There exist POVM measurements that contradict the quantum measurement postulate for discrete, degenerate systems

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Abstract

A 1986 experiment involving a two-particle entangled system is analyzed, and it is shown that: (1) the measurement results of that experiment are in contradiction to the discrete degenerate form of the quantum measurement postulate (DDQMP) and (2) the measurements done in the experiment are of the positive operator valued measure (POVM) type. Thus there exist POVM measurements which contradict the DDQMP. A modification to the DDQMP is provided which agrees with the experimental results. The modified DDQMP is then applied to a proposed experiment involving a three-particle Greenberger-Horne-Zeilinger state.

Introduction

The discrete degenerate form of the quantum measurement postulate (DDQMP, or postulate four in the text of Cohen-Tannoudji, Diu and Laloë [1]) states, in part, the following:

(DDQMP) *The probability of measuring eigenvalue k of observable A on a normalized system* $|\psi\rangle = \sum_{ij} c_{ij} |\varphi_i^j\rangle \in E(A)$ *= eigenspace of A is given by*

$$p_A(k) = \sum_{j=1}^{J} \left|\langle \varphi_k^j | \psi \rangle\right|^2 \tag{1}$$

where J is the degeneracy of the kth basis eigenstate $|\varphi_k^j\rangle$*.*

For a system of two or more particles, the DDQMP is used when only one part of the system is being considered during measurement. For example, consider a two-particle system $|\psi\rangle$, a superposition of discrete eigenstates $|\varphi_i, \theta_j\rangle =$ $|\varphi_i\rangle \otimes |\theta_j\rangle = |\varphi_i\rangle |\theta_j\rangle$ given by

$$\left|\psi\right\rangle = \sum_{i,j} c_{ij} \left|\varphi_i, \theta_j\right\rangle \tag{2}$$

where $\sum_{ij} \left|c_{ij}\right|^2 = 1$. Let there be two observers of the system $\left|\psi\right\rangle$, "Alice" and "Bob," and suppose Alice wishes to make a measurement associated to observable A on the system, whose eigenbasis is $\left\{\left|\varphi_i\right\rangle\right\}$. Now $\left|\psi\right\rangle$ is an element of the eigenspace $E\left(A \otimes B\right)$, where B is Bob's observable with eigenbasis $\left\{\left|\theta_j\right\rangle\right\}$, so Alice's observable is more properly $A \otimes I_B$, where I_B is the identity operator in the B eigenspace. The basis eigenvectors of Bob then are considered as degeneracies in Alice's eigensubspace $E\left(A \otimes I_B\right)$. That is, $E\left(A \otimes I_B\right)$ has eigenbasis $\left\{\left|\varphi_i, \theta_j\right\rangle\right\} = \left\{\left|\varphi_i^j\right\rangle\right\}$ where for every eigenvalue i there are J degenerate basis eigenvectors denoted by the j index. The DDQMP then prescribes the probability of Alice measuring eigenvalue k as:

$$p_A\left(k\right) = \sum_{j=1}^{J} \left|\left\langle \varphi_k, \theta_j \middle| \psi \right\rangle\right|^2 . \tag{3}$$

In other words, for a two particle system, we have the discrete form of the QMP for a system of two particles where one part of the system is measured:

(DQMP2) *The probability of Alice measuring eigenvalue k of observable* $A \otimes I_B$ *on a normalized two-particle system* $\left|\psi\right\rangle = \sum_{ij} c_{ij} \left|\varphi_i, \theta_j\right\rangle \in E\left(A \otimes B\right)$ *is given by equation* (3)*.*

At this point a question naturally arises: if Alice is uncertain as to which (degenerate) eigenvector gave rise to eigenvalue k, then why is the DQMP2 not

$$\tilde{p}_A\left(k\right) = \left|\sum_{j=1}^{J} \left\langle \varphi_k, \theta_j \middle| \psi \right\rangle\right|^2 ? \tag{4}$$

After all, for a single particle system, if a result is obtained from one of several *identical* intermediate states, (hence it cannot be determined which state gave the result) then interference effects arise and the probability formula, as with equation (4), involves a *sum inside the norm*, not the other way around, as in equation (3) and the DQMP2. On the other hand, for a single-particle system, the norm goes inside the sum only if the pathways are *distinguishable* (and hence the state which gave the result can in principle at least, be known). It is well understood that interference effects do *not* occur in such instances. Therefore it is suspected that equation (3) does *not* account for interference effects in two-particle systems, although such effects in those systems are well-known in experiment; see for example, Dopfer's PhD thesis [2].

Continuing with the Alice and Bob thought experiment above, it should then in some instances be possible for Alice to notice interference effects in her probability data. As will be shown below, it turns out that if Bob performs a projective measurement on his end of the system; *i.e.* determines an eigenvalue of $I_A \otimes B$, then equation (3) and the DQMP2, is applicable for Alice. On the other hand, if Bob destroys the eigenvalue or "which-way" information, then the DQMP2 fails for Alice. However, switching the sum and norm in (3) remedies the problem in that case:

$$\tilde{p}_A(k) = \frac{1}{N^2}\left|\sum_{j=1}^{J}\langle\varphi_k,\theta_j|\psi\rangle\right|^2, \tag{5}$$

together with the factor $N^2 = \sum_i \left|\sum_j \langle\varphi_i,\theta_j|\psi\rangle\right|^2$; a normalization factor. The normalization factor appears since there is no guarantee that such measurement has the completeness property. Equation (5) was originally proposed by Srikanth [3] in application to a proposed experiment.

Thus a modified DQMP2 is offered:

(MDQMP2) (i) *The probability of Alice measuring eigenvalue k of observable* $A \otimes I_B$ *on a normalized system* $|\psi\rangle = \sum_{ij} c_{ij}|\varphi_i,\theta_j\rangle \in E(A \otimes B)$ *is given by equation* (3) *if Bob performs a projective measurement, and consequently Alice can in principle at least, determine which degenerate eigenbasis vector gave rise to the result.*

(ii) *On the other hand, the probability is given by equation* (5) *if Bob destroys such information and so Alice cannot be certain which degenerate eigenbasis vector gave rise to the result. If, due to Bob's actions, Alice's measurements are given by operators constituting a POVM, then* $N^2 = 1$ *and equation* (4) *applies.*

Positive operator valued measures (POVM) are defined below. An important point to realize here is the following:

The old form of the quantum measurement postulate for two particles, the DQMP2, assuming consistency, cannot be proved or falsified, except by experimental evidence.

It is therefore *not* the objective of this article to falsify the DQMP2 on the basis of theoretical arguments, but rather:

The primary objective of this article is to show that there exists experimental evidence that is in contradiction to equation (3) and thus the DQMP2.

This primary objective is the subject of the next section. In the section following the next, it is shown that the measurements made in the experiment are of a more general type than the standard projective or von-Neumann measurements, the former usually referred to as POVM measurements, defined as follows [4]:

A set $\{E_i\}$ of observable positive semidefinite operators E_i in some space is a <u>*POVM*</u> *if the sum of the operators over the entire set is the identity in that space:*

$$\sum_i E_i = I \,. \tag{6}$$

Projection operators or "projectors" obey equation (6), and so are a subcategory of POVM operators. Eigenvalue probabilities of a system $|\psi\rangle$ are calculated from POVM operators E_i in the same manner as with projectors:

$$p(k) = \langle\psi|E_k|\psi\rangle \tag{7}$$

Only with projectors, there is the added restriction that they be idempotent:

$$P_i^2 = P_i \,. \tag{8}$$

Equation (8) does *not* necessarily hold for the more general POVM operators.

Since we will only work with POVM operators here, completeness holds and so $N^2 = 1$ in equation (5) for both experiments below. Therefore equation (4) applies.

Experimental evidence in contradiction to postulate four

The experimental evidence examined here is from Aspect, Grangier and Roger [5], hereafter referred to as the "Aspect experiment." In that experiment, systems of correlated photon pairs were produced from a single source. The photons, being correlated, are represented by a system of the following form:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1,\theta_1\rangle + e^{i\alpha}|\varphi_2,\theta_2\rangle\right) \tag{9}$$

where the subscripts 1 and 2 signify polarity and α is a relative phase factor. In the experiment, for each pair produced by the source, one photon passes through observer Alice's Mach-Zehnder (MZ) interferometer and the other serves as a registration gatekeeper, collected by Bob. The purpose of the latter is to eliminate noise from singles passing into the interferometer. The experimental set-up is shown in figure 1.

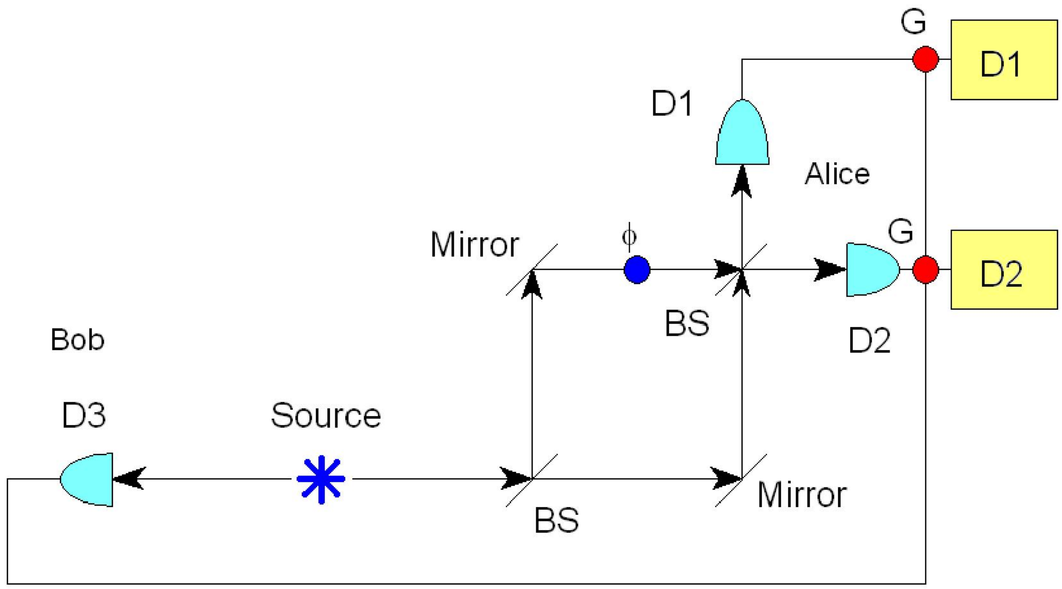

Figure 1. The experimental set-up of Grangier, Aspect and Roger [5]. A two-photon system is emitted by the source. Alice's apparatus on the right is a MZ interferometer together with detectors D1 and D2. Bob's apparatus consists of a single detector D3 whose photon serves as a gatekeeper for Alice's photon counting. The gates G cut off noise from singles and both only open when Bob's photon is detected. Otherwise both are shut.

For the Aspect experiment, let $|\varphi_i\rangle$ in equation (9) represent the states of Alice's MZ photon, and $|\theta_j\rangle$ the states of Bob's gatekeeper photon. There are only two polarity states *i*, *j* = 1, 2 for each photon, which correspond to reflection or transmission upon interaction with a half-silvered mirror respectively. For transforming Alice's $|\varphi_i\rangle$ into the basis $|\tilde{\varphi}_l\rangle$ of the detectors (*l* = 1, 2), the rotation transformation is used:

$$\Phi = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \tilde{\Phi}, \tag{10}$$

where $\Phi^{\mathrm{T}} = \left[|\varphi_1\rangle \quad |\varphi_2\rangle \right]$, $\tilde{\Phi}^{\mathrm{T}} = \left[|\tilde{\varphi}_1\rangle \quad |\tilde{\varphi}_2\rangle \right]$ and δ is the rotation angle, proportional to the phase shift between arms in MZ. Note that the rotation transform (10) is a generalization of the Hadamard transform in that the latter results from the former when $\delta = \pi / 4$. The transformation (10), applied to state (9) results in:

$$\begin{aligned} \left|\psi\right\rangle = \frac{1}{\sqrt{2}}\Big(&\cos\delta\left|\varphi_1\right\rangle\left|\theta_1\right\rangle - \sin\delta\left|\varphi_2\right\rangle\left|\theta_1\right\rangle \\ &+e^{i\alpha}\sin\delta\left|\varphi_1\right\rangle\left|\theta_2\right\rangle + e^{i\alpha}\cos\delta\left|\varphi_2\right\rangle\left|\theta_2\right\rangle\Big) \end{aligned}. \tag{11}$$

where the tilde has been dropped for notational simplicity. In state (11), it is evident that there are degenerate eigenstates present: $\left|\varphi_1\right\rangle\left|\theta_1\right\rangle, \left|\varphi_1\right\rangle\left|\theta_2\right\rangle$ for Alice's eigenvalue 1 which corresponds to registration by detector 1, and degenerate eigenstates $\left|\varphi_2\right\rangle\left|\theta_1\right\rangle, \left|\varphi_2\right\rangle\left|\theta_2\right\rangle$ for eigenvalue 2; *i.e.* detector 2 registration. In applying the quantum measurement postulate DQMP2 to state (11), the probability of detection in detector 1 can be calculated; *i.e.* the probability of Alice measuring eigenvalue 1:

$$\begin{aligned} p_A(1) &= \sum_{j=1}^{2}\left|\left\langle \varphi_1, \theta_j \middle| \psi \right\rangle\right|^2 \\ &= \frac{1}{2}. \end{aligned} \tag{12a}$$

Similarly, for detector 2:

$$p_A(2) = \frac{1}{2}. \tag{12b}$$

Clearly equations (12a) and (12b) predict that there is *no* intensity variation with respect to δ. This contradicts the experimental evidence, shown in figure 2. Hence:

The old form of the discrete quantum measurement postulate for two particles DQMP2 predicts that the detection probability in the Aspect experiment is independent of phase shift δ. This is in contradiction to the experimental evidence shown in figure 2.

Next, we apply the new form of the discrete quantum measurement postulate for two particles MDQMP2, using equation (4) since we assume Bob destroys which-way information and Alice performs a POVM measurement. The registration probability for detector 1 then becomes:

$$\tilde{p}_A(1) = \left| \sum_{j=1}^{2} \langle \varphi_1, \theta_j | \psi \rangle \right|^2$$
$$= \frac{1}{2} + \frac{1}{2}\sin 2\delta \cos\alpha. \quad (13a)$$

Similarly, for detector 2:

$$\tilde{p}_A(2) = \frac{1}{2} - \frac{1}{2}\sin 2\delta \cos\alpha\,. \quad (13b)$$

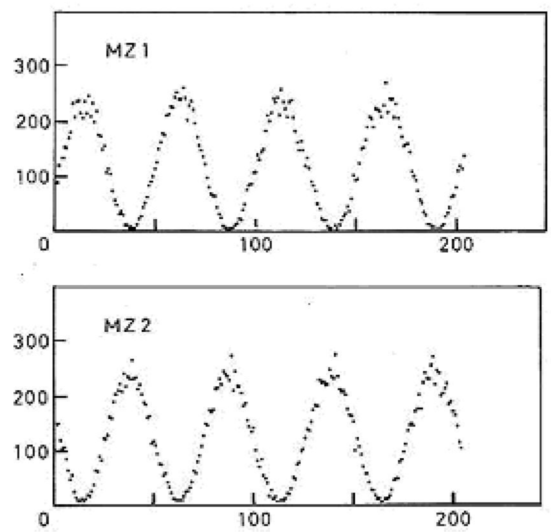

Figure 2. Data from the Aspect experiment [5]. The intensity variation with respect to relative phase δ between arms of MZ is in contradiction to DQMP2 and in concurrence with the MDQMP2. (c) 1986 Europhysics Letters. Reprint permission pending.

Unlike equations (12), equations (13) suggest an oppositely-modulated periodic dependence of detection probability with respect to phase difference δ between detectors, for at least some values of α, as seen in figure 2. In particular, the equations (13) match the normalized data when $\alpha = 0$. Thus:

The new form of the discrete quantum measurement postulate for two particles MDQMP2 is in concurrence with the results of the Aspect experiment shown in figure 2, *whereas the old form DQMP2 fails to predict the results. This follows, since Bob destroys his eigenvalue or "which-way" information in the experiment.*

Operations performed in the Aspect experiment are of the POVM type

Recall that equation (3) prescribes Alice's probability for eigenvalue *k* when Bob allows her to determine which degenerate eigenvector gave the result. It can be written in terms of operators as

$$P_A(k) = \langle \psi | P_k | \psi \rangle \tag{14}$$

where P_k is the projector

$$P_k = \sum_{j=1}^{J} \left| \varphi_k \theta_j \right\rangle \left\langle \varphi_k \theta_j \right|. \tag{15}$$

Since condition (6) holds:

$$\sum_{i=1}^{I} P_i = \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \varphi_i \theta_j \right\rangle \left\langle \varphi_i \theta_j \right| = \mathbf{I}, \tag{16}$$

where **I** is the identity of Alice's A I-dimensional eigensubspace, the projectors P_k of A are said to form a *complete basis*.

Recall also that equation (5) prescribes Alice's probability of eigenvalue k if Bob destroys information which could lead to Alice determining which degenerate eigenvector gave her result. It can be written in terms of operators as well:

$$\tilde{p}_A(k) = \frac{1}{N^2} \langle \psi | \tilde{P}_k | \psi \rangle \tag{17}$$

where

$$\tilde{P}_k = \sum_{j_1, j_2 = 1}^{J} \left| \varphi_k, \theta_{j_1} \right\rangle \left\langle \varphi_k, \theta_{j_2} \right|. \tag{18}$$

The operator (18) can be decomposed into two terms:

$$\tilde{P}_k = P_k + P'_k \tag{19}$$

where

$$P'_k = \sum_{\substack{j_1, j_2 = 1 \\ j_1 \neq j_2}}^{J} \left| \varphi_k, \theta_{j_1} \right\rangle \left\langle \varphi_k, \theta_{j_2} \right| \tag{20}$$

is the component of $\tilde{P}_k$ which gives rise to interference effects. From equations (16) and (19), it follows that

$$\sum_{i=1}^{I} \tilde{P}_i = \mathbf{I} + \sum_{i=1}^{I} P'_i. \tag{21}$$

Generally,

$$\sum_{i=1}^{I} P'_i \neq \mathbf{0}. \tag{22}$$

Thus there is no guarantee that the $\tilde{P}_k$ form a complete basis. In the case where inequality (22) applies, the measurement made by Alice is said to be an *incomplete measurement*. It may be the case that incomplete measurements are forbidden. But that is not a problem here, since both experiments analyzed here involve only complete measurements; in particular for the Aspect experiment, we have that:

$$\sum_{i=1}^{2} P_i' = \mathbf{0}. \tag{23}$$

Thus:

The measurements made by Alice in the Aspect experiment are made in a complete basis; i.e.:

$$\sum_{i=1}^{2} \tilde{P}_i = \mathbf{I}. \tag{24}$$

Further, the operators $\tilde{P}_k$ can be shown to be positive semidefinite observables. Hence:

The measurements done in the Aspect experiment are represented by operators which are elements of a POVM.

Since P_k is a projector, it is idempotent; *i.e.* $P_k^2 = P_k$. On the other hand, $\tilde{P}_k$ is not idempotent, but rather:

$$\tilde{P}_k^2 = J\tilde{P}_k, \tag{25}$$

where J is again, the degeneracy of the kth eigenvalue, and hence the dimension of the kth eigensubspace.

In the case of ensembles of several states $|\psi_i\rangle$ with frequency p_i, one works with density operators $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, and a more general form of equations (3) and (14) for calculating Alice's probabilities if Bob performs a projective measurement:

$$P_A(k) = \mathrm{tr}(P_k \rho) \tag{26}$$

where "tr" is the trace. If Bob destroys his which-way information, then equations (17) and (19) generalize to:

$$\tilde{p}_A(k) = \frac{1}{N^2}\left[\mathrm{tr}(P_k\rho) + \mathrm{tr}(P_k'\rho)\right]. \tag{27}$$

Thus the MDQMP2 for ensembles prescribes:

(MDQMP2E) *The probability of Alice measuring eigenvalue k of observable* $A \otimes I_B$ *on an ensemble of systems represented by ρ is given by equation* (26) *if Bob performs a projective measurement, and by equation* (27) *if Bob destroys which-way information.* $N^2 = 1$ *if Bob performs a POVM type operation.*

A Greenberger-Horne-Zeilinger (GHZ) experiment is analyzed in the next section, using an extension of MDQMP2 to three particles. Like with the Aspect experiment, measurements are done in a complete basis, whether or not Bob destroys which-way information.

Greenberger-Horne-Zeilinger experiment

GHZ states involve entanglements of more than two particles. Here we will consider a 3-photon state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|A_1\rangle|B_1\rangle|C_1\rangle + |A_2\rangle|B_2\rangle|C_2\rangle\right) \tag{28}$$

where the *A*, *B* and *C* represent entangled photons, and these can each occur in one of two states: 1 or 2, hence the subscripts. Here, the three photons are emitted by a common source along coplanar trajectories. Each encounters a beam splitter, which splits the beam within a plane with angle *α*, *β* or *γ* respectively, with respect to the normal vector **n** to the original plane. Alice receives photons *B* and *C* and Bob receives photon *A*. The set-up is similar to Mermin's gedanken experiment [6] and is sketched in figure 3. An important thing to realize is that in this experiment:

Alice measures correlation between her two photons vs. relative angle between her polarizers.

"Relative angle" is defined below. To the author's knowledge, this experiment has not yet been performed.

Upon transforming equation (28) into the basis of the beam splitters using the rotation transformation (10), the following state is obtained:

$$
\begin{aligned}
|\psi\rangle = \frac{1}{\sqrt{2}}\Big[&(\cos\alpha\cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma)|A_1\rangle|B_1\rangle|C_1\rangle \\
&+(\sin\alpha\sin\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma)|A_1\rangle|B_1\rangle|C_2\rangle \\
&+(\sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\cos\gamma)|A_1\rangle|B_2\rangle|C_1\rangle \\
&+(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma)|A_2\rangle|B_1\rangle|C_1\rangle \\
&+(\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\beta\cos\gamma)|A_1\rangle|B_2\rangle|C_2\rangle \\
&+(\sin\alpha\cos\beta\sin\gamma + \cos\alpha\sin\beta\cos\gamma)|A_2\rangle|B_1\rangle|C_2\rangle \\
&+(\sin\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma)|A_2\rangle|B_2\rangle|C_1\rangle \\
&+(\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma)|A_2\rangle|B_2\rangle|C_2\rangle\Big]
\end{aligned} \tag{29}
$$

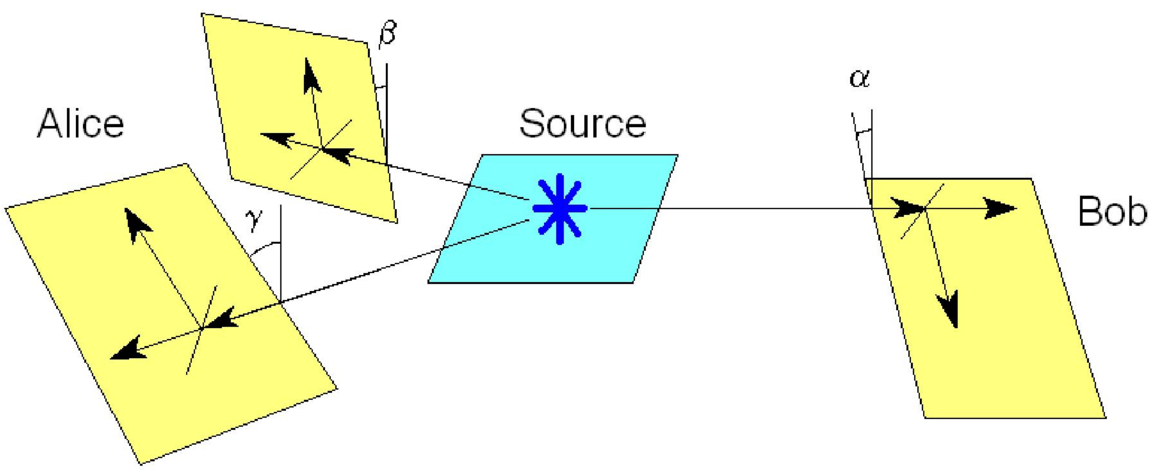

Figure 3. A sketch of the GHZ experimental set up. Three photons are emitted by a source and travel in the same blue plane. Two go to Alice, and one to Bob. Each photon reaches a beam splitter and afterwards may take one of two paths, coplanar with respect to a yellow plane. The yellow planes have relative angles α, β or γ with respect to the normal of the blue plane. In the experiment, Alice measures photon correlation.

For calculating the joint probability for Alice finding eigenvalues (detector numbers) b and c (= 1, 2), given angles α, β and γ, an extension of the MDQMP2 and equation (3) is used, since Bob will be measuring "which way" information on his end:

$$
\begin{aligned}
p(b \wedge c \,|\, \alpha \wedge \beta \wedge \gamma) &= \sum_j \left|\langle A_j B_b C_c|\psi\rangle\right|^2 \\
&= \langle\psi|P_{bc}|\psi\rangle
\end{aligned} \tag{30}
$$

where the projector $P_{bc} = \sum_j \left| A_j B_b C_c \right\rangle \left\langle A_j B_b C_c \right|$. Applying the state (29) to equation (30), one finds the probability for example, of both photons *B* and *C* being detected by detectors numbered 1, given angles α, β and γ:

$$p\left(1 \wedge 1 \,|\, \alpha \wedge \beta \wedge \gamma\right) = \frac{1}{2}\left(\cos^2 \beta \cos^2 \gamma + \sin^2 \beta \sin^2 \gamma\right). \tag{31}$$

Note that probability (31) is independent of Bob's polarizer angle α, therefore α is dropped hereafter. To continue, define the relative angle θ between Alice's angles β and γ:

$$\theta = \beta - \gamma \,. \tag{32}$$

Using equation (32), (31) becomes

$$\begin{aligned} p\left(1 \wedge 1 \,|\, \beta \wedge \gamma\right) &= p\left(1 \wedge 1 \,|\, \alpha \wedge \beta \wedge \theta\right) \\ &= \frac{1}{4} + \frac{1}{8}\left[\cos 2\theta + \cos\left(4\beta - 2\theta\right)\right]. \end{aligned} \tag{33}$$

To get the probability of measuring $b = c = 1$ given relative angle θ and *any* angle β, equation (25) is integrated over the domain of the uniform random variable β and normalized:

$$\begin{aligned} p\left(1 \wedge 1 \,|\, \theta\right) &= \frac{\int_0^\pi p\left(1 \wedge 1 \,|\, \alpha \wedge \beta \wedge \theta\right) d\beta}{\int_0^\pi d\beta} \\ &= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{4} + \frac{1}{8}\left[\cos 2\theta + \cos\left(4\beta - 2\theta\right)\right]\right) d\beta \\ &= \frac{1}{8} + \frac{1}{4}\cos^2 \theta. \end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned} p\left(1 \wedge 2 \,|\, \theta\right) &= p\left(2 \wedge 1 \,|\, \theta\right) \\ &= \frac{1}{8} + \frac{1}{4}\sin^2 \theta \end{aligned} \tag{35}$$

and

$$p\left(2\wedge 2\,|\,\theta\right)=\frac{1}{8}+\frac{1}{4}\cos^2\theta. \tag{36}$$

Note that

$$\sum_{b,c=1}^{2} p\left(b\wedge c\,|\,\theta\right)=1; \tag{37}$$

as expected, since earlier it was claimed that measurement has been done in a complete basis.

Next, we claim that equations (34) through (36) *obey* the Bell inequality [7]: given four variables b_1, b_2, c_1 and c_2 each with domain $\{1,\ -1\}$, the function

$$\Gamma = b_1c_1 + b_1c_2 + b_2c_1 - b_2c_2 \tag{38}$$

must have range $\{2,\ -2\}$ and hence the average $\langle\Gamma\rangle$ over many trials must obey

$$-2\le\langle\Gamma\rangle\le 2. \tag{39}$$

Suppose then that b_1, b_2, c_1 and c_2 are the outcomes of photons *B* and *C* at angles ($\beta=$) β_1, β_2, and ($\gamma=$) γ_1, γ_2 respectively. These outcomes each are either 1 or -1. Further, suppose that

$$\left|\beta_2-\gamma_1\right|=\varphi=\left|\beta_1-\gamma_1\right|=\left|\beta_1-\gamma_2\right|; \tag{40}$$

as illustrated in figure 4.

Consider the two angles β and γ set to β_1 and γ_1 respectively. For photon *B*, the outcome b_1 can be 1 or -1. Likewise, for photon *C*, the outcome c_1 can be 1 or -1. Then, the mean value of the product b_1c_1 is

$$\begin{aligned}\langle b_1c_1\rangle &= \sum_{b_1,c_1=-1}^{1} p\left(b_1,c_1\right)b_1c_1\\ &= p\left(-1,-1\right)-p\left(1,-1\right)-p\left(-1,1\right)+p\left(1,1\right)\\ &= p\left(2\wedge 2\,|\,\varphi\right)-p\left(1\wedge 2\,|\,\varphi\right)-p\left(2\wedge 1\,|\,\varphi\right)+p\left(1\wedge 1\,|\,\varphi\right)\\ &= \frac{1}{2}\cos 2\varphi.\end{aligned} \tag{41}$$

In calculating equation (41) outcome 1 is associated to detection by detector 1, and outcome -1, detector 2.

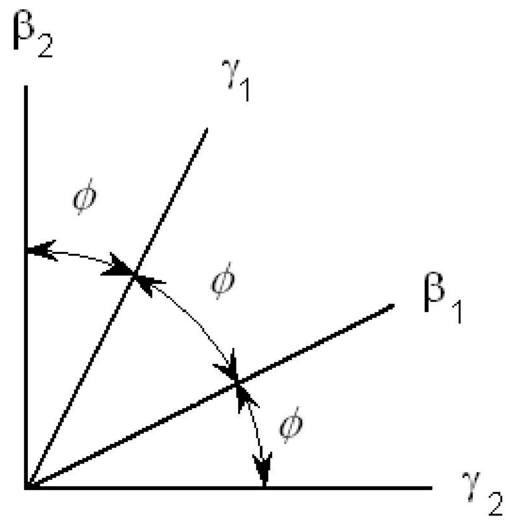

Figure 4. Diagram showing the relative angles ϕ between Alice's polarizer settings $\beta_1, \beta_2, \gamma_1$ and γ_2.

Similarly,

$$\langle b_1 c_2 \rangle = \frac{1}{2}\cos 2\varphi = \langle b_2 c_1 \rangle \tag{42}$$

and

$$\langle b_2 c_2 \rangle = \frac{1}{2}\cos 6\varphi \, . \tag{43}$$

Combining the average of equation (38) with (41) through (43) results in:

$$\begin{aligned} \langle \Gamma \rangle &= \langle b_1 c_1 \rangle + \langle b_1 c_2 \rangle + \langle b_2 c_1 \rangle - \langle b_2 c_2 \rangle \\ &= \frac{3}{2}\cos 2\varphi - \frac{1}{2}\cos 6\varphi. \end{aligned} \tag{44}$$

Since

$$-2 \le \frac{3}{2}\cos 2\varphi - \frac{1}{2}\cos 6\varphi \le 2 \, , \tag{45}$$

it follows that

In the GHZ experiment, Bell's inequality is satisfied for Alice if Bob measures his which-way information.

For the case where Bob destroys which way information, Alice's probabilities are predicted by an extension of equation (4). To make the calculations easier, Bob fixes his polarizer angle to $\alpha = 0$. (Since Bob erases which-way information, he might set up an MZ to accomplish it, with the plane of MZ set to this angle. It is left as an exercise for the reader, to determine what happens when Bob randomizes α.) This reduces equation (21) to

$$\begin{aligned}|\psi\rangle = \frac{1}{\sqrt{2}}\big[&\cos\beta\cos\gamma|A_1\rangle|B_1\rangle|C_1\rangle - \cos\beta\sin\gamma|A_1\rangle|B_1\rangle|C_2\rangle \\ &-\sin\beta\cos\gamma|A_1\rangle|B_2\rangle|C_1\rangle + \sin\beta\sin\gamma|A_2\rangle|B_1\rangle|C_1\rangle \\ &+\sin\beta\sin\gamma|A_1\rangle|B_2\rangle|C_2\rangle + \sin\beta\cos\gamma|A_2\rangle|B_1\rangle|C_2\rangle \\ &+\cos\beta\sin\gamma|A_2\rangle|B_2\rangle|C_1\rangle + \cos\beta\cos\gamma|A_2\rangle|B_2\rangle|C_2\rangle\big].\end{aligned} \tag{46}$$

From state (46), Alice's probability of measuring 1 from both photons given angles β and θ is thus:

$$\begin{aligned}\tilde{p}(1\wedge 1\,|\,\beta\wedge\theta) &= \left|\sum_j \langle A_j B_1 C_1|\psi\rangle\right|^2 \\ &= \frac{1}{2}\cos^2\theta.\end{aligned} \tag{47}$$

Since equation (47) is independent of angle β, it follows that

$$\tilde{p}(1\wedge 1\,|\,\theta) = \frac{1}{2}\cos^2\theta. \tag{48}$$

Similarly,

$$\begin{aligned}\tilde{p}(1\wedge 2\,|\,\theta) &= p(2\wedge 1\,|\,\theta) \\ &= \frac{1}{2}\sin^2\theta\end{aligned} \tag{49}$$

and

$$\tilde{p}(2\wedge 2\,|\,\theta) = \frac{1}{2}\cos^2\theta. \tag{50}$$

Note that probabilities (48) through (50) sum to unity. Further, these probabilities give a violation of the Bell inequality (39). In fact,

$$\langle\tilde{\Gamma}\rangle = 2\langle\Gamma\rangle \tag{51}$$

where $\langle\tilde{\Gamma}\rangle$ is the correlation function obtained from probabilities (48) through (50) and $\langle\Gamma\rangle$ was given in equation (44). Hence:

If Bob destroys which-way information, then Alice will determine a violation of Bell's inequality in the GHZ experiment.

Conclusion

It has been shown that there exists experimental evidence from two-particle systems showing a violation of the discrete degenerate quantum measurement postulate. A modification to the postulate, originally proposed by Srikanth, involving a positive operator valued measure is then used to describe the results successfully. This modification accounts for interference effects, which are the source of the contradiction. The modification of the postulate is then applied to a second proposed experiment involving a three-particle "Greenberger-Horne-Zeilinger" state.

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