Comments on the Entropy of the Wave Function Collapse

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Academically, among students, an apparent paradox may arise when one tries to interpret the second law of thermodynamics within the context of the quantum mechanical wave function collapse. This is so because a quantum mechanical system suddenly seems to undergo, from a less restrictive state constructed from a superposition of eigenstates of a given operator, to a more restrictive state: the collapsed state. This paper is intended to show how this picture turns out to be a misconception and, albeit briefly, furtherly discuss the scope of Max Born's probabilistic interpretation within the second law of thermodynamics.

THE BOLTZMANN FORMULA: A SOURCE OF MISCONCEPTION FOR A RECKLESS VISION

[1] At a first glance, one may think the wave function collapse violates the second law of thermodynamics, since a quantum system prepared as a superposition of eigenstates of a given operator suddenly undergoes to a more restrictive state. But this is not the case, in virtue of the fact that a superposition and a eigenstate are states on equal footing. The use of the Boltzmann formula:

$$S = k \ln w, \tag{1}$$

for the entropy S of a thermodinamically closed system, where $k = 1.38 \times 10^{-23}\, JK^{-1}$ is the Boltzmann constant, leads, at a first glance, to the impression that the entropy should have a greater value before the collapse, under an erroneous assumption that the initial number, w_0, of microstates, w, should be greater than the final number of microstates, w_f, in virtue of the needed quantity of eigenstates, $w_0 > 1$, used to construct the wave function before the collpse, in contrast to the apparent $w_f = 1$ after the collapse. We will see that the converse occurs. Furthermore, one should, firstly, define the thermodinamically closed system as consisting of two subsystems: the quantum object subsystem plus the classical apparatus subsystem.

A SIMPLE SOLUTION FOR THIS APPARENT PARADOX

[2] Consider a quantum subsystem Ψ: prepared as a superposition of the n eigenstates $\{\phi_k\}$, with $1 \leq k \leq n$, of a given operator Φ with finite non-degenerated spectrum:

$$\Psi = a_1\phi_1 + \cdots + a_n\phi_n = \sum_{k=1}^{n} a_k\phi_k, \tag{2}$$

where:

$$a_k = \int_V \phi_k^* \Psi dV, \tag{3}$$

is the inner product with which the Hilbert state space is equipped. The * denotes the complex conjugation and dV the elementar volume of the physical space V of a given representation.

Up to the measure, before the interaction between a classical apparatus subsystem, designed to obtain observable eingenvalues of the operator Φ, and a quantum subsystem Ψ given by eqn. (2), there exists just one microstate of the global system consisted by apparatus subsystem plus quantum subsystem, since these two subsystems are not initially correlated and the initial microstate of the quantum subsystem Ψ is just the unique state Ψ as well the initial microstate of the classical apparatus subsystem is the unique one in which it has no eigenvalue registered.

Hence, in virtue of the initial independence of the subsystems, the initial microstate of the global thermodinamically closed system has multiplicity $w_0 = 1 \times 1 = 1$, being the initial entropy of the global system given by:

$$S_0 = k \ln 1 = 0, \tag{4}$$

in virtue of the eqn. (1).

One may argue the initial state of the classical apparatus subsystem has got a multiplicity greater than 1, since this subsystem seems to have internal modes compatible with an empty memory. We emphasize this is not the case, since the state of the memory defines the apparatus state, being this state an empty one in spite of any apparatus internal modes before an accomplished measure [3]. The same comment is valid for the quantum subsystem, since the state of this subsystem is Ψ, previously defined by the superposition of a Φ operator eigenstates, $\{\phi_k\}$, being the object Ψ an unique one. These objects, by definition, are initially constrained to these defined states, and one does not need to take into account the different manners by which these subsystems should equally evolute to their respective initial states.

Once a measure is accomplished, there will exist n possible eigenvalues to be registered within the memory of the classical apparatus subsystem, viz., since there are

n different final situations for the global system, where n is the number of non-degenerated eigenvectors of the Φ operator. A reckless short-term analysis would lead to the conclusion that the final number of microstates of the global system, w_f, should be $w_f = n$, since it seems to be the number of ways by which a final collapsed state is reached. But such a conclusion is wrong, since the final state is not simply a collapsed one with a label on it. Differently from a case in which a pair of unbiased dice is thrown, where a particular result of a throw of dice is not physically different from any other result, except for the labels on them, a given collapsed state encapsulates physical content. Each collapsed state is a different final state with its characteristic multiplicity, and one should not enroll the possible collapsed states within a same bag with $w_f = n$ possible collapsed elements. Comparing with the throw of dice case, if you erased the dice numbers, their labels, you could not infer the difference between the results, but the physical content within the collapsed wave function result would lead one to infer the difference between different results, between different outcomes of collapse of Ψ.

- Different physical characteristics implying different outcomes for the wave function collapse define evolutions from the initial global system to new states of the global system, instead of different configurations for a same final state.

In the throw of dice example, the different outcomes are different configurations of a same final state. If the collapsed wave function was a state with n different possible configurations for this same collapsed state, the final number of microstates would be $w_f = n$, but this is not the case.

For the collapsed states, the multiplicities of the possible final results are not necessarily the same, since they depend on the outcome probabilities of their respective eigenvalues. Let p be the label of the eigenvalue with the least reliable ($\neq 0$) [4] outcome probability. The outcome probability of a given eigenvalue is given by the Max Born's rule, from which the least probability, of the p-labeled eigenvalue, is simply given by $a_p^* a_p$, where [see eqn. (3)]:

$$a_p^* a_p = \left| \int_V \phi_p^* \Psi dV \right|^2 \neq 0. \tag{5}$$

Applying a frequential interpretation for the probability, the least multiplicity of microstates is $N a_p^* a_p$, where N is the quantity of *state-balls* within an *a posteriori interpreted quantum-subsystem-urn* (we are emphasizing that the interaction with the classical apparatus subsystem permits a classical [5], under the frequential sense, a posteriori, interpretation of probabilities, since any quantum effects of probabilistic superposition of amplitudes cease after the collapse, permitting a frequential interpretation via Born's rule). Such a frequential interpretation requires $N \to \infty$, i.e., infinitely many measures to be accomplished on identical quantum subsystems by the classical apparatus subsystem, but we will back to this point later.

The least final entropy of the global system, related to the outcome probability of the p-labeled eigenvalue, reads:

$$S_f = k \ln \left(N a_p^* a_p \right). \tag{6}$$

From the eqns. (4) and (6), the least possible entropy variation turns out to be:

$$\Delta S = S_f - S_0 = k \ln \left(N a_p^* a_p \right). \tag{7}$$

From the eqn. (7), we infer that the second law of thermodynamics holds *iff*:

$$N a_p^* a_p \geq 1 \Rightarrow a_p^* a_p \geq \frac{1}{N}, \tag{8}$$

since $N > 0$. Now, we will prove the following theorem:

Theorem: *The second law of thermodynamics holds for the wave function collapse under a frequential interpretation via Max Born's rule and, once accomplished the collapse, the collapse is an irreversible phenomenon.*

Proof: • Suppose the converse, i.e., that the second law of thermodynamics does not hold for the wave function collapse under a frequential interpretation via Max Born's rule. In virtue of eqn. (7), one has:

$$\Delta S = S_f - S_0 = k \ln \left(N a_p^* a_p \right) < 0 \Rightarrow N a_p^* a_p < 1. \tag{9}$$

Since $a_p \neq 0$ [6], $N \geq 1/(a_p^* a_p)$ violates the condition stated by the eqn. (9). But $N \to \infty$, in virtue of the frequential interpretation, hence $N > 1/(a_p^* a_p)$, and the eqn. (9) is an absurd. We conclude the second law of thermodynamics holds within the terms of this theorem. The proof the collapse is an irreversible phenomenon follows as a corollary of this theorem. In fact:

$$N > 1/(a_p^* a_p) \Rightarrow N a_p^* a_p > 1 \therefore$$

$$\Delta S = k \ln \left(N a_p^* a_p \right) > 0, \tag{10}$$

and the collapse of the wave function is an irreversible phenomenon, being $\Delta S > 0$ the entropy variation of the thermodinamically closed system: quantum subsystem plus classical apparatus subsystem. •

The law of large numbers states the probability of an event p, P_p, is given by the limit:

$$\lim_{N \to \infty} \frac{\sum_{l=1}^{N} \xi_l^p}{N} = P_p, \tag{11}$$

where ξ_l^p assumes the value 1 when the event p occurs, or zero otherwise. If $a_k^* a_k \equiv P_p \neq 0$, the limit must obey:

$$\lim_{N\to\infty} \frac{\sum_{l=1}^{N} \xi_l^p}{N} = \frac{\lim_{N\to\infty} \sum_{l=1}^{N} \xi_l^p}{\lim_{N\to\infty} N} \neq 0. \tag{12}$$

From eqn. (12), we conclude $\lim_{N\to\infty} \sum_{l=1}^{N} \xi_l^p$ cannot be finite, since N grows without limit. Hence:

$$\lim_{N\to\infty} \sum_{l=1}^{N} \xi_l^p > 1. \tag{13}$$

Particularly, the eqn. (13) gives the number of microstates of the p-labeled eigenstate, proving the above theorem. Rigorously, one should substitute:

$$N \to N + \frac{f(N)}{a_k^* a_k}, \tag{14}$$

within the above theorem proof, with:

$$\lim_{N\to\infty} \frac{f(N)}{N} = 0. \tag{15}$$

Such choice leads to:

$$\sum_{l=1}^{N} \xi_l^p = N a_k^* a_k = \left(N + \frac{f(N)}{a_k^* a_k}\right) a_k^* a_k = N \left(a_k^* a_k + \frac{f(N)}{N}\right) \;\therefore \tag{16}$$

$$\frac{\sum_{l=1}^{N} \xi_l^p}{N} = a_k^* a_k + \frac{f(N)}{N}. \tag{17}$$

Taking the limit $N \to \infty$ in eqn. (17), we recover the law of large numbers. Taking the limit $N \to \infty$ in eqn. (16), one obtains in virtue of the eqn. (13):

$$\lim_{N\to\infty} \sum_{l=1}^{N} \xi_l^p = \lim_{N\to\infty} \left(N + \frac{f(N)}{a_k^* a_k}\right) a_k^* a_k > 1 \;\therefore \tag{18}$$

$$\lim_{N\to\infty} \left(N + \frac{f(N)}{a_k^* a_k}\right) > \frac{1}{a_k^* a_k}. \tag{19}$$

Eqn. (19) is the argument used to prove the theorem, as one infers from the eqn. (14).

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[1] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (John Wiley and Sons, 1985), second edition ed.

[2] S. Gasiorowicz, *Quantum Physics* (John Wiley and Sons, 2003), third edition ed.

[3] The irrelevance of the apparatus internal modes compatible with a given apparatus memory state asserts the hypothesis of an unbiased apparatus subsystem. Any result to be measured by the apparatus subsystem must have the same number of equally like apparatus microstates. If some result was related to a different number of apparatus compatible microstates, the results with the maximal number of apparatus compatible microstates would be biased. The collapse should not be caused by apparata biases. In virtue of this hypothesis, one may neglect the apparatus internal modes compatible with a particular apparatus memory state, since the same number of internal modes is common to all the memory states, and the variation of entropy cancels out the same common number (say w_a): $\Delta S = S_f - S_0 = k \ln (w_f \times w_a) - k \ln (1 \times 1 \times w_a) = k \ln w_f - k \ln (1 \times 1)$, where w_f is the number of microstates of a given final state of the global isolated system in which the apparatus has registered the respective collapsed state, considering the apparatus memory state as its unique degree of freedom.

[4] If $a_p = 0$, the respective eigenstate ϕ_p, within the superposition representing Ψ [see eqn. (2)], turns out to be an impossible collapsed state. Such consideration would be totally void, since the final microstate associated to it would never occur, being $\Delta S = k \ln 0 - k \ln 1 = -\infty$ [see eqns. (1) and (4)] a violation of the second law of thermodynamics, in accordance with the impossibility of a final microstate with $a_p = 0$.

[5] Here, the classical designation resides within the counting process after the collapse. We are not saying the final collapsed state leads to a classical interpretation of the quantum object, we are enphasizing that the dialectics after the collapse to interpret frequency of a given collapsed state is the classical one via Born's rule. One does not count quantum waves, but the discret signals of a collapsed object. Surelly, alluding, e.g., to the double-slit canonical example, the diffraction pattern on the screen has not a discrete counterpart, but the points on the screen, when

the intensity of the source is reduced, have and may be counted.
[6] Remember the reliability defining the p-labeled eigenstate, see eqn. (5) again and its inherent paragraph.