Pseudo-Smarandache Functions of First and Second kind*

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Abstract

In this paper we define two kinds of Pseudo-Smarandache functions. We have investigated more than fifty terms of each pseudo-Smarandache function. We have proved some interesting results and properties of these functions.

1 Introduction

The Pseudo-Smarandache function $Z(n)$ was introduced by Kashihara [4] as follows

Definition 1.1. *For any integer* $n \geq 1, Z(n)$ *is the smallest positive integer* m *such that* $1 + 2 + 3 + \ldots m$ *is divisible by* n.

Alternately, $Z(n) = min\{m : m \in N : n \mid \dfrac{m(m+1)}{2}\}$.

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The main results and properties of Pseudo-Smarandache functions are available in [3] [4],[5].

We noticed that the sum $1 + 2 + 3 + \ldots + m$ can be replaced by the series of squares of first m natural numbers and the cubes of first m natural numbers respectively, to get the Pseudo-Smarandache functions of first kind and second kind.

In the following we define Pseudo-Smarandache functions of first kind and second kind.

Definition 1.2. *For any integer $n \geq 1$, the Pseudo-Smarandache function of first kind, $Z_1(n)$ is the smallest positive integer m such that $1^2+2^2+3^2 \ldots +m^2$ is divisible by n.*

Alternately, $Z_1(n) = min\{m : m \in N : n \mid \dfrac{m(m+1)(2m+1)}{6}\}$.

Definition 1.3. *For any integer $n \geq 1$, the Pseudo-Smarandache function of second kind, $Z_2(n)$ is the smallest positive integer m such that $1^3 + 2^3 + 3^3 \ldots + m^3$ is divisible by n.*

Alternately, $Z_2(n) = min\{m : m \in N : n \mid \dfrac{m^2(m+1)^2}{4}\}$.

For ready reference we give below some values of $S(m)$s and $Z_1(n)$s, where $S(m)$ stands for the sum of the squares of first m natural numbers and $Z_1(n)$ stands for the Pseudo-Smarandache function of first kind for the value n for $n \in N$.

Values of $S(m)$

$S(1) = 1$	$S(15) = 1240$	$S(29) = 8555$	$S(43) = 27434$
$S(2) = 5$	$S(16) = 1496$	$S(30) = 9455$	$S(44) = 29370$
$S(3) = 14$	$S(17) = 1785$	$S(31) = 10416$	$S(45) = 31395$
$S(4) = 30$	$S(18) = 2109$	$S(32) = 11440$	$S(46) = 33511$
$S(5) = 55$	$S(19) = 2470$	$S(33) = 12529$	$S(47) = 35726$
$S(6) = 91$	$S(20) = 2870$	$S(34) = 13685$	$S(48) = 38024$
$S(7) = 140$	$S(21) = 3311$	$S(35) = 14910$	$S(49) = 40425$
$S(8) = 204$	$S(22) = 3795$	$S(36) = 16206$	$S(50) = 42925$
$S(9) = 285$	$S(23) = 4324$	$S(37) = 17575$	$S(51) = 50882$
$S(10) = 385$	$S(24) = 4900$	$S(38) = 19019$	$S(52) = 48230$
$S(11) = 506$	$S(25) = 5525$	$S(39) = 20540$	$S(53) = 51039$
$S(12) = 650$	$S(26) = 6201$	$S(40) = 22140$	$S(54) = 53955$
$S(13) = 819$	$S(27) = 6930$	$S(41) = 23821$	$S(55) = 56980$
$S(14) = 1015$	$S(28) = 7714$	$S(42) = 25585$	$S(56) = 60116$

Values of $Z_1(n)$

$Z_1(1) = 1$	$Z_1(14) = 3$	$Z_1(27) = 40$	$Z_1(40) = 15$
$Z_1(2) = 3$	$Z_1(15) = 4$	$Z_1(28) = 7$	$Z_1(41) = 20$
$Z_1(3) = 4$	$Z_1(16) = 31$	$Z_1(29) = 14$	$Z_1(42) = 27$
$Z_1(4) = 7$	$Z_1(43) = 21$	$Z_1(17) = 8$	$Z_1(30) = 4$
$Z_1(5) = 2$	$Z_1(18) = 27$	$Z_1(31) = 15$	$Z_1(44) = 16$
$Z_1(6) = 4$	$Z_1(19) = 9$	$Z_1(32) = 63$	$Z_1(45) = 27$
$Z_1(7) = 3$	$Z_1(20) = 7$	$Z_1(33) = 22$	$Z_1(46) = 11$
$Z_1(8) = 15$	$Z_1(21) = 17$	$Z_1(34) = 8$	$Z_1(47) = 23$
$Z_1(9) = 13$	$Z_1(22) = 11$	$Z_1(35) = 7$	$Z_1(48) = 31$
$Z_1(10) = 4$	$Z_1(23) = 11$	$Z_1(36) = 40$	$Z_1(49) = 24$
$Z_1(11) = 5$	$Z_1(24) = 31$	$Z_1(37) = 18$	$Z_1(50) = 12$
$Z_1(12) = 8$	$Z_1(25) = 12$	$Z_1(38) = 19$	$Z_1(51) = 8$
$Z_1(13) = 6$	$Z_1(26) = 12$	$Z_1(39) = 13$	$Z_1(52) = 32$

2 Some Results for Pseudo-Smarandache functions of first kind.

Following results can be directly verified from the table given above.

1. $Z_1(n) = 1$ only if $n = 1$.

2. $Z_1(n) \geq 1$ for all $n \in N$.

3. $Z_1(p) \leq p$, where p is a prime.

4. If $Z_1(p) = n$, $p \neq 3$, then $p > n$.

Lemma 2.1. *If p is a prime then $Z_1(p) = p + 1$, for $p = 2$ or 3. Also, $Z_1(p) = \dfrac{p-1}{2}$ for $p \geq 5$.*

Proof. For $p = 2$ and 3, the verification is direct from the above table of $Z_1(n)$.
Let $S = 1^2 + 2^2 + 3^2 + \ldots + (\frac{p-1}{2})^2$. Then $S = \dfrac{p(p+1)(p-1)}{24}$. Hence p divides S. Also $p \nmid \dfrac{p-1}{2}$ as $\dfrac{p-1}{2} < p$. Let if possible (assumption) $p \mid 1^2 + 2^2 + \ldots + m^2$ where $m < \dfrac{p-1}{2}$. But in that case p divides every summand of the sum S. But $p \nmid (\frac{p-1}{2})^2$. Hence our assumption is wrong. Thus S is the minimum. Thus $Z_1(p) = \dfrac{p-1}{2}$

□

Lemma 2.2. *For $p = 2$, $Z_1(p^k) = p^{k+1} - 1$.*

Proof. Straight verification confirms the result.

□

Lemma 2.3. *$Z_1(n) \geq max\{Z_1(N) : N \mid n\}$.*

Proof. Notice that in this case values of N are less than or equal to n and are divisors of n. Hence values of $Z_1(N)$ can not exceed $Z_1(n)$. □

Lemma 2.4. *Let $n = \dfrac{k(k+1)(2k+1)}{6}$ for some $k \in N$, then $Z_1(n) = k$.*

Proof. The result is the immediate consequence of the fact that no previous value of $S(n)$ is divisible by k.

□

Lemma 2.5. *It is not possible that* $Z_1(m) = m$ *for any* $m \in N$.

Proof. Let if possible $Z_1(m) = m$. Then by definition m is the smallest of the positive integer which divides $1^2 + 2^2 + 3^2 + \ldots m^2$. Hence m does not divide $1^2 + 2^2 + 3^2 + \ldots (m-1)^2$. Let $1^2 + 2^2 + 3^2 + \ldots (m-1)^2 = k$. So, m divides $k + m^2$. Hence m divides k, a contradiction.

□

Lemma 2.6. $S(m) = k$ *then* $S(m) = Z_1(2k+1)$.

Here $S(n)$ will stand for the sum of the cubes of first n natural numbers. Please find the table on the next page.

Values of $S(n)$

$S(1) = 1$	$S(15) = 14400$	$S(29) = 189225$	$S(43) = 894916$
$S(2) = 9$	$S(16) = 18496$	$S(30) = 216225$	$S(44) = 980100$
$S(3) = 36$	$S(17) = 23409$	$S(31) = 246016$	$S(45) = 1071225$
$S(4) = 100$	$S(18) = 29241$	$S(32) = 278784$	$S(46) = 1168561$
$S(5) = 225$	$S(19) = 36100$	$S(33) = 314721$	$S(47) = 1272384$
$S(6) = 441$	$S(20) = 44100$	$S(34) = 354025$	$S(48) = 1382976$
$S(7) = 784$	$S(21) = 53361$	$S(35) = 396900$	$S(49) = 1500625$
$S(8) = 1296$	$S(22) = 64009$	$S(36) = 443556$	$S(50) = 1625625$
$S(9) = 2025$	$S(23) = 76176$	$S(37) = 494209$	
$S(10) = 3025$	$S(24) = 90000$	$S(38) = 549081$	
$S(11) = 4356$	$S(25) = 105625$	$S(39) = 608400$	
$S(12) = 6084$	$S(26) = 123201$	$S(40) = 672400$	
$S(13) = 8281$	$S(27) = 142884$	$S(41) = 741321$	
$S(14) = 11025$	$S(28) = 164836$	$S(42) = 815409$	

Values of $Z_2(n)$

$Z_2(1) = 1$	$Z_2(14) = 7$	$Z_2(27) = 8$	$Z_2(40) = 15$
$Z_2(2) = 3$	$Z_2(15) = 5$	$Z_2(28) = 7$	$Z_2(41) = 40$
$Z_2(3) = 2$	$Z_2(16) = 7$	$Z_2(29) = 28$	$Z_2(42) = 20$
$Z_2(4) = 3$	$Z_2(17) = 16$	$Z_2(30) = 15$	$Z_2(43) = 42$
$Z_2(5) = 4$	$Z_2(18) = 3$	$Z_2(31) = 30$	$Z_2(44) = 111$
$Z_2(6) = 3$	$Z_2(19) = 18$	$Z_2(32) = 15$	$Z_2(45) = 5$
$Z_2(7) = 6$	$Z_2(20) = 4$	$Z_2(33) = 11$	$Z_2(46) = 23$
$Z_2(8) = 7$	$Z_2(21) = 6$	$Z_2(34) = 16$	$Z_2(47) = 46$
$Z_2(9) = 2$	$Z_2(22) = 11$	$Z_2(35) = 14$	$Z_2(48) = 8$
$Z_2(10) = 4$	$Z_2(23) = 22$	$Z_2(36) = 3$	$Z_2(49) = 6$
$Z_2(11) = 10$	$Z_2(24) = 15$	$Z_2(37) = 36$	$Z_2(50) = 4$
$Z_2(12) = 3$	$Z_2(25) = 4$	$Z_2(38) = 19$	
$Z_2(13) = 12$	$Z_2(26) = 12$	$Z_2(39) = 12$	

3 Some Results on Pseudo-Smarandache function of second kind

Following properties are result of direct verification from the above tables.

1. $Z_2(n) = n$ only for $n = 1$.

2. $Z_2(p) = p - 1$, $p \neq 2$. $Z_2(p) = p + 1$ for $p = 2$.

3. $Z_2(n) \geq max\{Z_2(N) : N \mid n\}$.

Following are some of the important results.

Lemma 3.1. *If* $S(n) = k$ *then* $Z_2(k) = n$.

Proof. The proof follows from the definition of $Z_2(n)$. □

Open Problem:
What is the relation between $Z_1(n)$ and $Z_2(n)$?

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