* F = 0$ is the Maxwell's equations for the conjugate) we arrive at

$$\frac{n}{r}J_n \sin n\varphi \sin \omega_0 t\, e^{i\varphi} + \frac{\partial J_n}{\partial r}\cos n\varphi \sin \omega_0 t\, ie^{i\varphi} - TiJ_n \cos n\varphi \cos \omega_0 t$$

We recognize here the unit vectors $e_r = e^{i\varphi}$ and $e_\varphi = ie^{i\varphi}$, polar coordinates on the r, φ plane.

(*)
I consider the case with n even to write$J_{-n} = +J_n$.

So finally we get Ramo Whinnery fields for TE

$$E_r = \frac{n}{r} J_n \sin n\varphi \sin \omega_0 t$$
$$E_\varphi = \frac{\partial J_n}{\partial r} \cos n\varphi \sin \omega_0 t$$
$$H_z = -J_n \cos n\varphi \cos \omega_0 t$$

A2

Restart from the beginning looking at other ways to reach the solution of waveguide equations (and take advantage of it, just to muddy the waters, looking for the solution $e^{+i\omega_0\tau}$).
We search F as eigenfunction of the operator J_z with eigenvalue $(n+\frac{1}{2})$.
You have to solve the equation

$$\partial^* F = 0$$

namely

$$e^{i\varphi}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}\right)F + T\frac{\partial}{\partial\tau}F = 0$$

Put $F = F_1(r,\varphi)e^{+i\omega_0\tau}$ and can thus simplify the exponential from right to obtain

$$e^{i\varphi}\frac{\partial F_1}{\partial r}+e^{i\varphi}\frac{i}{r}\frac{\partial F_1}{\partial\varphi}+T\omega_0 F_1 i = 0$$

I separate the variables r,φ in F_1, but I can not use such an expression $-i\frac{\partial}{\partial\varphi}F_1 = nF_1$, as $-i\frac{\partial}{\partial\varphi}F_1$ he has no eigenvalue, it's equal to …. to anything (as <u>we already know</u> .... how is F_1).
But (since we already know that $J_z F = (n+\frac{1}{2})F$, and then also $J_z F_1 = (n+\frac{1}{2})F_1$) say:
"<u>I try to put</u> $J_z F_1 = (n+\frac{1}{2})F_1$".
To do this ... we must first bring up this operator which is not there. Because instead of $-i\frac{\partial}{\partial\varphi}F_1$ we need $J_z F_1 = -\frac{\partial}{\partial\varphi}F_1 i - \frac{1}{2}iF_1 i$, we must first rewrite the equation with a multiplication by i from the right,

$$e^{i\varphi}\frac{\partial F_1}{\partial r}i+e^{i\varphi}\frac{i}{r}\frac{\partial F_1}{\partial\varphi}i - T\omega_0 F_1 = 0$$

and then adding and subtracting aptly $\frac{1}{2}iF_1 i$;

$$e^{i\varphi}\frac{\partial F_1}{\partial r}i+e^{i\varphi}\frac{i}{r}\frac{\partial F_1}{\partial\varphi}i+e^{i\varphi}\frac{i}{r}\frac{1}{2}iF_1 i - e^{i\varphi}\frac{i}{r}\frac{1}{2}iF_1 i - T\omega_0 F_1 = 0$$

$$e^{i\varphi}\frac{\partial F_1}{\partial r}i - e^{i\varphi}\frac{i}{r}\left(-\frac{\partial F_1}{\partial\varphi}i-\frac{1}{2}iF_1 i\right)+e^{i\varphi}\frac{1}{r}\frac{1}{2}F_1 i - T\omega_0 F_1 = 0$$

I did so appear $J_z F_1$. Now "I try to put" $J_z F_1 = (n+\frac{1}{2})F_1$ and I get:

$$e^{i\varphi}\frac{\partial F_1}{\partial r}i - e^{i\varphi}\frac{i}{r}\left(n+\frac{1}{2}\right)F_1+\frac{e^{i\varphi}}{r}\frac{1}{2}F_1 i - T\omega_0 F_1 = 0$$

The equation contains $F_1(r,\varphi)$ with variables r,φ that are not yet separated (in fact, if we look at the equation we see that there is a $e^{i\varphi}$ non-simplified), however, can be rewritten by multiplying $e^{-i\varphi/2}$ from the left (taking advantage of the fact that $e^{-i\varphi/2}T = Te^{i\varphi/2}$) and it shows that it is so

$$e^{i\varphi/2}\frac{\partial F_1}{\partial r}i - e^{i\varphi/2}\frac{i}{r}\left(n+\frac{1}{2}\right)F_1+\frac{e^{i\varphi/2}}{r}\frac{1}{2}F_1 i - T\omega_0 e^{i\varphi/2}F_1 = 0$$

<u>This makes you want to take as unknown $e^{i\varphi/2}F_1$ instead of F_1.</u>

So I assume as unknown $e^{i\varphi/2}F_1$ and separate the variables in the form

$$e^{i\varphi/2}F_1 = \mathfrak{F}(r)\Phi(\varphi)$$

Substituting and simplifying $\Phi(\varphi)$ we arrive to this equation for $\mathfrak{F}(r)$

$$\boxed{\frac{\partial\mathfrak{F}}{\partial r}i-\frac{i}{r}\left(n+\frac{1}{2}\right)\mathfrak{F}+\frac{1}{2r}\mathfrak{F}i-T\omega_0\mathfrak{F}=0}$$

<u>This differential equation is the 1 st order equation of the “hypercomplex Bessel function” $\mathfrak{F}(r)=J_n+TiJ_{n+1}$ (which presents herself as a single entity).</u>
If I ask $\mathfrak{F}=R_E+TR_H$ I get

$$\frac{\partial R_E}{\partial r}-\frac{n}{r}R_E+i\omega_0R_H=0$$
$$\frac{\partial R_H}{\partial r}+\frac{n+1}{r}R_H+i\omega_0R_E=0$$

that is the usual system of equations with the usual solutions (it is written differently from the previous one because this time I tried solutions $e^{+i\omega_0\tau}$).
Having so found $\mathfrak{F}(r)=J_n+TiJ_{n+1}$ we can finish by computing $\Phi(\varphi)$.

$$e^{i\varphi/2}F_1=(J_n+TiJ_{n+1})\Phi(\varphi)$$
$$F_1=e^{-\frac{i\varphi}{2}}(J_n+TiJ_{n+1})\Phi(\varphi)$$

and the condition $J_zF_1=(n+\frac{1}{2})F_1$ with long but obvious passages gives as it must be

$$\Phi(\varphi)=e^{i(n+{}^{1}/{}_{2})\varphi}$$

A3

The expression I need

$$-\Gamma^* T \frac{z}{r}\psi_l^m = -(l+2)T\frac{z}{r}\psi_l^m$$

I find, ready, in C. Doran, A. Lasenby, S. Gull, S. Somaroo, A. Challinor, “Spacetime Algebra and Electron Physics” Adv. Imag. & Elect. Phys. (1996). It is written

$$-\vec{x}\wedge\vec{\nabla}\widehat{e_r}\psi_l = -(l+2)\widehat{e_r}\psi_l$$

being

$$-\vec{x}\wedge\vec{\nabla}\psi_l = l\psi_l$$

It is written: “so, without loss of generality, we can choose l to be positive and recover the negative $-l$ states through multiplying by $\widehat{e_r}$”.
Translate from Cambridge’s notations. Is:

$$-\vec{x}\wedge\vec{\nabla} = -\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\wedge\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)$$
$$= -i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)-j\left(x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\right)-ji\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right) = -\Gamma^*$$

(see paragraph 4). It is then:

$$\widehat{e_r} = \left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\frac{1}{r} = \frac{(x-iy-jz)}{r}\hat{\imath} = \frac{z^*}{r}\hat{\imath}$$

decoding the whole you get

$$-\Gamma^*\frac{z^*}{r}\hat{\imath}\psi_l^m = -(l+2)\frac{z^*}{r}\hat{\imath}\psi_l^m$$

Finally you can write, by multiplying by $\hat{T}$ ($T=\hat{\imath}\hat{T}$, $\frac{z^*}{r}\hat{\imath}=\hat{\imath}\frac{z}{r}$), as:

$$-\Gamma^* T \frac{z}{r}\psi_l^m = -(l+2)T\frac{z}{r}\psi_l^m$$

A4 (*)

For those who have doubts about whether these are actually analytic functions in 4 dimensions, observe from (12) we found in the final

$$\begin{cases} E_1 = j_l \psi_l^m \\ H_1 = -j_{l+1} \frac{z}{r} \psi_l^m i \end{cases}$$

We can verify, by replacing, if they really satisfy the (1) or the (5) or the system (7). From the first of (7) we have

$$\frac{z}{r}\frac{\partial j_l \psi_l^m}{\partial r} + \frac{z}{r}\frac{1}{r}\Gamma^* j_l \psi_l^m - j_{l+1}\frac{z}{r}\psi_l^m i \omega_0 i = 0$$

then immediately $\frac{z}{r}$ is simplified from left. The term ψ_l^m is simplified from right, $\omega_0 = 1$, so the equation reduces to

$$\frac{\partial j_l}{\partial r} - \frac{l}{r} j_l + j_{l+1} = 0$$

and this is true as it is the recursive equation for j_l QED.
We pass to the second of (7)

$$\frac{z}{r}\frac{\partial T(j_{l+1}\frac{z}{r}\psi_l^m i)}{\partial r} + \frac{z}{r}\frac{1}{r}\Gamma^* T\left(j_{l+1}\frac{z}{r}\psi_l^m i\right) - T j_l \psi_l^m \omega_0 i = 0$$

First I change the second term using the relationship

$$-\Gamma^* T \frac{z}{r}\psi_l^m = -(l+2) T \frac{z}{r}\psi_l^m$$

I get so

$$\frac{z}{r}\frac{\partial T(j_{l+1}\frac{z}{r}\psi_l^m i)}{\partial r} + \frac{z}{r}\frac{1}{r}(l+2) T \frac{z}{r}\psi_l^m (j_{l+1} i) - T j_l \psi_l^m \omega_0 i = 0$$

Now in the 1 st term I move $T\frac{z}{r}$ out of the sign of the derivative (does not depend on r but only on angles) and I simplify wherever a term $T\frac{zz^*}{r^2} = T$.

$$\frac{\partial (j_{l+1}\psi_l^m i)}{\partial r} + \frac{1}{r}(l+2)\psi_l^m (j_{l+1} i) - j_l \psi_l^m \omega_0 i = 0$$

Take also $\omega_0 = 1$ and simplify a $\psi_l^m i$ from right

$$\frac{\partial j_{l+1}}{\partial r} + \frac{(l+2)}{r} j_{l+1} - j_l$$

and this is true as recursive equation for j_l QED.

(*)
I added a direct verification in this Appendix, in retrospect, that actually (12) satisfies the original equation (1) (n.d.r.).

A5

It is also possible to write the 1 st order equations for the spherical Bessel functions as a single complex equation.

$$\frac{\partial \mathfrak{f}}{\partial r} + \frac{1}{r}(l+1)\hat{T}\mathfrak{f}\hat{T} + \frac{1}{r}\mathfrak{f} + T\mathfrak{f}\omega_0 i = 0$$

This is the 1 st order equation of “hypercomplex spherical Bessel function” $\mathfrak{f}(r) = j_l + iTj_{l+1}$ (which presents herself as a single entity).

In fact, if I put $\mathfrak{f} = R_E + iTR_H$, also considering that $\hat{T}\mathfrak{f}\hat{T} = -R_E + iTR_H$,, I get the following two equations

$$\frac{\partial R_E}{\partial r} - \frac{l}{r}R_E + R_H = 0$$
$$\frac{\partial R_H}{\partial r} + \frac{(l+2)}{r}R_H - R_E = 0$$

These, taking into account the recursive equations for the spherical Bessel functions

$$\frac{\partial j_l}{\partial r} - \frac{l}{r}j_l + j_{l+1} = 0$$
$$\frac{\partial j_{l+1}}{\partial r} + \frac{(l+2)}{r}j_{l+1} - j_l = 0$$

are properly verified by $R_E = j_l$ and $R_H = j_{l+1}$.
The final solution is therefore $\mathfrak{f}(r) = j_l + iTj_{l+1}$.

REFERENCES

[1] David Hestenes, ”… you must be very careful when and where and how you voice such a crackpot idea“.

[2] Chandogya Upanisad 6.15.3 “Whatever this subtle essence, all the universe is made of it, it is the true reality, it is the Atman. It is you, Svetaketu”.

[3] Sommerfeld, Lectures on theoretical physics: “It is not exaggerated claim that the theory of analytic functions of a complex variable is identical with two-dimensional potential theory or, in terms of hydrodynamics, with two-dimensional theory of potential flow”.

[4] Maxwell, A Treatise on Electricity & Magnetism: “… the method … is much more powerful than any known method applicable to three dimensions. The method depends on the properties of conjugate functions of two variables”.

[5] David Hestenes, ”But if you try to explain it to most physicists, they are likely to dismiss you as some kind of crackpot“.

[6] W. Pauli, “"is long-standing desire to bring all the mechanical properties of the electron in electromagnetic principles” (1921).

[7] Sommerfeld, “Gustav Mie took the first step in this direction in 1912 in his famous papers “Foundations of a Theory of Matter”. Their goal is no less than the generalization of the Maxwell equations so that they include the <u>existence of the electron</u>”, Electrodynamics (1948).

[8] W. Pauli, “however, is easy to see (...) necessarily (...) the existence of an energy of non-electromagnetic. The Lorentz electromagnetic energy of the electron at rest is equal to three-quarters of its total energy” (1921

[9] Sommerfeld, “Poincaré introduced (1906) a membrane (…). The missing quarter (…) was supposed to be hidden herein”.

[10] Doran (et al.) “It is not surprising, therefore, that the ∂^* operator should play a fundamental role in the STA formulation of the Dirac theory. <u>What is less obvious is that the same operator should also play a fundamental role in the STA formulation of the Maxwell equations</u>”.
(The notation ∂^* is mine. The emphasis is mine. The reason I mention the phrase is that it is <u>natural</u> to me, with these notations, which the operator ∂^* appears in both and which Ψ and F are attributable to the same entities. Maybe that I am something wrong).