Notes on quantum mechanics and general covariance.

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Abstract

In this letter we study two different aspects of general covariance, first we quantize a reparametrization invariant theory, the free particle in Minkowski spacetime and point out in detail where this theory fails (notably these comments appear to be missing in the literature). Second we study the covariance of quantum field theory and show how it connects to causality, the outcome of this study is that QFT is what we shall call ultra weakly covariant with respect to the background spacetime. Third, we treat the question of whether evolution in quantum theory (apart from the measurement act) needs to be unitary, it is easily shown that a perfectly satisfying probabilistic interpretation exists which does not require unitary evolution. Fourth, we speculate on some modifications quantum theory should undergo in order for it to be generally covariant. This paper is primarily written for the student who wishes to study quantum gravity.

1 Introduction

In this letter, we study in detail the extend to which the principles underlying quantum theory and general relativity agree or disagree. Since different people mean different things when speaking about general covariance [1], it is perhaps good to name them differently: strong covariance means that the equations of motion do not depend upon the coordinate system at hand and moreover, all fields involved are dynamical[1], weak covariance means that all predictions of the theory do not depend upon the coordinate system at hand, but some fields are background fields and ultra weak covariance indicates that the predictions of the theory remain invariant for coordinate systems defining foliations of spacetime which are spacelike with respect to the background structure. As we shall prove, quantum field theory turns out to be ultra weakly but not strongly covariant; moreover, ultra weak covariance is all we need to ensure classical causality, that is spacelike separated field operators do (anti) commute with each other. It

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[1]Meaning that they satisfy nontrivial equations of motion.

is argued that progress in quantum theory can be made if we know how to reformulate quantum field theory into a weakly covariant fashion.

2 Covariant quantization of the free relativistic particle.

Henceforth the metric structure on spacetime will have signature $(+---)$ and the action principle at hand is given by

$$S = m\int_{\tau_0}^{\tau_1} \sqrt{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}d\tau.$$

The canonical momenta p_μ are given by

$$p_\mu = m\frac{\eta_{\mu\nu}\dot{x}^{\nu}}{\sqrt{\eta_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}$$

and satisfy the constraint

$$\eta^{\mu\nu}p_\mu p_\nu = m^2.$$

Denoting by $v(\tau) = \sqrt{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$, the inverse Legendre transformation is given by

$$\dot{x}^{\mu} = \frac{v(\tau)\eta^{\mu\nu}p_\nu}{m}$$

and a[2] Hamiltonian giving the correct equations of motion is given by

$$\mathcal{H}(\tau) = \frac{1}{2m}v(\tau)\left(\eta^{\mu\nu}p_\mu p_\nu - m^2\right).$$

Clasically, we have that $\dot{p}_\mu(\tau) = 0$ and

$$x^{\mu}(\tau) = x^{\mu}_0 + \frac{\eta^{\mu\nu}p_\nu}{m}\int_{\tau_0}^{\tau} v(s)ds$$

meaning that the $x^{\mu}(\tau)$ are not predictions of the theory (since they depend on v), but nevertheless observables and they are called *partial* observables by some authors. What we do know however is that for $p_0 > 0$, $t(\tau)$ is an increasing function of τ and for suitable v the entire t range will be covered, hence we can ask the question, what is $x^i(t)$? It is given by

$$x^i(\tau) = x^i_0 - \frac{p_i}{p_0}\left(t(\tau) - t_0\right)$$

and is a prediction of the theory since it does not depend upon v. Therefore, the theory forcasts the spatial coordinates at some fixed time coordinate t,

[2] There exist several Hamiltonians, for example $\mathcal{H}(\tau) = v(\tau)\left(\frac{1}{m}\eta^{\mu\nu}p_\mu p_\nu - m\sqrt{\eta^{\mu\nu}p_\mu p_\nu}\right)$ is another one.

but the statement doesn't involve τ in any way. Now we come to a delicate point, the right hand side of the above expression is nonlocal in time, it involves dynamical quantities evaluated at times τ_0 and τ, and therefore it is not a Dirac observable[3]. Since the constraint $\mathcal{C}(\tau) = \eta^{\mu\nu} p_\mu(\tau) p_\nu(\tau) - m^2$ is first class, it is a generator of gauge transformations, moreover $\dot{\mathcal{C}}(\tau) = 0$ weakly[4]. Since $x^i(\tau) + \frac{p_i(\tau)}{p_0(\tau)} t(\tau)$ is a Dirac observable of the theory, the Poisson bracket with $\mathcal{C}(\tau)$ should vanish. Indeed,

$$\begin{aligned} \{x^i(\tau) + \frac{p_i(\tau)}{p_0(\tau)} t(\tau), \mathcal{C}(\tau)\} &= \{x^i(\tau), -p_i^2(\tau)\} + \frac{p_i(\tau)}{p_0(\tau)} \{t(\tau), p_0^2(\tau)\} \\ &= -2p_i(\tau) + 2\frac{p_i(\tau)}{p_0(\tau)} p_0(\tau) \\ &= 0. \end{aligned}$$

However, to measure this observable (constant of motion) we need four different measurements; classically this is fine, but what is the meaning quantum mechanically? At any rate, classically, it is generically the case that there is a huge sensitivity of the number of nontrivial observables (not Dirac observables) on the initial conditions. If one thinks about general relativity, one has spacetimes where one can find global coordinate charts formed by curvature invariants (and invariants not included in this set are *predicted* from these coordinates) while for example in Minkowski spacetime, no local statements based upon these invariants can be made. In what follows about the quantum theory, our comments shall be focused around the lack of localizability, the scalar product, and the problem of time. The momentum operators are constant and can for all times τ be given by the operators

$$p_\mu = -i\hbar \partial_\mu$$

while the $x^\mu(\tau_0)$ are given by

$$x^\mu(\tau_0) = \widehat{x}^\mu$$

where $\widehat{x}^\mu$ is the multiplication operator by x^μ. $x^\mu(\tau)$ is then given by

$$x^\mu(\tau) = \widehat{x}^\mu - i\hbar \frac{\eta^{\mu\nu}\partial_\nu}{m} \int_{\tau_0}^{\tau} v(s) ds.$$

These operators have a unique Self-Adjoint extension on $L^2(\mathbb{R}^4, d^4x)$ and the constraint operator is given by

$$\hbar^2 \eta^{\mu\nu} \partial_\mu \partial_\nu + m^2.$$

Physical states Ψ must then satisfy

$$\hbar^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \Psi + m^2 \Psi = 0.$$

[3]For instance, we don't know how to evaluate Poisson brackets of the type $\{a(\tau), b(\tilde{\tau})\}$. This is different in quantum mechanics where we can commute operators at different times.

[4]This is precisely so in gravity.

A first problem one encounters is that there exist no normalizable physical states, in the literature one proposes to consider the Klein-Gordon scalar product

$$(\Phi, \Psi) = \int d^3x \, (i\partial_t \Phi^* \Psi - i\Phi^* \partial_t \Psi)_{|t=0}$$

on the subspace of positive frequency solutions. However, the time operator $\widehat{t}$ is not Hermitian with respect to this scalar product! This problem is circumvented by considering the time independent scalar product

$$(\Phi, \Psi) = \int d^3x \Phi^* \Psi_{|t=0}$$

on the same subspace. Let us now treat the problem of time; the time evolution operator $U(\tau)$ is given by

$$U(\tau) = \exp^{-\frac{i}{2m\hbar} \int_{\tau_0}^{\tau} v(s)ds \left(\hbar^2 \eta^{\mu\nu} \partial_\mu \partial_\nu + m^2\right)}$$

and

$$x^\mu(\tau) = U(-\tau) \widehat{x}^\mu U(\tau).$$

Now, there happens something funny, it is so that

$$(U(\tau)\Phi, \widehat{x}^\mu U(\tau)\Psi) \neq (\Phi, U(-\tau)\widehat{x}^\mu U(\tau)\Psi)$$

and the reason is that the operator $-i\hbar\partial_t$ is only Hermitian when both arguments in the scalar product are positive frequency physical states. The operator $\widehat{x}^\mu$ however, maps physical states to nonphysical states and hence the Hermiticity property does not apply. This means that there is a disagreement between the Schroedinger picture and the Heisenberg picture; the former has no time evolution of physical matrix elements, while the latter has! So, the problem of time appears to be absent in the Heisenberg picture. Now, we come to the most serious problem, the lack of localization: the operators $x^\mu(\tau)$ do not commute with the constraint operator and hence both cannot be simultaneously diagonalized. Therefore, it is not possible to make sharp measurements of space *and* time according to Von Neumann measurement theory. Funny enough, the expectation values of (powers of) position operators $\widehat{x}^\mu$ indicate localization (due to the fact that the scalar product takes one time slice, expectation values of operators do depend upon the slice). For example $(\Phi, t^n \Phi) = 0$ hence $\delta t = 0$, however this perfect localization fails for $t(\tau)$, $\tau \neq \tau_0$, and realistic wave packages[5]! Since von Neumann measurement theory fails, I think it is fair to say that the above quantization is inadequate and therefore one might conclude that quantum mechanics, with its current measurement theory, is not strongly covariant. Similar

[5]Therefore it is not fair to say that time is fixed to some definite value. Furthermore $\delta t(\tau) = a \int_{\tau_0}^{\tau} v(s)ds$ with $a > 0$ with average $< t(\tau) >= b \int_{\tau_0}^{\tau} v(s)ds$, $b > 0$. One could at each stage where δt exceeds some treshold value, replace the state Ψ by one which is more localized in time (smaller a). But then it *appears* to become impossible to localize the wave package in space.

problems should be expected for quantum gravity; actually it turns out that the situation gets even worse. Although it is possible to (formally) construct an appropriate Klein Gordon like scalar product on superspace, it turns out to be impossible to find a subspace of positive frequency solutions with positive norm [2]. This has lead to the suggestion of third quantization.

3 Ultra weak covariance of quantum field theory.

What follows is fairly obvious but it is nevertheless good do at least once in your lifetime. We are going to prove that quantum field theory is ultra weakly covariant and that therefore causality always holds. For sake of simplicity we shall argue from the viewpoint of free massive Klein Gordon theory given by the action

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\left(\,g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - m^2\psi^2\right)$$

in a general curved spacetime. Let (t, x^i) be a foliation of spacetime; note first that Hamiltonian evolution is only well defined if the hypersurfaces of constant t do not become null. Indeed, g^{tt} is proportional to the determinant of g_{ij} which vanishes if and only if the ∂_i span a null hypersurface. A vanishing g^{tt} means that the Legendre transform and therefore the Hamiltonian becomes singular, which effectively stops evolution. Consider two foliations (t, x^i) and $(\tilde{t}, \tilde{x}^i)$ of spacetime with spacelike leaves which coincide for $t \in [t_0, t_0 + \epsilon]$. Choosing intial operators on some Hilbert space $\mathcal{H}$ so that the intial value conditions

$$\begin{aligned}
\left[\psi(t_0, x^i), \psi(t_0, y^i)\right] &= 0 \\
\left[\psi(t_0, x^i), \pi(t_0, y^i)\right] &= i\delta(x^i - y^i) \\
\left[\pi(t_0, x^i), \pi(t_0, y^i)\right] &= 0
\end{aligned}$$

are satisfied, covariance of the equations of motion guarantees that the solutions will be the same and in particular one has that

$$\left[\psi(t(\tilde{t}, \tilde{x}^i), x^i(\tilde{t}, \tilde{x}^j)), \psi(t(\tilde{t}, \tilde{y}^i), x^i(\tilde{t}, \tilde{y}^j))\right] = 0.$$

One can get rid of the common initial value tube by repeating the same argument for $(\tilde{t}, \tilde{x}^i)$ "backwards" in time. It is now easy to see that the previous (initial value) conditions hold for any coordinate system with spacelike leaves. This shows that the *solution* $\psi(t, x^i)$ behaves in a covariant fashion with respect to spacelike foliations. This does not show of course that were one to choose another initial representation on $\mathcal{H}'$, that both theories are equivalent in some sense; it is well known that inequivalent representations of the CCR algebra exist. But what we have shown here is that any of these theories behaves covariantly and hence causality is respected. It is perhaps illuminating to show this explicitely for massive Klein Gordon theory in flat Minkowski space time, since the calculation reveals the importance of spacelike folations (it doesn't work for timelike ones). This is left as a not entirely trivial exercise for the reader.

4 Unitarity and quantum mechanics

It is often repeated that quantum mechanics *needs* to be unitary because one needs a coherent probability interpretation. What we shall show here is that one can consistently define experimental probabilities as outcomes of measurements without reference to unitary evolution at all. This is accomplished in the Heisenberg picture. Let A denote an observable at time $t = 0$ and $U(t) = \exp^{-iHt}$ be the unitary evolution operator. Clearly the spectrum of A is the same as that of $A(t) = U(-t)AU(t)$ and the orthogonal projector on the subspace of eigenvalue λ evolves as

$$P^A_\lambda(t) = U(-t)P^A_\lambda(0)U(t).$$

Denote by Ψ the initial state, then the probability that at time t we measure for the observable A the eigenvalue λ is given by

$$\text{Prob}(A, \lambda, t) = \frac{\big(\Psi, P^A_\lambda(t)\Psi\big)}{(\Psi, \Psi)}.$$

This can be extended to an arbitrary number of arguments: the probability that O is measured to be μ at time s given that A was measured to be λ at time t is given by

$$\text{Prob}(O, \mu, s|A, \lambda, t) = \frac{\big(\Psi, P^A_\lambda(t)P^O_\mu(s)P^A_\lambda(t)\Psi\big)}{\big(\Psi, P^A_\lambda(t)\Psi\big)}.$$

One simply notices that all above probabilities are (a) well defined in the sense that they sum up to one if all alternatives are taken into account and (b) their expression does not require the evolution operator $U(t)$, all that is needed are time dependent operators. Of course unitarity serves also other purposes, one of which we spoke about before, such as the preservation of the commutation relations in time. So, one needs to study what will happen to those in a nonunitary quantum theory.

5 Conclusions and outlook.

We have learned that the Schroedinger and Heisenberg picture are two inequivalent theories and that the latter is the only plausible one. However, it is not sufficient to simply adapt the Heisenberg picture, the Von Neumann theory of measurement is not valid anymore and a substitute for an adquate probabilistic interpretation needs to be found. Of course, it may just be that something else is wrong at a more fundamental level. One way to make progress would be to redefine quantum field theory in a weakly covariant fashion, to make it really four dimensional as to speak. Certainly such endeavour has already been undertaken by the algebraic formulation of free quantum field theory, but this appears not to be sufficient to me. One notices that the commutation relations are of singular nature leading to distribution valued operators. This would lead to serious difficulties if all measurements were assumed to be dynamical (ie, Dirac observables) and in section two we have already dismissed such point of view. We have shown that the CCR algebra is compatible with ultra weak covariance; however, the computations suggested at the end of section three show that they do not in a weakly covariant framework. Therefore, we have to look for a completely new way of devising a quantum theory in which the role of causality is an emergent and not a fundamental property: measurement influences should be allowed to tunnel moderately through the light cone as one would expect in a full theory of quantum gravity[6]. It could be that if these issues were resolved, some Von Neumann type measurement rule would persist and therefore, our first priority consists in understanding the emergence of the (approximate) canonical (anti)commutation relations.

References

[1] R.M. Wald, General relativity, Chicago University Press, 1984.

[2] C.J. Isham, Canonical quantum gravity and the problem of time, gr-qc/9210011.

[6]The same conclusion can be reached from a different perspective, see [2].