Asymptotic Moments of Near Neighbor Distances for the Gaussian Distribution

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Abstract

We study the moments $E[d_{1,k}^{\alpha}]$ of the k-th nearest neighbor distance for independent identically distributed points in $\Re^n$. In the earlier literature, the case $\alpha > n$ has been analyzed by assuming a bounded support for the underlying density. The boundedness assumption is removed by assuming the multivariate Gaussian distribution. In this case, the nearest neighbor distances show very different behavior in comparison to earlier results. In the unbounded case, it is shown that $E[d_{1,k}^{\alpha}]$ is asymptotically proportional to $M^{-1}\log^{n-1-\alpha/2} M$ instead of $M^{-\alpha/n}$ as in the previous literature.

keywords: nearest neighbor; moments; gaussian; random geometry

1 Introduction

Consider a set of independent identically distributed (i.i.d.) random variables $(X_i)_{i=1}^M$ with a common density $p(x)$ on $\Re^n$. We study the moments of the nearest neighbor distance

$$E[d_{1,k}^{\alpha}] \tag{1}$$

in the limit $M \to \infty$. The quantity (1) appears commonly in the literature on random geometric graphs, where directed and undirected nearest neighbor graphs are analyzed as special cases of more general frameworks [9, 10, 13]. In this paper, the nearest neighbor distance serves as the quantity of interest with the hope that in the future, the ideas can be represented in a more abstract form.

The expectation (1) is also of interest in its own right and tends to appear under various scientific contexts. A significant application is found in the non-parametric estimation of Rényi entropies, where asymptotic analysis provides theoretically sound estimators [5, 6, 8]. Moreover, nearest neighbor distances and distributions play a major role in the understanding of nonparametric estimation in general [1, 4]. Finally, it should be mentioned that quantities related to (1) are encountered in physics, especially statistical mechanics and the theory of gases and liquids [11, 3].

In the earlier literature, it has been shown that under general conditions (Γ denotes the Gamma function)

$$E[d_{1,k}^{\alpha}] \to V_n^{-\alpha/n} \frac{\Gamma(k+\alpha/n)}{\Gamma(k)} \int_{\Re^n} p(x)^{1-\alpha/n} dx$$

in the limit $M \to \infty$ if $0 < \alpha < n$ [12, 2]. However, the case $\alpha > n$ is quite different and usually a boundedness condition must be imposed on the support of $p(x)$. As the contribution of this paper, we analyze what happens if $\alpha > n$, while $p(x)$ is unbounded. To simplify matters, we examine only the multivariate Gaussian distribution

$$p(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}\|x\|^2}$$

with the long term goal of extending the results to more general classes of densities. It turns out that the asymptotic behavior is very different to the case $0 < \alpha < n$. We show that in the limit $M \to \infty$,

$$(M \log^{\alpha/2+1-n} M) E[d_{1,k}^{\alpha}] \to \frac{2^{n-\alpha/2-1} n V_n}{(k-1)!} \int_0^{\infty} g\left(\frac{1}{y}\right) dy,$$

where the definition of g depends on n, k and α (see Section 3).

2 Definitions

We start with some basic definitions. V_n denotes the volume of the unit ball $B(0,1)$ in $\Re^n$ in the Euclidean norm (which will be used all the time in this paper). $I(\cdot)$ refers to the indicator function of a random event. For a vector $x \in \Re^n$, $x^{(j)}$ denotes component j of that vector. The volume of a set A with respect to the Lebesgue measure is denoted by $\lambda(A)$. If $g(r)$ is a function defined on an open subset of $\Re$, we denote the derivative of g by Dg.

$(X_i)_{i=1}^{M}$ is taken as an i.i.d. sample with $X_i \in \Re^n$. Each X_i follows a common density $p(x)$; our work concerns the Gaussian case

$$p(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}\|x\|^2}. \tag{2}$$

The first nearest neighbor of X_i is defined by

$$N[i,1] = \text{argmin}_{1 \le j \le M, j \ne i} \|X_j - X_i\|$$

and by recursion, the k-th nearest neighbor is

$$N[i,k] = \text{argmin}_{1 \le j \le M, j \notin \{i, N[i,1], \ldots, N[i,k-1]\}} \|X_j - X_i\|.$$

The corresponding k-th nearest neighbor distance is $d_{i,k} = \|X_{N[i,k]} - X_i\|$. The goal of the paper is to analyze

$$E[d_{i,k}^\alpha] \tag{3}$$

in the limit $M \to \infty$ with everything else fixed. Because the sample is independent identically distributed (i.i.d), we set $i = 1$.

Throughout the paper there will be constants, which depend on some variables, but not on the others. Such variables are denoted by $c(\ldots)$, where inside the parentheses we indicate the dependency. Strictly speaking, c is a function of some variables, but in the standard convention, it will be called a constant. During the course of our proofs, several different unknown constants will emerge. To keep them separate, lower indices (in the form c_i) are used.

General error terms, which can be bounded but not written in closed form, will be denoted by R (or R_i with a lower index i). After the appearance of each such term, we write an equation of the form

$$|R| \le c(\ldots)f(\ldots),$$

where c is a constant and f is a function of M or some other variables. Inside proofs, the Big-Oh notation will be invoked as another way to express unknown but negligible terms.

3 Main Results and Previous Work

The analysis of nearest neighbor distances can be viewed as part of the general framework of random geometric graphs. In this field, results are established for quantities of the form

$$\xi(X_1, (X_i)_{i=1}^M),$$

where ξ has some locality properties. By imposing higher levels of abstraction, very general functions can be analyzed as long as locality arguments are available. We refer to [9, 10, 13] as a starting point to understand the issues arising in the field.

However, abstract theories do not directly give exact information about the asymptotic behavior of the moments (3). The step towards concretizing the results concerning nearest neighbor graphs was taken in [12]. The following has been proven:

Theorem 1. *Suppose that* $0 < \alpha < n$*,* $p(x)$ *is a density with*

$$\int_{\Re^n} p(x)^{1-\alpha/n} dx < \infty$$

and

$$\int_{\Re^n} \|x\|^r p(x) dx < \infty$$

for some $r > n/(n-\alpha)$*. Then*

$$M^{\alpha/n} E[d_{1,k}^{\alpha}] \to V_n^{-\alpha/n} \frac{\Gamma(k+\alpha/n)}{\Gamma(k)} \int_{\Re^n} p(x)^{1-\alpha/n} dx$$

in the limit $M \to \infty$*.* $\Gamma(\cdot)$ *refers to the Gamma function. If* $\alpha \geq n$*, the limit holds if* $p(x)$ *is bounded from below and above on a bounded convex set* $\mathcal{X}$*.*

As a downside, Theorem 1 has the convexity requirement on $\mathcal{X}$ if $\alpha > n$. Furthermore, it does not provide a rate of convergence. These issues have been addressed by the concrete approach in [2], where it was shown that if $\inf_{x \in \mathcal{X}} p(x) > 0$ and $p(x)$ has a bounded gradient on $\mathcal{X}$, then under rather weak conditions on the space $\mathcal{X}$, we have

$$M^{\alpha/n} E[d_{1,k}^{\alpha}] = V_n^{-\alpha/n} \frac{\Gamma(k+\alpha/n)}{\Gamma(k)} \int_{\mathcal{X}} p(x)^{1-\alpha/n} dx + O(M^{-1/n+\rho})$$

for any $\rho > 0$ removing the convexity requirement.

As a common factor between the results, observe that in the case $\alpha > n$, two requirements must be satisfied:

1. The set $\mathcal{X}$ must be bounded.
2. $\inf_{x \in \mathcal{X}} p(x) > 0$.

In this paper we ask, what happens when neither 1. nor 2. hold but $\alpha > n$ (the case $\alpha = n$ is not addressed). The early works in random geometry took the uniform distributions as a case of special interest. Analogously, we choose the Gaussian density

$$p(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}\|x\|^2}$$

as our target of study.

It turns out that the behavior for $\alpha > n$ is very different to Theorem 1 for the Gaussian distribution. As the main contribution of the paper, we prove the following.

Theorem 2. *Suppose that Equation* (2) *holds and* $\alpha > n$*. Then*

$$(M \log^{\alpha/2+1-n} M) E[d_{1,k}^{\alpha}] \to \frac{2^{n-\alpha/2-1} n V_n}{(k-1)!} \int_0^{\infty} g\left(\frac{1}{y}\right) dy.$$

in the limit $M \to \infty$ *with*

$$g(t) = \int_0^{\infty} \omega^{k-1} e^{-\omega} f^{-1}(\omega t)^{\alpha} d\omega,$$

where f^{-1} *refers to the inverse function of*

$$f(t) = t^n \int_{B(0,1)} e^{t y^{(1)}} dy.$$

The main difference to Theorem 1 is that now $E[d_{1,k}^\alpha]$ is of order $M(\log M)^{n-\alpha/2-1}$ instead of $M^{-\alpha/n}$. Theorem 2 can be further developed by analyzing the rate of convergence and possible applications. This remains a topic of future research. Another open question is the extension to a general density p, which the author believes is possible. This could possibly unify the case with boundary effect [7] and the more general unbounded case

4 Outline of the Proof

We will use the small ball probability

$$\omega_x(r) = \int_{B(x,r)} p(y)dy$$

due to its useful distribution free properties. In fact, [2] shows that the distribution of the quantity $\omega_{X_1}(d_{1,k})$ does not depend on the density p and moreover, tends to take values of order M^{-1}. Another useful fact is that conditionalization on X_1 does not change the distribution of $\omega_{X_1}(d_{1,k})$. We approximate

$$\begin{aligned} \omega_x(r) &= (2\pi)^{-n/2} \int_{B(x,r)} e^{-\frac{1}{2}\|y\|^2} dy \\ &= (2\pi)^{-n/2} \int_{B(x,r)} e^{-\frac{1}{2}\|x\|^2 - x^T(y-x) - \frac{1}{2}\|y-x\|^2} dy \\ &\approx p(x) \int_{B(0,r)} e^{-x^T y} dy = p(x) r^n \int_{B(0,1)} e^{-r x^T y} dy \end{aligned} \tag{4}$$

assuming that $e^{-\frac{1}{2}r^2}$ is close to 1. By a change of variables (rotation inside the last integral in (4)) we have

$$\omega_x(r) \approx p(x) r^n \int_{B(0,1)} e^{-r\|x\|y^{(1)}} dy.$$

Now if we take $f(t) = t^n \int_{B(0,1)} e^{-ty^{(1)}} dy$, then $\|x\|^n \omega_x(r) \approx p(x) f(\|x\| r)$ and we solve

$$r \approx \frac{f^{-1}\left(\frac{\|x\|^n \omega_x(r)}{p(x)}\right)}{\|x\|}.$$

f^{-1} refers to the inverse of f. By substituting $d_{1,k}$ in place of r and $\|X_1\|$ in place of $\|x\|$, we get conditionally on X_1

$$E[d_{1,k}^\alpha] \approx E[E[\frac{f^{-1}\left(\frac{\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)}\right)^\alpha}{\|X_1\|^\alpha} | X_1]].$$

The argument for f^{-1} looks rather complicated. However, because the conditional distribution of $\omega_{X_1}(d_{1,k})$ does not depend on the density $p(x)$ or X_1, it would be sufficient to somehow control the dependency on X_1.

Our strategy can be summarized as dividing $\Re^n$ into the three regions S_1, S_2 and S_3 together with decomposing

$$E[d_{1,k}^\alpha] = \int_{S_1} E[d_{1,k}^\alpha | X_1 = x] p(x) dx + \int_{S_2} E[d_{1,k}^\alpha | X_1 = x] p(x) dx$$
$$+ \int_{S_3} E[d_{1,k}^\alpha | X_1 = x] p(x) dx.$$

The three sets depend on a variable $0 < \epsilon < 1$ and the number of samples M. We think $\epsilon > 0$ as a parameter, which at the end of the analysis is set to approach zero after first taking the limit $M \to \infty$. As a sidenote, it should be clear at this point that the parameters (n, k, α) are assumed to stay fixed all the time.

The motivation for S_1 might be seen in the idea of performing a Taylor expansion of $f^{-1}(\cdot)^\alpha$ at zero, which might render the analysis into the well-known case [2]. Keeping in mind that $\omega_{X_1}(d_{1,k})$ is of order of magnitude M^{-1}, we take (the definition applies for any $n \geq 1$)

$$\begin{aligned} S_1 &= \{x \in \Re^n : \ p(x) > \frac{\log^{n/2} M}{\epsilon M}\} \\ &= \{x \in \Re^n : \ \|x\| < \sqrt{2 \log M - n \log \log M + 2 \log \epsilon}\}; \end{aligned} \tag{5}$$

then for large M, $\|X_1\| = O(\sqrt{\log M})$ and

$$\frac{\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)} = O(\epsilon)$$

by substituting $\omega_{X_1}(d_{1,k}) = \frac{1}{M}$ to analyze the order of magnitude. If ϵ is small, then this shows that the argument of f^{-1} is small suggesting that a Taylor expansion might be possible. However, during the course of the proof, it turns out that points in S_1 contribute little in comparison to the set

$$S_2 = \{x \in \Re^n : \ \frac{\epsilon \log^{n/2} M}{M} \leq p(x) \leq \frac{\log^{n/2} M}{\epsilon M}\}. \tag{6}$$

In this case, a Taylor expansion does not seem possible. Fortunately, we are able to show that conditionally on $X_1 \in S_2$, the variable

$$Y = \frac{M p(X_1)}{\log^{n/2} M} \tag{7}$$

is approximately uniformly distributed on $[\epsilon, \epsilon^{-1}]$ and moreover, it is independent of $\omega_{X_1}(d_{1,k})$. This is useful, because for large M, $\|X_1\| \approx \sqrt{2 \log M}$ and we get

$$E[d_{1,k}^\alpha | X_1 \in S_2] \approx E[\frac{f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})}{Y}\right)^\alpha}{(2 \log M)^{\alpha/2}} | X_1 \in S_2]. \tag{8}$$

Because the probability $P(X_1 \in S_2)$ turns out to admit a convenient asymptotic expression, it is possible to use Equation (8) to estimate the quantity

$$\int_{S_2} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx = E[d_{1,k}^{\alpha}|X_1 \in S_2]P(X_1 \in S_2).$$

In addition to S_1 and S_2, there is the set

$$S_3 = \{x \in \Re^n : \ p(x) < \frac{\epsilon \log^{n/2} M}{M}\}. \tag{9}$$

However, similarly as S_1, nearest neighbor distances corresponding to $X_1 \in S_3$ turn out to have a neglible effect if ϵ is small.

5 Auxiliary Results

In this section, we give some results and applications for $\omega_{X_1}(d_{1,k})$, where

$$\omega_x(r) = \int_{B(x,r)} p(x)dx.$$

The following result characterizes the distribution of $\omega_{X_1}(d_{1,k})$, which conveniently does not depend on X_1 or the density $p(x)$.

Lemma 1. *Given X_1, the conditional density of $\omega_{X_1}(d_{1,k})$ is given by*

$$p(\omega|X_1) = p(\omega) = k\binom{M-1}{k}\omega^{k-1}(1-\omega)^{M-k-1}. \tag{10}$$

Moreover,

$$E[\omega_{X_1}(d_{1,k})^{\alpha}|X_1] = \frac{\Gamma(k+\alpha/n)\Gamma(M)}{\Gamma(k)\Gamma(M+\alpha/n)}. \tag{11}$$

Proof. In [2], it has been shown that $d_{1,k}$ has the conditional density

$$P(d_{1,k} \in [r_1, r_2]|X_1 = x) = k\binom{M-1}{k}\int_{[r_1,r_2]} \omega_x(r)^{k-1}(1-\omega_x(r))^{M-k-1}d\omega_x(r).$$

Here $d\omega_x(r)$ refers to the Lebesgue-Stieltjes measure, where $\omega_x(r)$ is considered a function of r. Because $\omega_x(r)$ is differentiable, we have

$$\begin{aligned} P(d_{1,k} \in &[r_1, r_2]|X_1 = x) \\ &= k\binom{M-1}{k}\int_{[r_1,r_2]} \omega_x(r)^{k-1}(1-\omega_x(r))^{M-k-1}D\omega_x(r)dr. \end{aligned} \tag{12}$$

By monotonicity of $\omega_x(r)$ we have

$$P(\omega_{X_1}(d_{1,k}) \in [\omega_{X_1}(r_1), \omega_{X_1}(r_2)]|X_1 = x) = P(d_{1,k} \in [r_1, r_2]|X_1 = x).$$

Using the change of variables $y = \omega_x(r)$ in (12) now yields

$$\begin{aligned} P(\omega_{X_1}(d_{1,k}) &\in [\omega_{X_1}(r_1), \omega_{X_1}(r_2)] | X_1 = x) \\ &= k \binom{M-1}{k} \int_{[\omega_x(r_1), \omega_x(r_2)]} y^{k-1}(1-y)^{M-k-1} dy, \end{aligned}$$

which is sufficient to verify (10). The moments are computed using the formula for Beta functions

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad (a, b > 0)$$

together with

$$k \binom{M-1}{k} = \frac{\Gamma(M)}{\Gamma(k)\Gamma(M-k)}.$$

□

It is useful to observe that for any $\beta > 0$,

$$\frac{\Gamma(M+\beta)}{\Gamma(M)} = M^\beta + O(M^{\beta-1}) \tag{13}$$

to understand better the moments (11). The following is useful for technical reasons:

Lemma 2. *Assume that Equation (2) holds. Then for* $0 < r < 1$ *and* $x \in \Re^n$,

$$\omega_x(r) \geq c p(x) r^n$$

for some constant $c(n) > 0$.

Proof. We compute straightforwardly:

$$\begin{aligned} \omega_x(r) &= (2\pi)^{-n/2} \int_{B(x,r)} e^{-\frac{1}{2}\|y\|^2} dy \\ &= (2\pi)^{-n/2} e^{-\frac{1}{2}\|x\|^2} \int_{B(x,r)} e^{-x^T(y-x) - \frac{1}{2}\|y-x\|^2} dy \\ &\geq p(x) e^{-\frac{1}{2}r^2} \int_{B(x,r)} e^{-x^T(y-x)} dy \geq \frac{e^{-\frac{1}{2}}}{2} V_n p(x) r^n. \end{aligned}$$

□

The moments $E[d_{1,k}^\alpha | X_1]$ do not get too large if $\|X_1\|$ does not get too large:

Lemma 3. *Assume that (2) holds. Then for* $x \in \Re^n$, $M > 2k$ *and* $\alpha > 0$

$$E[d_{1,k}^\alpha | X_1 = x] \leq c(\|x\|^\alpha + 1)$$

for some constant $c(n, k, \alpha)$.

Proof. If $(X_i)_{i=2}^M$ is partitioned into k parts and from each we take the smallest distance to X_1, then $d_{1,k}^\alpha$ is smaller than the maximum of these distances. Consequently, it is also smaller than the sum of the distances and by the i.i.d. assumption, for any $x \in \Re^n$

$$\begin{aligned} E[d_{1,k}^\alpha | X_1 = x] &\leq kE[\min_{2 \leq i < (M-1)/k} \|X_i - x\|^\alpha] \\ &\leq kE[\|X_2 - x\|^\alpha] \leq cE[\|X_2\|^\alpha + \|x\|^\alpha] \end{aligned}$$

for some constant $c(k, \alpha)$. Observing that the α moments of X_2 are finite completes the proof. □

Next we show that the α-moments are at most of order $(p(x)M)^{-\alpha/n}$ if the quantity inside the parentheses does not get too small. The result is an application of Lemmas 1-2.

Lemma 4. *Assume that Equation (2) holds and fix any $\delta > 0$. Then if $p(x) > \frac{\delta \log^{n/2} M}{M}$, we find a threshold $M_0(n, k, \alpha, \delta)$ such that for all $M > M_0$, we have almost surely,*

$$E[d_{1,k}^\alpha | X_1 = x] \leq c(p(x)M)^{-\alpha/n}$$

for some constant $c(n, k, \alpha)$.

Proof. We decompose

$$E[d_{1,k}^\alpha | X_1 = x] = E[d_{1,k}^\alpha I(d_{1,k} \leq 1) | X_1 = x] + E[d_{1,k}^\alpha I(d_{1,k} > 1) | X_1 = x]. \quad (14)$$

We consider next the first term in the right side. By Lemma 2,

$$\frac{d_{1,k}^n}{\omega_{X_1}(d_{1,k})} I(d_{1,k} \leq 1) \leq \frac{c_1}{p(X_1)} \quad (15)$$

(for some constant $c_1(n)$) and using this we have by Lemma 1 together with Equations (13) and (15),

$$\begin{aligned} &E[d_{1,k}^\alpha I(d_{1,k} \leq 1) | X_1 = x] \\ &\quad = E[\left(\frac{d_{1,k}^n}{\omega_{X_1}(d_{1,k})}\right)^{\alpha/n} \omega_{X_1}(d_{1,k})^{\alpha/n} I(d_{1,k} \leq 1) | X_1 = x] \\ &\quad \leq c_1^{\alpha/n} p(x)^{-\alpha/n} E[\omega_{X_1}(d_{1,k})^{\alpha/n} | X_1 = x] \\ &\quad \leq c_2 (p(x)M)^{-\alpha/n} \end{aligned} \quad (16)$$

for some constant $c_2(n, k, \alpha)$. We have proven the claim for the first term in (14). For the second term, we apply Hölder's inequality:

$$E[d_{1,k}^\alpha I(d_{1,k} > 1) | X_1 = x] \leq \sqrt{P(d_{1,k} > 1 | X_1 = x)} \sqrt{E[d_{1,k}^{2\alpha} | X_1 = x]}. \quad (17)$$

$\omega_x(r)$ is a strictly increasing function with respect to r. Using this fact and the inequalities $k\binom{M-1}{k} \leq M^k$ and $1-\omega \leq e^{-\omega}$ together with Lemma 1, we have

$$\begin{aligned} P(d_{1,k} > 1|X_1 = x) &= P(\omega_{X_1}(d_{1,k}) > \omega_{X_1}(1)|X_1 = x) \\ &= k\binom{M-1}{k}\int_{\omega_x(1)}^1 \omega^{k-1}(1-\omega)^{M-k-1}d\omega \\ &\leq M^k \int_{\omega_x(1)}^1 \omega^{k-1}e^{-(M-k-1)\omega}d\omega \\ &= \left(\frac{M}{M-k-1}\right)^k \int_{(M-k-1)\omega_x(1)}^{M-k-1} \omega^{k-1}e^{-\omega}d\omega \\ &= \left(\frac{M}{M-k-1}\right)^k \int_{(M-k-1)\omega_x(1)}^{M-k-1} e^{-\omega+(k-1)\log\omega}d\omega. \quad (18) \end{aligned}$$

Now

$$\omega_x(1) \geq c_1 p(x) \quad (19)$$

by Equation (15) and if $k > 1$,

$$\log\omega \leq \frac{1}{2(k-1)}\omega + \log(1+2(k-1)). \quad (20)$$

The previous equation can be proven by moving the terms in the right side to the left and finding the zero point of the first derivative. The derivation is not very relevant the main point being the slow increase of the logarithm in comparison to the term ω. Using the two facts (19) and (20), we have

$$\begin{aligned} &P(d_{1,k} > 1|X_1 = x) \\ &\quad\leq \left(\frac{M}{M-k-1}\right)^k \int_{c_1p(x)(M-k-1)}^{M} e^{-\frac{1}{2}\omega+(k-1)\log(1+2(k-1))}d\omega \\ &\quad= 2(1+2(k-1))^{k-1}\left(\frac{M}{M-k-1}\right)^k (e^{-\frac{1}{2}c_1p(x)(M-k-1)} - e^{-\frac{1}{2}M}) \\ &\quad\leq c_3e^{-\frac{1}{4}c_1p(x)M} \quad (21) \end{aligned}$$

for some $c_3(n,k)$ assuming that $M > 2k+2$. By the assumptions of the lemma $p(x) > \frac{\delta\log^{n/2}M}{M}$, which implies

$$\|x\| \leq \sqrt{2\log M - n\log\log M - 2\log\delta - n\log(2\pi)} \leq \sqrt{3\log M}$$

after some threshold $M_0(n,\delta)$ and $M > M_0$. By Lemma 3 we then have

$$E[d_{1,k}^{2\alpha}|X_1 = x] \leq c_4\log^{\alpha} M \quad (22)$$

for some constant $c_4(n,k,\alpha)$ (assuming trivially $M > 1$). Equations (21) and (22) together with (17) now imply

$$E[d_{1,k}^{\alpha}I(d_{1,k} > 1)|X_1 = x] \leq \sqrt{c_3c_4}e^{-\frac{1}{8}c_1p(x)M}\log^{\alpha/2}M. \quad (23)$$

The assumption $p(x)M \geq \delta \log^{n/2} M$ implies that for any $j > 0$,

$$e^{-\frac{1}{8}c_1 p(x)M} \leq \frac{8^j j!}{c_1^j \delta^j (p(x)M)^j}$$

showing that in the limit $M \to \infty$, (23) approaches zero faster than $(p(x)M)^{-\alpha/n}$ in Equation (16). $\square$

We formalize the argument in Section 4, which connects $\omega_x(r)$ to the function f:

Lemma 5. *Assume that Equation (2) holds. Then*

$$\|x\|^n \omega_x(r) = p(x) f(\|x\| r) - R$$

with

$$f(t) = t^n \int_{B(0,1)} e^{t y^{(1)}} dy$$

and

$$0 \leq R \leq p(x) r^2 f(\|x\| r).$$

f is defined and continuous on $[0, \infty)$ and it has the range $[0, \infty)$. It is also strictly increasing implying the existence of an inverse function $f^{-1} : [0, \infty) \mapsto [0, \infty)$.

Proof. The proof involves extracting the error term and bounding it.

$$\begin{aligned}
\|x\|^n \omega_x(r) &= (2\pi)^{-n/2} \|x\|^n \int_{B(x,r)} e^{-\frac{1}{2}\|y\|^2} dy \\
&= (2\pi)^{-n/2} (\|x\| r)^n \int_{B(0,1)} e^{-\frac{1}{2}\|ry - x\|^2} dy \\
&= (2\pi)^{-n/2} (\|x\| r)^n e^{-\frac{1}{2}\|x\|^2} \int_{B(0,1)} e^{r x^T y} dy - A \\
&= p(x) (\|x\| r)^n \int_{B(0,1)} e^{r x^T y} dy - A \qquad (24)
\end{aligned}$$

with

$$A = p(x)(\|x\| r)^n \int_{B(0,1)} e^{r x^T y} (1 - e^{-\frac{1}{2} r^2 \|y\|^2}) dy. \qquad (25)$$

The main task is to bound A. This is achieved by the mean-value theorem: for $\|y\| \leq 1$ and $r > 0$,

$$1 - e^{-\frac{1}{2} r^2 \|y\|^2} = \frac{1}{2} r^2 \|y\|^2 e^{-\delta} \leq r^2$$

for some $\delta \in [0, \infty]$. This inequality implies that

$$\begin{aligned}
0 \leq A &\leq p(x)(\|x\| r)^n r^2 \int_{B(0,1)} e^{r x^T y} dy \\
&\leq p(x)(\|x\| r)^n r^2 \int_{B(0,1)} e^{r \|x\| y^{(1)}} dy = p(x) r^2 f(\|x\| r).
\end{aligned}$$

In the last inequality, the vectors have been conveniently rotated. The same rotation shows that in (24), we have

$$p(x)(\|x\|r)^n \int_{B(0,1)} e^{rx^T y} dy = p(x) f(\|x\|r).$$

□

For $t > 0$, we define

$$g(t) = \int_0^\infty \omega^{k-1} e^{-\omega} f^{-1}(\omega t)^\alpha d\omega. \tag{26}$$

We show that g approaches zero at least as fast as $t^{\alpha/n}$ and grows at most logarithmically if $t \to \infty$. The same holds for $f^{-1}(t)^\alpha$:

Lemma 6. *The function* (26) *satisfies*

$$0 \leq g(t) + f^{-1}(t)^\alpha \leq ct^{\alpha/n}$$

on $(0, 1]$ *for some constant* $c(n, k, \alpha)$. *On* $(1, \infty)$ *we have*

$$0 \leq g(t) + f^{-1}(t)^\alpha \leq c(1 + \log^\alpha t).$$

Proof. 1. Bounds on f^{-1}

Consider $t \in (0, 1)$. For any

$$z > \left(\frac{2t}{V_n}\right)^{1/n},$$

we have

$$f(z) > \frac{2t \int_{B(0,1)} e^{zy^{(1)}} dy}{V_n} > t.$$

This implies that

$$f^{-1}(t) \leq \left(\frac{2t}{V_n}\right)^{1/n}.$$

Next assume that $t > 1$. Take $z > 2\log t + A + 1$ with

$$A = \lambda(B(0, 1) \cap \{x \in \Re^n : \ x^{(1)} > \frac{1}{2}\})^{-1}.$$

Then

$$\begin{aligned} f(z) &> A \int_{B(0,1)\cap\{x:\ x^{(1)} > \frac{1}{2}\}} e^{2y^{(1)} \log t} dy \\ &> A \int_{B(0,1)\cap\{x:\ x^{(1)} > \frac{1}{2}\}} e^{\log t} dy = t. \end{aligned}$$

This means that

$$f^{-1}(t) \leq 2\log t + A + 1.$$

The outcome for $f^{-1}(t)^\alpha$ follows by recalling that $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$ for any $a, b > 0$.

2. The function g

We proceed to bounds on the function g. We take $t \in (0,1)$. Then using the results for f^{-1} yield

$$\begin{aligned}
g(t) &= \int_0^{1/t} \omega^{k-1} e^{-\omega} f^{-1}(\omega t)^\alpha d\omega + \int_{1/t}^{\infty} \omega^{k-1} e^{-\omega} f^{-1}(\omega t)^\alpha d\omega \\
&\leq c_1 t^{\alpha/n} \int_0^{1/t} \omega^{k-1+\alpha/n} e^{-\omega} d\omega + c_1 \int_{1/t}^{\infty} \omega^{k-1} e^{-\omega} \log^\alpha((2+\omega)(2+t)) d\omega \\
&\leq c_2 t^{\alpha/n} \int_0^{1/t} \omega^{k-1+\alpha/n} e^{-\omega} d\omega + c_2 \log^\alpha(2+t) \int_{1/t}^{\infty} \omega^{k-1} e^{-\omega} d\omega \\
&\quad + c_2 \int_{1/t}^{\infty} \omega^{k-1} e^{-\omega} \log^\alpha(2+\omega) d\omega \\
&= I_1 + I_2 + I_3 \qquad (27)
\end{aligned}$$

for some constants $c_1(n,\alpha)$ and $c_2(n,\alpha)$. The shorthand notation I_i $(i = 1,2,3)$ was adopted for the three terms. The argument $(2+t)(2+\omega)$ for the logarithm was chosen in order to ensure that the upper bound can be assumed to hold also for $t > 1$.

Now

$$I_1 \leq c_2 t^{\alpha/n} \int_0^{\infty} \omega^{k-1+\alpha/n} e^{-\omega} d\omega. \qquad (28)$$

Also, for example by partial integration (the point being the fast decrease of $e^{-\omega}$),

$$I_2 \leq k! c_2 e^{-1/t} t^{-k} \log^\alpha(2+t) \leq c_3 t^{\alpha/n} \qquad (29)$$

for some constant $c_3(n,k,\alpha)$. Of course, the last inequality is not tight, because $e^{-1/t}$ approaches zero very fast in the limit $t \to 0$, but nevertheless it fits our purpose. Similarly, using $\log(2+\omega) \leq \omega$ for $\omega \geq 1$ gives

$$|I_3| \leq c_2 \int_{1/t}^{\infty} \omega^{k+\alpha-1} e^{-\omega} d\omega \leq c_4 t^{1/n} \qquad (30)$$

(for some $c_4(n,k,\alpha)$) by the same proof as for I_2. In summary, Equations (27)-(30) show that for $0 < t < 1$,

$$g(t) \leq c_5 t^{\alpha/n}$$

for some constant $c_5(n,k,\alpha)$. There is still the case $t > 1$. We again use the decomposition (27):

$$
\begin{aligned}
I_1 &\le c_2 t^{1-k} \int_0^{1/t} d\omega = t^{-k} \\
I_2 &\le c_2 \log^\alpha(2+t) \int_0^\infty \omega^{k-1} e^{-\omega} d\omega \\
I_3 &\le c_2 \int_0^\infty \omega^{k-1} e^{-\omega} \log^\alpha(2+\omega) d\omega
\end{aligned}
$$

The only term that grows with respect to t is I_2, which grows proportionally to $\log^\alpha(2+t)$; in the final claim, we use t instead of $2+t$. □

6 Region S_1

Recall that region S_1 is defined by

$$
\begin{aligned}
S_1 &= \{x \in \Re^n : \ p(x) > \frac{\log^{n/2} M}{\epsilon M}\} \\
&= \{x \in \Re^n : \ \|x\| < \sqrt{2\log M - n \log\log M + 2\log\epsilon - n\log(2\pi)}\}.
\end{aligned} \tag{31}
$$

It may happen that S_1 is an empty set; from now on we always assume that M is large enough in comparison to ϵ^{-1} and n in order to ensure that S_1 is non-empty with a positive volume. Similar convention is adopted for the sets S_2 and S_3.

As stated in Section 4, $0 < \epsilon < 1$ is a fixed constant until the end, where the limit $\epsilon \to 0$ is taken after the limit $M \to \infty$. We define (assuming that $\alpha > n$)

$$
i^* = [\log^{-1} 2 \frac{n \log\log M}{\alpha - n}] + 1.
$$

$[\cdot]$ refers to the integer part of the number inside the bracket. As our proof strategy, S_1 is divided into smaller subsets, which are easier to control with the tools we have available this far:

$$
\begin{aligned}
\tilde{S}_{1,i} &= \{x \in \Re^n : \ 2^i \frac{\log^{n/2} M}{\epsilon M} \le p(x) < 2^{i+1} \frac{\log^{n/2} M}{\epsilon M}\} \\
&= \{x \in \Re^n : \ \|x\| \in [a_i, b_i)\}
\end{aligned} \tag{32}
$$

$(0 \le i \le i^*)$ with

$$
\begin{aligned}
a_i &= \sqrt{2\log M - n\log\log M - 2(i+1)\log 2 + 2\log\epsilon - n\log(2\pi)} \\
b_i &= \sqrt{2\log M - n\log\log M - 2i\log 2 + 2\log\epsilon - n\log(2\pi)}.
\end{aligned}
$$

The remaining part is denoted by

$$S_{1,C} = S_1 \setminus \cup_{i=0}^{i^*} \tilde{S}_{1,i}.$$

The following bounds the nearest neighbor distance when $X_1 \in \tilde{S}_{1,i}$. Without losing generality, we prove the claim after some threshold M_0, which is natural as in any case later the limit $M \to \infty$ is taken. As a somewhat subtle detail, we will generally adopt this way of expressing our statements in those cases, where proving the claim for all $M > 0$ is not an obvious task.

Lemma 7. *Assume that* (2) *holds and* $\alpha > n$. *Then there exists a threshold* $M_0(n, k, \alpha, \epsilon) > 0$ *such that for* $0 \leq i \leq i^*$ *and* $M > M_0$,

$$\int_{\tilde{S}_{1,i}} E[d_{1,k}^{\alpha} | X_1 = x] p(x) dx \leq 2^{i(1-\alpha/n)} c \epsilon^{\alpha/n-1} \frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n, k, \alpha)$.

Proof. By Lemma 4,

$$\begin{aligned}
\int_{\tilde{S}_{1,i}} E[d_{1,k}^{\alpha} | X_1 = x] p(x) dx &\leq c_1 M^{-\alpha/n} \int_{\tilde{S}_{1,i}} p(x)^{1-\alpha/n} dx \\
&\leq 2^{i(1-\alpha/n)} c_1 M^{-\alpha/n} \left(\frac{\log^{n/2} M}{\epsilon M} \right)^{1-\alpha/n} \lambda(\tilde{S}_{1,i}) \\
&\leq 2^{i(1-\alpha/n)} c_1 \epsilon^{\alpha/n-1} \frac{\log^{n/2-\alpha/2} M}{M} \lambda(\tilde{S}_{1,i}) \qquad (33)
\end{aligned}$$

for some constant $c_1(n, k, \alpha)$ and $M_0(n, k, \alpha, \epsilon)$. We should now compute the volume $\lambda(\tilde{S}_{1,i})$. The set $\tilde{S}_{1,i}$ consists of points $x \in \Re^n$ with $\|x\|$ in the interval $[a_i, b_i)$. Then $\lambda(\tilde{S}_{1,i}) = V_n(b_i^n - a_i^n)$. By a Taylor expansion,

$$\begin{aligned}
a_i^n = 2^{n/2} \log^{n/2} M &\left(1 - \frac{n^2 \log\log M + 2n(i+1)\log 2 - 2n \log \epsilon + n^2 \log(2\pi)}{4 \log M} \right) \\
&+ R \qquad (34)
\end{aligned}$$

in the limit $M \to \infty$ with everything else fixed and

$$|R| \leq c_2 \frac{\log^2 \log M}{\log^{2-n/2} M}$$

with $c_2(n, k, \alpha, \epsilon)$ independent of i. Similar approximation holds for b_i^n. Using the expansion,

$$\begin{aligned}
\lambda(\tilde{S}_{1,i}) &= V_n(b_i^n - a_i^n) \\
&= (2^{n/2-1} \log 2) n V_n \log^{n/2-1} M + O\left(\frac{\log^2 \log M}{\log^{2-n/2} M} \right). \qquad (35)
\end{aligned}$$

By substitution of (35) into (33), we have

$$\int_{\tilde{S}_{1,i}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \leq 2^{i(1-\alpha/n)+n/2-1} c_1 n V_n \epsilon^{\alpha/n-1} \frac{\log^{n-\alpha/2-1} M}{M} + c_2 \frac{\log^{n-\alpha/2-2} M}{M}$$

for some constants $c_3(n, k, \alpha, \epsilon)$. Of the two terms in the right side, the latter converges to zero faster with respect to M and consequently becomes smaller after some threshold $M_0(n, k, \alpha, \epsilon)$. □

After removing the sets $\tilde{S}_{1,i}$, we are left with $\tilde{S}_{1,C}$. However, it does not pose problems.

Lemma 8. *Assume that* (2) *holds and* $\alpha > n$. *Then there exists a threshold* $M_0(n, k, \alpha, \epsilon)$ *such that for any* $M > M_0$, *we have*

$$\int_{\tilde{S}_{1,C}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \leq c\epsilon^{\alpha/n-1} \frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n, k, \alpha)$.

Proof. By Lemma 4 and the definition of $\tilde{S}_{1,C}$,

$$\begin{aligned}\int_{\tilde{S}_{1,C}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx &\leq c_1 M^{-\alpha/n} \int_{\tilde{S}_{1,C}} p(x)^{1-\alpha/n} dx \\ &\leq 2^{i^*(1-\alpha/n)} c_1 \epsilon^{\alpha/n-1} \frac{\log^{n/2-\alpha/2} M}{M} \lambda(\tilde{S}_{1,C}) \quad (36)\end{aligned}$$

for some constant $c_1(n, k, \alpha)$. $\tilde{S}_{1,C}$ consists of points $x \in \Re^n$ with

$$\|x\| \leq \sqrt{2\log M - n\log\log M - 2i^* \log 2 + 2\log\epsilon - n\log(2\pi)} \leq \sqrt{3\log M}$$

once M exceeds some threshold depending on n, k, α and ϵ. This implies that

$$\lambda(\tilde{S}_{1,C}) \leq 3^{n/2} V_n \log^{n/2} M. \quad (37)$$

Also, we compute

$$2^{i^*(1-\alpha/n)} = e^{i^*(1-\alpha/n)\log 2} \leq \log^{-1} M. \quad (38)$$

Substituting Equations (37) and (38) into (36) yields

$$\int_{\tilde{S}_{1,C}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \leq 3^{n/2} c_1 V_n \epsilon^{\alpha/n-1} \frac{\log^{n-\alpha/2-1} M}{M}.$$

□

Lemmas 7 and 8 imply that for $\alpha > n$,

$$\begin{aligned}\int_{S_1} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx &= \int_{\tilde{S}_{1,C}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \\ &+ \sum_{i=0}^{i^*} \int_{\tilde{S}_{1,i}} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \\ &\leq c\epsilon^{\alpha/n-1}\frac{\log^{n-\alpha/2-1} M}{M} \\ &+ \sum_{i=0}^{i^*} 2^{i(1-\alpha/n)} c\epsilon^{\alpha/n-1}\frac{\log^{n-\alpha/2-1} M}{M} \\ &\leq c\epsilon^{\alpha/n-1}\frac{\log^{n-\alpha/2-1} M}{M}(1 + \sum_{i=0}^{\infty} 2^{i(1-\alpha/n)}). \quad (39)\end{aligned}$$

for some constant $c(n, k, \alpha)$ and $M > M_0$. We conclude

Lemma 9. *Assume that* (2) *holds and* $\alpha > n$. *Then there exists a threshold* $M_0(n, k, \alpha, \epsilon)$ *such that for any* $M > M_0$, *we have*

$$\int_{S_1} E[d_{1,k}^{\alpha}|X_1 = x]p(x)dx \leq c\epsilon^{\alpha/n-1}\frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n, k, \alpha)$.

7 Region S_2

Region 2 is defined by

$$S_2 = \{x \in \Re^n : \frac{\epsilon \log^{n/2} M}{M} \leq p(x) \leq \frac{\log^{n/2} M}{\epsilon M}\}. \quad (40)$$

As mentioned earlier, M is assumed to be large enough to ensure that S_2 has a positive volume. It is necessary to obtain an approximation to $P(X_1 \in S_2)$. This can be done rather straightforwardly:

Lemma 10. *Assuming* (2), *it holds that*

$$P(X_1 \in S_2) = \frac{2^{n/2-1} n V_n \log^{n-1} M}{\epsilon M}(1 - \epsilon^2) + R$$

with

$$|R| \leq c\frac{\log^2 \log M \log^{n-2} M}{M}$$

for some constant $c(n, \epsilon)$.

Proof. S_2 consists of points x with

$$\begin{aligned}
&\|x\| \in [a,b] \\
&a = \sqrt{2\log M - n\log\log M + 2\log\epsilon - n\log(2\pi)} \\
&b = \sqrt{2\log M - n\log\log M - 2\log\epsilon - n\log(2\pi)}.
\end{aligned}$$

We compute

$$\begin{aligned}
P(X_1 \in S_2) &= (2\pi)^{-n/2}\int_{S_2} p(x)dx = (2\pi)^{-n/2} nV_n \int_a^b x^{n-1}e^{-\frac{1}{2}x^2}dx \\
&= (2\pi)^{-n/2} nV_n \int_a^b x^{n-1}e^{-\frac{1}{2}a^2-(x-a)a-\frac{1}{2}(x-a)^2}dx \\
&= nV_n p(a)a^{n-1}\int_a^b e^{-(x-a)a}dx \\
&\quad + nV_n p(a)\int_a^b (x^{n-1}-a^{n-1})e^{-(x-a)a-\frac{1}{2}(x-a)^2}dx \\
&\quad + nV_n p(a)a^{n-1}\int_a^b (e^{-\frac{1}{2}(x-a)^2}-1)e^{-(x-a)a}dx \\
&= I_1 + I_2 + I_3
\end{aligned}$$

with

$$\begin{aligned}
I_1 &= \frac{nV_n \log^{n/2} M}{\epsilon M} a^{n-1}\int_a^b e^{-(x-a)a}dx \\
&= \frac{nV_n \log^{n/2} M}{\epsilon M} a^{n-2}(1-e^{-(b-a)a}) \\
I_2 &= \frac{nV_n \log^{n/2} M}{\epsilon M}\int_a^b (x^{n-1}-a^{n-1})e^{-(x-a)a-\frac{1}{2}(x-a)^2}dx \\
I_3 &= \frac{nV_n \log^{n/2} M}{\epsilon M} a^{n-1}\int_a^b (e^{-\frac{1}{2}(x-a)^2}-1)e^{-(x-a)a}dx.
\end{aligned}$$

During the proof it is easiest to employ the Big-Oh notation. Such error terms depend here on n and ϵ.

1. *The term* I_1

By a Taylor expansion, in analog to Equation (34),

$$a = \sqrt{2\log M} - \frac{n\log\log M}{2\sqrt{2\log M}} + \frac{\log\epsilon}{\sqrt{2\log M}} - \frac{n\log(2\pi)}{2\sqrt{2\log M}} + O\left(\frac{\log^2\log M}{\log^{3/2} M}\right) \tag{41}$$

$$b = \sqrt{2\log M} - \frac{n\log\log M}{2\sqrt{2\log M}} - \frac{\log\epsilon}{\sqrt{2\log M}} - \frac{n\log(2\pi)}{2\sqrt{2\log M}} + O\left(\frac{\log^2\log M}{\log^{3/2} M}\right). \tag{42}$$

By (41)-(42),

$$b - a = \frac{\sqrt{2}\log \epsilon^{-1}}{\sqrt{\log M}} + O(\frac{\log^2 \log M}{\log^{3/2} M}) \tag{43}$$

and

$$\begin{aligned} 1 - e^{-(b-a)a} &= e^{[\frac{\sqrt{2}\log \epsilon}{\sqrt{\log M}} + O(\frac{\log^2 \log M}{\log^{3/2} M})][\sqrt{2\log M} + O(\frac{\log\log M}{\sqrt{\log M}})]} - 1 \\ &= 1 - e^{2\log\epsilon + O(\frac{\log^2 \log M}{\log M})} = 1 - \epsilon^2 + O\left(\frac{\log^2 \log M}{\log M}\right). \end{aligned} \tag{44}$$

Also,

$$a^{n-2} = 2^{n/2-1}\log^{n/2-1} M + O\left(\frac{\log\log M}{\log^{2-n/2} M}\right). \tag{45}$$

Using Equations (44) and (45) in the expression for I_1 yields

$$\begin{aligned} I_1 &= \frac{nV_n \log^{n/2} M}{\epsilon M}\left[2^{n/2-1}\log^{n/2-1} M + O\left(\frac{\log\log M}{\log^{2-n/2} M}\right)\right][1 - \epsilon^2 \\ &+ O(\frac{\log^2 \log M}{\log M})] \\ &= \frac{2^{n/2-1} nV_n \log^{n-1} M}{\epsilon M}(1-\epsilon^2) + O\left(\frac{\log^2 \log M \log^{n-2} M}{M}\right). \end{aligned}$$

2. *The term I_2*

By the mean-value theorem,

$$|x^{n-1} - a^{n-1}| \leq |b^{n-1} - a^{n-1}| \leq c_1 \log^{n/2-3/2} M$$

for some constant $c_1(n,\epsilon)$. Also, $a^{-1} \leq c_2 \log^{-1/2} M$ for some $c_2(n,\epsilon)$. We have

$$\begin{aligned} I_2 &\leq \frac{c_1 nV_n \log^{n-3/2} M}{\epsilon M}\int_a^b e^{-(x-a)a}dx \\ &\leq \frac{c_1 c_2 nV_n \log^{n-2} M}{\epsilon M}. \end{aligned}$$

3. *The term I_3*

Now

$$|I_3| \leq \frac{nV_n \log^{n/2} M}{\epsilon M} a^{n-1}(1 - e^{-\frac{1}{2}(b-a)^2})\int_a^b e^{-(x-a)a}dx. \tag{46}$$

Again,

$$\int_a^b e^{-(x-a)a}dx = a^{-1}(1 - e^{-(b-a)a}) \leq a^{-1}. \tag{47}$$

Moreover, by the expansion for $b-a$ appearing in Equation (43),

$$1 - e^{-\frac{1}{2}(b-a)^2} = -\frac{1}{2}(b-a)^2 + O((b-a)^4) \le \frac{c_3}{\log M} \tag{48}$$

for some constant $c_3(n,\epsilon)$ Finally,

$$a^{n-2} \le c_4 \log^{n/2-1} M. \tag{49}$$

for some constant $c_4(n,\epsilon)$. Substituting (47)-(49) into (46) yields

$$|I_3| \le \frac{c_3 c_4 n V_n \log^{n-2} M}{\epsilon M}.$$

The proof is finished since the terms I_1,I_2 and I_3 have been addressed. □

In general, to establish asymptotics, it is useful to truncate $d_{1,k}$ to avoid too large values. To this end, we choose some $L > 0$ (recall that at this point, α, n, k and ϵ stay fixed) and define

$$I_L = I(d_{1,k} < \frac{L}{\epsilon^{1/n}\sqrt{\log M}}).$$

The power for $\log M$ is carefully chosen to ensure the correct order of magnitude with large L rendering the event $1 - I_L$ neglible. The following lemma verifies this fact; the bound is designed to hold after some threshold M_0, which depends on L itself. However, after the threshold we get an upper bound which goes exponentially to zero with respect to L.

Lemma 11. *Under* (2) *and for any* $L > 0$*, there exists a threshold* $M_0(n,k,\alpha,\epsilon,L)$ *such that for all* $M > M_0$*, it holds that*

$$E[d_{1,k}^{\alpha}(1-I_L)|X_1 \in S_2] \le c(n,k,\alpha,\epsilon) \log^{-\alpha/2} M e^{-c(n,k,\alpha,\epsilon)^{-1} L^n}$$

for some positive constant $c(n,k,\alpha,\epsilon)$.

Proof. The proof employs Hölder's inequality:

$$\begin{aligned} &E[d_{1,k}^{\alpha}(1-I_L)|X_1 \in S_2] \\ &\quad \le \sqrt{E[d_{1,k}^{2\alpha}|X_1 \in S_2]}\sqrt{P(d_{1,k} > \frac{L}{\epsilon^{1/n}\sqrt{\log M}}|X_1 \in S_2)}. \end{aligned} \tag{50}$$

By Lemma 4 and the definition of S_2, there exists $M_0(n,k,\alpha,\epsilon)$ such that

$$\begin{aligned} E[d_{1,k}^{2\alpha}|X_1 \in S_2] &= E[E[d_{1,k}^{2\alpha}|X_1]|X_1 \in S_2] \\ &\le c_1 E[(p(X_1)M)^{-2\alpha/n}|X_1 \in S_2] \\ &\le c_1 \epsilon^{-2\alpha/n} \log^{-\alpha} M \end{aligned} \tag{51}$$

for some constant $c_1(n,k,\alpha)$ and all $M > M_0$. We want to bound $P(d_{1,k} > L\epsilon^{-1/n}\log^{-1/2} M | X_1 \in S_2)$ in order to finish the proof. By Lemma 2, we have for $0 < r < 1$ and $x \in S_2$,

$$\omega_x(r) \geq c_2 p(x) r^n \geq \frac{c_2 \epsilon r^n \log^{n/2} M}{M} \tag{52}$$

for some constant $c_2(n)$. Then because $\omega_x(r)$ is strictly increasing with respect to r, using Lemma 1 we have

$$\begin{aligned}
&P(d_{1,k} > \frac{L}{\epsilon^{1/n}\sqrt{\log M}} | X_1 \in S_2) \\
&\quad = P(\omega_{X_1}(d_{1,k}) > \omega_{X_1}\left(\frac{L}{\epsilon^{1/n}\sqrt{\log M}}\right) | X_1 \in S_2) \\
&\quad \leq P(\omega_{X_1}(d_{1,k}) > \frac{c_2 L^n}{M} | X_1 \in S_2) \\
&\quad = k \binom{M-1}{k} \int_{c_2 L^n M^{-1}}^{1} \omega^{k-1}(1-\omega)^{M-k-1} d\omega
\end{aligned}$$

with $c_2 L^n M^{-1} < 1$ (which can be imposed by taking a sufficiently large threshold M_0). We use

$$\binom{M-1}{k} \leq \frac{M^k}{k!}$$

and

$$(1-\omega) \leq e^{-\omega}$$

to obtain for $M > c_2 L^n + 4k$,

$$\begin{aligned}
&k \binom{M-1}{k} \int_{c_2 L^n M^{-1}}^{1} \omega^{k-1}(1-\omega)^{M-k-1} d\omega \\
&\quad \leq M^k \int_{c_2 L^n M^{-1}}^{1} \omega^{k-1} e^{-(M-k-1)\omega} d\omega \\
&\quad \leq M^k \int_{c_2 L^n M^{-1}}^{1} \omega^{k-1} e^{-\frac{1}{2}M\omega} d\omega.
\end{aligned}$$

The last integral can be solved by partial integration or alternatively, we approximate

$$\begin{aligned}
\int_{c_2 L^n M^{-1}}^{1} \omega^{k-1} e^{-\frac{1}{2}M\omega} d\omega &\leq \sum_{i=0}^{\infty} \int_{2^i c_2 L^n M^{-1}}^{2^{i+1} c_2 L^n M^{-1}} \omega^{k-1} e^{-\frac{1}{2}M\omega} d\omega \\
&\leq \sum_{i=0}^{\infty} 2^{(i+1)k} c_2^k L^{nk} M^{-k} e^{-2^{i-1} c_2 L^n}.
\end{aligned}$$

Furthermore,

$$\sum_{i=0}^{\infty} 2^{(i+1)k} e^{-2^{i-1} c_2 L^n} \leq \sum_{i=0}^{\infty} e^{-2^{i-1} c_2 L^n + (i+1)k \log 2}$$
$$\leq \sum_{i=0}^{\infty} e^{-2^{i-2} c_2 L^n} \leq e^{-c_3 L^n}$$

for some constant $c_3(n, k, \epsilon)$ assuming without losing generality that $c_2 L^n \geq 4k \log 2$ (using $i + 1 \leq 2^i$). We conclude that

$$P(d_{1,k} > \frac{L}{\epsilon^{1/n} \sqrt{\log M}} | X_1 = x) \leq c_2^k L^{nk} e^{-c_3 L^n}. \tag{53}$$

In light of (50), (51) and (53) we have arrived to the conclusion

$$E[d_{1,k}^{\alpha}(1 - I_L) | X_1 \in S_2] \leq c_4 L^{nk/2} e^{-c_4^{-1} L^n} \log^{-\alpha/2} M$$

for some constant $c_4(n, k, \alpha, \epsilon)$. The term $L^{nk/2}$ can be dropped, as it is neglible compared to the exponential decay with respect to L. □

The variable Y emerged in Equation (7). It was defined by

$$Y = \frac{M p(x)}{\log^{n/2} M}. \tag{54}$$

A major idea behind our proofs is the asymptotic uniformity of Y as shown by

Lemma 12. *Suppose that* (2) *holds. Let* $h(y)$ *be a measurable function* $[\epsilon, \epsilon^{-1}] \mapsto [0, 1]$. *Then*

$$E[h(Y) | X_1 \in S_2] \to \frac{\epsilon}{1 - \epsilon^2} \int_{\epsilon}^{\epsilon^{-1}} h(y) dy.$$

in the limit $M \to \infty$.

Proof. The function

$$s(y) = \frac{M e^{-\frac{1}{2} y^2}}{(2\pi)^{n/2} \log^{n/2} M} \tag{55}$$

is strictly decreasing on $y \in [a, b]$ with a and b defined in Equations (41) and (42). It has the inverse $s^{-1} : [\epsilon, \epsilon^{-1}] \mapsto [a, b]$:

$$s^{-1}(y) = \sqrt{-2 \log y - n \log \log M + 2 \log M - n \log 2\pi}$$

with the first derivative denoted by Ds^{-1}. Conditionally on $X_1 \in S_2$, the variable $\|X_1\|$ has the density

$$p_{\|X_1\|}(y) = \frac{n V_n}{(2\pi)^{n/2} P(X_1 \in S_2)} y^{n-1} e^{-\frac{1}{2} y^2}$$

and Y has the density (on $[a,b]$)

$$\begin{aligned}
&p_{\|X_1\|}(s^{-1}(y))|Ds^{-1}(y)| \\
&\quad = \frac{nV_n s^{-1}(y)^{n-1}}{(2\pi)^{n/2}P(X_1 \in S_2)}|Ds^{-1}(y)|e^{\log y + \frac{n}{2}\log\log M - \log M + \frac{n}{2}\log 2\pi} \\
&\quad = \frac{nV_n y s^{-1}(y)^{n-1}\log^{n/2} M}{MP(X_1 \in S_2)}|Ds^{-1}(y)|. \qquad (56)
\end{aligned}$$

Because $y \in [\epsilon, \epsilon^{-1}]$, we have in the limit $M \to \infty$ with everything else fixed,

$$\begin{aligned}
s^{-1}(y)^{n-1} &= (2\log M)^{n/2-1/2}\left(1 - \frac{\log y}{\log M} - \frac{n\log\log M}{2\log M} - \frac{\log 2\pi}{2\log M}\right)^{n/2-1/2} \\
&= (2\log M)^{n/2-1/2}\left(1 + O\left(\frac{\log\log M}{\log M}\right)\right). \qquad (57)
\end{aligned}$$

Also, by Lemma 10,

$$P(X_1 \in S_2) = \frac{2^{n/2-1}nV_n \log^{n-1} M}{\epsilon M}(1-\epsilon^2)\left(1 + O\left(\frac{\log^2\log M}{\log M}\right)\right) \qquad (58)$$

and

$$\begin{aligned}
|Ds^{-1}(y)| &= \frac{1}{y\sqrt{-2\log y - n\log\log M + 2\log M - n\log 2\pi}} \\
&= \frac{1}{y\sqrt{2\log M}}\left(1 + O\left(\frac{\log\log M}{\log M}\right)\right). \qquad (59)
\end{aligned}$$

By Equations (56)-(59) we have

$$\begin{aligned}
p_{\|X_1\|}(s^{-1}(y))|Ds^{-1}(y)| &= \frac{\epsilon}{1-\epsilon^2}\left(1 + O\left(\frac{\log^2\log M}{\log M}\right)\right)^3 \\
&= \frac{\epsilon}{1-\epsilon^2} + O\left(\frac{\log^2\log M}{\log M}\right). \qquad (60)
\end{aligned}$$

By this approximation,

$$\begin{aligned}
E[h(Y)|X_1 \in S_2] &= \frac{\epsilon}{1-\epsilon^2}\int_\epsilon^{\epsilon^{-1}} h(y)dy + O\left(\frac{\log^2\log M}{\log M}\right) \\
&\to \frac{\epsilon}{1-\epsilon^2}\int_\epsilon^{\epsilon^{-1}} h(y)dy
\end{aligned}$$

in the limit $M \to \infty$. □

Next we will find out the asymptotic behavior of $E[d_{1,k}^\alpha|X_1 \in S_2]$, which together with the approximation for $P(X_1 \in S_2)$ takes care of region S_2. The key to the analysis is Lemma 12. The following represents the nearest neighbor distance in terms of the small ball probability and the variable Y. We invoke the event I_L to bound $d_{1,k}$; L stays fixed in this considerations the idea being the limit $L \to \infty$ after taking the limit $M \to \infty$.

Lemma 13. *Assume that (2) holds and $\alpha > n$. Then*

$$E[d_{1,k}^{\alpha} I_L | X_1 \in S_2] = \frac{E[f^{-1}\left(\frac{2^{n/2} M \omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha} I_L | X_1 \in S_2]}{2^{\alpha/2} \log^{\alpha/2} M} + R_1,$$

where Y is defined in Equation (54) and

$$|R_1| \leq c \log^{-\alpha/2-1} M$$

for some constant $c(n, \alpha, \epsilon, L)$.

Proof. We first collect a few useful facts. If $x \in S_2$, then by Lemma 5

$$\|x\|^n \omega_x(r) = p(x) f(\|x\| r) - p(x) R_1 \tag{61}$$

or equivalently

$$r = \frac{f^{-1}\left(\frac{\|x\|^n \omega_x(r)}{p(x)} + R_1\right)}{\|x\|}$$

with

$$0 \leq R_1 \leq r^2 f(\|x\| r).$$

$x \in S_2$ implies

$$\frac{\sqrt{\log M}}{c_1} \leq \|x\| \leq c_1 \sqrt{\log M} \tag{62}$$

for some constant $c_1(n, \epsilon)$. The indicator function I_L ensures that we only need to consider

$$0 < r < \frac{L}{\epsilon^{1/n} \sqrt{\log M}}.$$

Then by (62)

$$\|x\| r \leq \frac{L c_1}{\epsilon^{1/n}}. \tag{63}$$

By a Taylor expansion, for any real number $\beta \in \Re$ and $x \in S_2$,

$$|\|x\|^{\beta} - (2 \log M)^{\beta/2}| \leq c_2 \log^{\beta/2-1} M \tag{64}$$

for some constant $c_2(n, \epsilon, \beta)$. Moreover, f is an increasing continuous function allowing a bound on R_1:

$$R_1 \leq r^2 f(\|x\| r) \leq \frac{L^2 f(\frac{L c_1}{\epsilon^{1/n}})}{\epsilon^{2/n} \log M} \leq \frac{c_3}{\log M} \tag{65}$$

for

$$c_3 = \frac{L^2 f(\frac{Lc_1}{\epsilon^{1/n}})}{\epsilon^{2/n}}.$$

Having made the preliminary observations, we are ready for the first step towards completing of the proof. We have for $x \in S_2$ by Equation (61)

$$E[d_{1,k}^{\alpha} I_L | X_1 = x] = E[\frac{f^{-1}\left(\frac{\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)} + R_2\right)^{\alpha}}{\|X_1\|^{\alpha}} I_L | X_1 = x]$$

with

$$0 \leq R_2 \leq \frac{c_3}{\log M} \tag{66}$$

(R_2 is R_1 with $d_{1,k}$ instead of r multiplied by I_L). The challenging part is to modify the argument for f^{-1}. We first tackle the easier task of replacing $\|x\|^{\alpha}$ with a function of M. To this end, we observe that

$$E[d_{1,k}^{\alpha} I_L | X_1 = x] = E[\frac{f^{-1}\left(\frac{\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)} + R_2\right)^{\alpha}}{2^{\alpha/2} \log^{\alpha/2} M} I_L | X_1 = x] + R_3 \tag{67}$$

with

$$R_3 = E[f^{-1}\left(\frac{\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)} + R_2\right)^{\alpha} (\|X_1\|^{-\alpha} - 2^{-\alpha/2} \log^{-\alpha/2} M) I_L | X_1 = x].$$

By Lemma 5 and Equations (61), (63) and (65) we find a constant $c_4(n, \epsilon, L)$ such that

$$\frac{\|x\|^n \omega_x(r)}{p(x)} + \frac{c_3}{\log M} \leq f(\|x\| r) + \frac{c_3}{\log M} \leq f\left(\frac{Lc_1}{\epsilon^{1/n}}\right) + \frac{c_3}{\log M} \leq c_4 \tag{68}$$

for $x \in S_2$ and $0 < r < L\epsilon^{-1/n} \log^{-1/2} M$. Using the previous inequality and the fact that f^{-1} is an increasing function together with Equation (64) allows us to bound

$$|R_3| \leq c_2(n, \epsilon, -\alpha) f^{-1}(c_4) \log^{-\alpha/2 - 1} M. \tag{69}$$

We move to the argument for f^{-1}. Again, it would be useful to get rid of the norm $\|x\|^n$. This is facilitated by modifying the argument appearing in (67) (due to conditionalization, we may use x instead of X_1 in the expressions):

$$\frac{\|x\|^n \omega_x(d_{1,k})}{p(x)} I_L = \frac{2^{n/2} \omega_x(d_{1,k}) \log^{n/2} M}{p(x)} I_L + R_4,$$

where by Equation (64) (to bound $\omega_x(d_{1,k})$, we use Equations (68) and (62))

$$|R_4| = \frac{|\|x\|^n - 2^{n/2} \log^{n/2} M| \omega_x(d_{1,k})}{p(x)} I_L \leq \frac{c_5}{\log M} \tag{70}$$

for some constant $c_5(n, \epsilon, L)$.

In summary, this far we have shown that

$$E[d_{1,k}^\alpha I_L | X_1 = x]$$
$$= E[\frac{f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})\log^{n/2} M}{p(X_1)} + R_2 + R_4\right)^\alpha}{2^{\alpha/2}\log^{\alpha/2} M} I_L | X_1 = x] + R_3, \tag{71}$$

where (66), (69) and (70) bound the three correction terms.

While the correction terms R_2 and R_4 are small, they appear inside the argument for f^{-1}. The best we can say about their effect is

$$E[|f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})\log^{n/2} M}{p(X_1)} + R_2 + R_4\right)^\alpha$$
$$- f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})\log^{n/2} M}{p(X_1)}\right)^\alpha |I_L | X_1 = x]$$
$$\leq (R_2 + R_4) \sup_{t \in [0, f^{-1}(2^{n/2+1} c_4)]} |D(f^{-1}(t)^\alpha)| \tag{72}$$

assuming without losing generality that $|R_2 + R_4| \leq c_4$. So, we need to bound the derivative of the function $f^{-1}(t)^\alpha$ on bounded intervals. We observe that

$$D(f^{-1}(t)^\alpha) = \frac{\alpha f^{-1}(t)^{\alpha-1}}{Df(f^{-1}(t))}. \tag{73}$$

Furthermore,

$$Df(t) = nt^{n-1} \int_{B(0,1)} e^{ty^{(1)}} dy + t^n \int_{B(0,1)} y^{(1)} e^{ty^{(1)}} dy$$
$$\geq nt^{n-1} \int_{B(0,1)} e^{ty^{(1)}} dy \geq \frac{1}{2} n V_n t^{n-1}, \tag{74}$$

because

$$\int_{B(0,1)} y^{(1)} e^{ty^{(1)}} dy = \int_0^t \int_{B(0,1)} (y^{(1)})^2 e^{ty^{(1)}} dy dt \geq 0.$$

Using (74) in (73) yields

$$D(f^{-1}(t)^\alpha) \leq \frac{2\alpha}{nV_n} f^{-1}(t)^{\alpha-n}.$$

$n > \alpha$ and f^{-1} is an increasing function implying that

$$\sup_{t \in [0, f^{-1}(2^{n/2+1} c_4)]} f^{-1}(t)^{n-\alpha} \leq f^{-1}(2^{n/2+1} c_4)^{n-\alpha}.$$

Using the upper bound in (72) shows that for $x \in S_2$,

$$E[\frac{f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})\log^{n/2}M}{p(X_1)} + R_2 + R_4\right)^{\alpha}}{2^{\alpha/2}\log^{\alpha/2}M} I_L | X_1 = x]$$
$$= E[\frac{f^{-1}\left(\frac{2^{n/2}\omega_{X_1}(d_{1,k})\log^{n/2}M}{p(X_1)}\right)^{\alpha}}{2^{\alpha/2}\log^{\alpha/2}M} I_L | X_1 = x] + R_5$$
$$= E[\frac{f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha}}{2^{\alpha/2}\log^{\alpha/2}M} I_L | X_1 = x] + R_5$$

with $|R_5| \leq c_6 \log^{-\alpha/2-1} M$ for some constant $c_6(n, \alpha, \epsilon, L)$. The proof is finished by recalling the earlier observation (71). The final form of the claim is achieved via the tower rule $E[\dots|X_1 \in S_2] = E[E[\dots|X_1]X_1 \in S_2]$. □

In Lemma 13, we find the term Y, which has the asymptotic uniformity property as proven in Lemma 12. Connecting the two results mainly involves removing the truncation I_L, but takes some technical effort. The function g was defined in Equation (26).

Lemma 14. *Assume that (2) holds and $\alpha > n$. Then*

$$(2\log M)^{\alpha/2} E[d_{1,k}^{\alpha} | X_1 \in S_2] \to \frac{\epsilon}{(k-1)!(1-\epsilon^2)} \int_{\epsilon}^{\epsilon^{-1}} g\left(\frac{2^{n/2}}{y}\right) dy$$

in the limit $M \to \infty$.

Proof. By Lemma 13, we know that

$$(2\log M)^{\alpha/2} E[d_{1,k}^{\alpha} I_L | X_1 \in S_2] - E[f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha} I_L | X_1 \in S_2] \to 0$$

in the limit $M \to \infty$ with $(n, k, \alpha, \epsilon, L)$ fixed. We write

$$E[f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha} I_L | X_1 \in S_2] = \frac{\int_{S_2} E[f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha} I_L | X_1 = x] p(x) dx}{P(X_1 \in S_2)}.$$

Using Equation (55) and Lemma 1 (recall that Y depends only on X_1),

$$E[f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^{\alpha} I_L | X_1 = x] = k\binom{M-1}{k} \int_0^{\omega_x(L\epsilon^{-1/n}\log^{-1/2}M)} \omega^{k-1} \times (1-\omega)^{M-k-1} f^{-1}\left(\frac{2^{n/2}M\omega}{s(\|x\|)}\right)^{\alpha} d\omega.$$

Now

$$k\binom{M-1}{k} = \frac{(M-1)!}{(k-1)!(M-1-k)!} = \frac{M^k}{(k-1)!} + R_1 \tag{75}$$

with $|R_1| \leq c_1 M^{k-1}$ for some constant $c_1(k)$. Also, because $\|x\|$ behaves asymptotically as $\sqrt{2\log M}$ and $p(x) > \frac{\log^{n/2} M}{\epsilon M}$ on S_2, Equation (68) shows that

$$\omega_x(\frac{L}{\epsilon^{1/n}\sqrt{\log M}}) \leq \frac{c_2}{M} \tag{76}$$

for some constant $c_2(n, \epsilon, L)$. This implies that for $\omega < \omega_x(L\epsilon^{-1/n}\log^{-1/2} M)$,

$$(1-\omega)^{M-k-1} = e^{-(M-k-1)\omega} + R_2 = e^{-M\omega} + R_2 + R_3 \tag{77}$$

with

$$\begin{aligned}|R_2| &\leq |(1-\omega)^{M-k-1} - e^{-(M-k-1)\omega}| = e^{(M-k-1)(\log(1-\omega)+\omega)} - 1 \\ &\leq \frac{c_3}{M}\end{aligned}$$

and

$$|R_3| \leq e^{-M\omega}(e^{(k-1)\omega} - 1) \leq \frac{c_3}{M}$$

for some constant $c_3(n, k, \alpha, \epsilon, L)$. By Equations (75)-(77) together with the fact that f^{-1} is an increasing function,

$$\begin{aligned}&k\binom{M-1}{k}\int_0^{\omega_x(L\epsilon^{-1/n}\log^{-1/2} M)} \omega^{k-1}(1-\omega)^{M-k-1} f^{-1}\left(\frac{2^{n/2}M\omega}{s(\|x\|)}\right)^{\alpha} d\omega \\ &= \frac{M^k}{(k-1)!}\int_0^{\omega_x(L\epsilon^{-1/n}\log^{-1/2} M)} \omega^{k-1}e^{-M\omega} f^{-1}\left(\frac{2^{n/2}M\omega}{s(\|x\|)}\right)^{\alpha} d\omega \\ &\quad + R_4 + R_5\end{aligned}$$

with

$$\begin{aligned}|R_4| &\leq |k\binom{M-1}{k} - \frac{M^k}{(k-1)!}|\int_0^{\omega_x(L\epsilon^{-1/n}\log^{-1/2} M)} \omega^{k-1} \\ &\qquad \times (1-\omega)^{M-k-1} f^{-1}\left(\frac{2^{n/2}M\omega}{\epsilon}\right)^{\alpha} d\omega \\ &\leq c_1 M^{k-1}\int_0^{c_2 M^{-1}} \omega^{k-1} f^{-1}\left(\frac{2^{n/2}M\omega}{\epsilon}\right)^{\alpha} d\omega \\ &\leq \frac{c_1 c_2^k f^{-1}(\frac{2^{n/2}c_2}{\epsilon})^{\alpha}}{kM}\end{aligned}$$

and

$$|R_5| \leq \frac{M^k}{(k-1)!} \int_0^{c_2 M^{-1}} \omega^{k-1} |e^{-M\omega} - (1-\omega)^{M-k-1}| f^{-1}\left(\frac{2^{n/2} M\omega}{\epsilon}\right)^{\alpha} d\omega$$
$$\leq \frac{c_2^k c_3 f^{-1}\left(\frac{2^{n/2} c_2}{\epsilon}\right)^{\alpha}}{M}.$$

Observe that the bounds for R_4 and R_5 hold for any $x \in S_2$. By a change of variables,

$$\frac{M^k}{(k-1)!} \int_0^{\omega_x(L\epsilon^{-1/2} \log^{-1/2} M)} \omega^{k-1} e^{-M\omega} f^{-1}\left(\frac{2^{n/2} M\omega}{s(\|x\|)}\right)^{\alpha} d\omega$$
$$= \frac{1}{(k-1)!} \int_0^{\infty} \omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^{\alpha} d\omega$$
$$- \frac{1}{(k-1)!} \int_{M\omega_x(L\epsilon^{-1/n} \log^{-1/2} M)}^{\infty} \omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^{\alpha} d\omega$$
$$= \frac{1}{(k-1)!} \int_0^{\infty} \omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^{\alpha} d\omega + R_6.$$

We would like to show that

$$\lim_{L\to\infty} \limsup_{M\to\infty} \sup_{x\in S_2} R_6$$
$$= \lim_{L\to\infty} \limsup_{M\to\infty} \sup_{x\in S_2} \int_{M\omega_x(L\epsilon^{-1/n} \log^{-1/2} M)}^{\infty} \omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^{\alpha} d\omega = 0. \tag{78}$$

To see that this is true, we observe that by Lemma 6, for some constant $c_4(n, k, \alpha, \epsilon)$ there is the bound

$$\omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^{\alpha} \leq \omega^{k-1} e^{-\omega} f^{-1}\left(\frac{2^{n/2}\omega}{\epsilon}\right)^{\alpha}$$
$$\leq c_4 \omega^{k-1} (1+\omega) e^{-\omega}$$

with the upper bound integrable on $[0, \infty)$ and independent of $x \in S_2$. Moreover, by Equation (52)

$$\lim_{L\to\infty} \liminf_{M\to\infty} \sup_{x\in S_2} M\omega_x\left(\frac{L}{\epsilon^{1/n}\sqrt{\log M}}\right) = \infty$$

showing that (78) indeed holds.

In summary, we have shown that

$$\lim_{L\to\infty}\limsup_{M\to\infty} E[f^{-1}\left(\frac{2^{n/2}M\omega_{X_1}(d_{1,k})}{Y}\right)^\alpha I_L|X_1\in S_2]$$
$$= \lim_{L\to\infty}\limsup_{M\to\infty}\frac{\frac{1}{(k-1)!}\int_{S_2}\int_0^\infty \omega^{k-1}e^{-\omega}f^{-1}\left(\frac{2^{n/2}\omega}{s(\|x\|)}\right)^\alpha p(x)d\omega dx}{P(X_1\in S_2)}$$
$$+\frac{\int_{S_2}(R_4+R_5+R_6)p(x)dx}{P(X_1\in S_2)}$$
$$= \lim_{M\to\infty}\frac{1}{(k-1)!}E[g\left(\frac{2^{n/2}}{Y}\right)|X_1\in S_2]$$

and similarly with lim inf instead of lim sup; the last limit exists by Lemma 12, which shows that

$$E[g\left(\frac{2^{n/2}}{Y}\right)|X_1\in S_2]\to\frac{\epsilon}{1-\epsilon^2}\int_\epsilon^{\epsilon^{-1}} g\left(\frac{2^{n/2}}{y}\right)dy$$

in the limit $M\to\infty$. We have established that

$$\lim_{L\to\infty}\limsup_{M\to\infty}(2\log M)^{\alpha/2}E[d_{1,k}^\alpha I_L|X_1\in S_2] = \frac{\epsilon}{(k-1)!(1-\epsilon^2)}\int_\epsilon^{\epsilon^{-1}} g\left(\frac{2^{n/2}}{y}\right)dy.$$

On the other hand, Lemma 11 shows that

$$\lim_{L\to\infty}\limsup_{M\to\infty}(2\log M)^{\alpha/2}|E[d_{1,k}^\alpha I_L|X_1\in S_2]-E[d_{1,k}^\alpha|X_1\in S_2]| = 0$$

finalizing the proof. □

Now we are able to put everything together to conclude region S_2:

Lemma 15. *Assume that (2) holds and $\alpha > n$. Then*

$$\lim_{\epsilon\to 0}\lim_{M\to\infty} M\log^{\alpha/2+1-n}M\int_{S_2}E[d_{1,k}^\alpha|X_1=x]p(x)dx$$
$$=\frac{2^{n-\alpha/2-1}nV_n}{(k-1)!}\int_0^\infty g\left(\frac{1}{y}\right)dy<\infty.$$

Proof. The claim follows from Lemmas 10 and 14:

$$M\log^{\alpha/2+1-n}M\int_{S_2}E[d_{1,k}^\alpha|X_1=x]p(x)dx$$
$$= E[d_{1,k}^\alpha|X_1\in S_2]P(X_1\in S_2)$$
$$\to\frac{2^{n/2-\alpha/2-1}nV_n}{(k-1)!(1-\epsilon^2)}\int_\epsilon^{\epsilon^{-1}} g\left(\frac{2^{n/2}}{y}\right)dy$$
$$=\frac{2^{n-\alpha/2-1}nV_n}{(k-1)!(1-\epsilon^2)}\int_{2^{-n/2}\epsilon}^{2^{-n/2}\epsilon^{-1}} g\left(\frac{1}{y}\right)dy$$

in the limit $M \to \infty$. We would like to show that

$$\int_{2^{-n/2}\epsilon}^{2^{-n/2}\epsilon^{-1}} g\left(\frac{1}{y}\right) dy \to \int_0^\infty g\left(\frac{1}{y}\right) dy$$

in the limit $\epsilon \to 0$, which amounts to showing that $g(y^{-1})$ is an integrable function. This is best done using Lemma 6, which shows that

$$\begin{aligned}\int_0^\infty g\left(\frac{1}{y}\right) dy &= \int_0^1 g\left(\frac{1}{y}\right) dy + \int_1^\infty g\left(\frac{1}{y}\right) dy \\ &\leq c\int_0^1 (1 + \log^\alpha y^{-1}) dy + c\int_1^\infty y^{-\alpha/n} dy\end{aligned}$$

for some constant $c(n, k, \alpha)$. Both terms in the right side are finite (the second one because $\alpha > n$) verifying the integrability requirement. □

8 Region S_3

S_3 consists of points, where the density p takes small values:

$$S_3 = \{x \in \Re^n : \ p(x) < \frac{\epsilon \log^{n/2} M}{M}\}.$$

To bound nearest neighbor distances on S_3, we need similar tools as for S_2, but only upper bounds are needed providing some more flexibility. The sets $\tilde{S}_{3,i}$ are defined analogously to (32):

$$\tilde{S}_{3,i} = \{x \in \Re^n : \ 2^{-i-1}\frac{\epsilon \log^{n/2} M}{M} < p(x) < 2^{-i}\frac{\epsilon \log^{n/2} M}{M}\}$$

for $0 \leq i \leq i^*$ with

$$i^* = [\frac{(\alpha+1)}{\log 2} \log\log M] + 1.$$

Moreover, $\tilde{S}_{3,C} = S_3 \setminus \cup_{i=0}^{i^*} \tilde{S}_{3,i}$. Then we have

Lemma 16. *Under* (2)*, it holds that for some threshold* $M_0(n, \epsilon)$*, we have for* $M > M_0$ *and* $0 \leq i \leq i^*$ *that*

$$P(X_1 \in \tilde{S}_{3,i}) \leq 2^{-i} c \frac{\epsilon \log^{n-1} M}{M}$$

for some constant $c(n)$*.*

Proof. The set $\tilde{S}_{3,i}$ consists of points $x \in \Re^n$ with

$$\begin{aligned}&\|x\| \in [a, b] \\ &a = \sqrt{2\log M - n\log\log M - 2\log\epsilon + i\log 4 - n\log(2\pi)} \\ &b = \sqrt{2\log M - n\log\log M - 2\log\epsilon + (i+1)\log 4 - n\log(2\pi)}. \qquad (79)\end{aligned}$$

Moreover,

$$p(x) \leq 2^{-i} \frac{\epsilon \log^{n/2} M}{M} \tag{80}$$

for $x \in \tilde{S}_{3,i}$. Using the mean value theorem for a and b we have for $0 \leq i \leq i^*$,

$$b - a \leq \frac{4}{\sqrt{\log M}} \tag{81}$$

after some threshold $M_0(n, \epsilon)$. Also, we may take $\|x\| \leq \sqrt{3 \log M}$ for $0 \leq i \leq i^*$ as the term $2 \log M$ inside the square root (79) grows faster than the other terms. Then

$$\begin{aligned} \lambda(\tilde{S}_{3,i}) = nV_n \int_a^b x^{n-1} dx &\leq 3^{n/2-1/2} nV_n (b-a) \log^{n/2-1/2} M \\ &\leq 3^{n/2+3/2} nV_n \log^{n/2-1} M. \end{aligned} \tag{82}$$

Combining Equations (80)-(82), we have

$$\begin{aligned} P(X_1 \in \tilde{S}_{3,i}) = \int_{\tilde{S}_{3,i}} p(x) dx &\leq 2^{-i} \lambda(\tilde{S}_{3,i}) \frac{\epsilon \log^{n/2} M}{M} \\ &\leq 2^{-i} 3^{n/2+3/2} nV_n \frac{\epsilon \log^{n-1} M}{M}. \end{aligned}$$

□

Assessing the contributions from $\tilde{S}_{3,i}$ is convenient by using the function f together with the small ball probability. The proof idea is essentially similar to that used for S_2 in Section 7, but because we need only an upper bound, the proof is easier.

Lemma 17. *Suppose that* (2) *holds and* $\alpha > n$. *Then for some threshold* $M_0(n, \alpha, k, \epsilon)$, *we have for* $M > M_0$ *and* $0 \leq i \leq i^*$ *that*

$$\int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \leq c 2^{-i} \epsilon (\log \epsilon^{-1} + i + 1) \frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n, k, \alpha)$.

Proof. We decompose

$$\begin{aligned} \int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx = &\int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha I(d_{1,k} \leq 1) | X_1 = x] p(x) dx \\ &+ \int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha I(d_{1,k} > 1) | X_1 = x] p(x) dx \\ &= (I_1 + I_2) P(X_1 \in \tilde{S}_{3,i}) \end{aligned}$$

with

$$I_1 = E[d_{1,k}^\alpha I(d_{1,k} \le 1) | X_1 \in \tilde{S}_{3,i}]$$
$$I_2 = E[d_{1,k}^\alpha I(d_{1,k} > 1) | X_1 \in \tilde{S}_{3,i}].$$

$P(X_1 \in \tilde{S}_{3,i})$ was computed in Lemma 16.

1. *The term I_1*

If $0 < r < 1$, we have

$$\begin{aligned} \|x\|^n \omega_x(r) &= (2\pi)^{-n/2} \|x\|^n \int_{B(x,r)} e^{-\frac{1}{2}\|y\|^2} dy \\ &= (2\pi)^{-n/2} \|x\|^n \int_{B(x,r)} e^{-\frac{1}{2}\|y-x\|^2 - \frac{1}{2}\|x\|^2 - x^T(y-x)} dy \\ &\ge e^{-\frac{1}{2}} \|x\|^n p(x) \int_{B(0,r)} e^{-x^T y} dy = e^{-\frac{1}{2}} p(x) f(\|x\| r), \end{aligned} \tag{83}$$

where the function f was defined in Lemma 5. This implies that

$$d_{1,k} \le \frac{f^{-1}\left(\frac{e^{\frac{1}{2}}\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)}\right)}{\|X_1\|}. \tag{84}$$

By taking M_0 large enough, we may ensure that

$$\sqrt{\log M} \le \|x\| \le \sqrt{3 \log M} \tag{85}$$

on $x \in \tilde{S}_{3,i}$ for $0 \le i \le i^*$. Then by Lemma 6 and Equations (84)-(85),

$$\begin{aligned} E[d_{1,k}^\alpha I(d_{1,k} \le 1) | X_1 = x] &\le E[\frac{f^{-1}\left(\frac{e^{1/2}\|X_1\|^n \omega_{X_1}(d_{1,k})}{p(X_1)}\right)^\alpha}{\log^{\alpha/2} M} | X_1 = x] \\ &\le c_1 E[\frac{1 + \log^\alpha\left(1 + \frac{2^{i+n+2} M \omega_{X_1}(d_{1,k})}{\epsilon}\right)}{\log^{\alpha/2} M} | X_1 = x] \end{aligned}$$

for some constant $c_1(n, \alpha)$. Now

$$\begin{aligned} \log(1 + 2^{i+n+2}\epsilon^{-1} M \omega_{X_1}(d_{1,k})) &\le (i+n+2)\log 2 + \log \epsilon^{-1} + \log(2^{-i-n-2}\epsilon \\ &\quad + M\omega_{X_1}(d_{1,k})) \\ &\le (i+n+2)\log 2 + \log \epsilon^{-1} + \log(1 + M\omega_{X_1}(d_{1,k})) \\ &\le (i+n+2)\log 2 + \log \epsilon^{-1} + M\omega_{X_1}(d_{1,k}) \end{aligned}$$

recalling that $0 < \epsilon < 1$. The α-moment of the conditional expectation of the last expression is bounded by $c_2(\log \epsilon^{-1} + i + 1)$ for some constant $c_2(n, k, \alpha)$ by Lemma 1 and Equation (13).

2. *The term I_2*
By Hölder's inequality, Lemma 3 and Equation (85),

$$I_2 \leq \sqrt{E[d_{1,k}^{2\alpha}|X_1 \in \tilde{S}_{3,i}]}\sqrt{P(d_{1,k} > 1|X_1 \in \tilde{S}_{3,i})}$$
$$\leq c_4 \log^{\alpha/2} M \sqrt{P(d_{1,k} > 1|X_1 \in \tilde{S}_{3,i})}$$

for some constant $c_4(n,k,\alpha)$. Equation (18) applies here: for $x \in \tilde{S}_{3,i}$,

$$P(d_{1,k} > 1|X_1 = x) \leq \left(\frac{M}{M-k-1}\right)^k \int_{(M-k-1)\omega_x(1)}^{M-k-1} e^{-y+(k-1)\log y} dy$$
$$\leq c_5(n,k) e^{-c_5(n,k)^{-1} M}$$

for some constant $c_5(n,k)$. It would be sufficient to show that for any $j > 0$,

$$\sup_{0 \leq i \leq i^*, x \in \tilde{S}_{3,i}} \omega_x(1) M \log^{-j} M \to \infty \tag{86}$$

in the limit $M \to \infty$. By Equations (83) and (85) taking into account that on $\tilde{S}_{3,i}$,

$$p(x) \geq 2^{-i^*} \frac{\epsilon \log^{n/2} M}{M} \geq \frac{\epsilon \log^{n/2-\alpha-1} M}{2M},$$

we have

$$\omega_x(1) \geq e^{-\frac{1}{2}} \frac{p(x) f(\|x\|)}{\|x\|^n} \geq e^{-\frac{1}{2}} \frac{\epsilon}{4M \log^{\alpha+1} M} f(\sqrt{\log M})$$
$$\geq e^{-\frac{1}{2}} \frac{\epsilon}{4M \log^{\alpha+1-n/2} M} \int_{B(0,1)} e^{\sqrt{\log M} y^{(1)}} dy$$
$$\geq e^{-\frac{1}{2}} \frac{\epsilon}{4M \log^{\alpha+1-n/2} M} \lambda(B(0,1) \cap \{y \in \Re^n : \ y^{(1)} > \frac{1}{2}\}) e^{\frac{1}{2}\sqrt{\log M}}.$$

The term $e^{\frac{1}{2}\sqrt{\log M}}$ approaches infinity faster than $\log^j M$ for any $j > 0$. This shows (86) and we conclude that I_2 approaches ∞ faster than $\log^j M$ (for any $j > 0$) in the limit $M \to \infty$. □

The region $\tilde{S}_{3,C}$ is easier, because by taking i^* as a large number, we are able to control the probability of this set.

Lemma 18. *Suppose that* (2) *holds and* $\alpha > n$. *Then for some threshold* $M_0(n,k,\alpha,\epsilon)$ *we have for* $M > M_0$, *that*

$$\int_{\tilde{S}_{3,C}} E[d_{1,k}^{\alpha}|X_1 = x] p(x) dx \leq c\epsilon \frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n,k,\alpha)$.

Proof. On $\tilde{S}_{3,C}$ we have

$$p(x) \leq 2^{-i^*} \frac{\epsilon \log^{n/2} M}{M} \leq \frac{\epsilon \log^{n/2-\alpha-1} M}{M}.$$

We define

$$R_i = 2^{i+1}\tilde{S}_{3,C} \setminus 2^i \tilde{S}_{3,C},$$

where

$$2^i \tilde{S}_{3,C} = \{x \in \Re^n : \ 2^{-i}x \in \tilde{S}_{3,C}\}.$$

We may assume that $\|x\| \leq 2\sqrt{\log M}$ on R_1 and consequently

$$\|x\| \leq 2^i \sqrt{\log M}$$

on R_i for any $i > 0$. Now by Lemma 3,

$$\begin{aligned}
\int_{R_i} E[d_{1,k}^\alpha | X_1 = x] p(x) dx &\leq c_1 \int_{R_i} (\|x\|^\alpha + 1) p(x) dx \\
&\leq c_1 (2^{i\alpha} \log^{\alpha/2} M + 1) \int_{R_i} p(x) dx \\
&\leq 2^{i\alpha} c_2 \log^{\alpha/2} M \left(\frac{(2\pi)^{n/2} \epsilon \log^{n/2-\alpha-1} M}{M} \right)^{2^{2i}} \lambda(R_i), \qquad (87)
\end{aligned}$$

where $c_1(n)$ is some constant, and to be exact,

$$c_2 = 2(2\pi)^{-n/2} c_1.$$

The factor 2 comes from the fact that $\log^{\alpha/2} M > 1$ for $M > 3$ (which trivially can be assumed without losing generality). $\lambda(R_i)$ is roughly bounded by

$$\lambda(R_i) \leq V_n \sup_{x \in R_i} \|x\|^n \leq 2^{ni} V_n \log^{n/2} M.$$

By substitution into (87), we find out that

$$\begin{aligned}
\int_{\tilde{S}_{3,C}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx &\leq \sum_{i=0}^{\infty} \int_{R_i} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \\
&\leq c_2 V_n \epsilon \frac{\log^{n-\alpha/2-1} M}{M} \sum_{i=0}^{\infty} 2^{(n+\alpha)i} \left(\frac{(2\pi)^{n/2} \epsilon \log^{n/2-\alpha-1} M}{M} \right)^{2^{2i}-1}
\end{aligned}$$

and now it is rather obvious that the sum does not pose problems. □

Lemma 19. *Assume that* (2) *holds,* $\alpha > n$ *and* $\epsilon < 1/2$ *(only small values of* ϵ *matter in any case). Then there exists a threshold* $M_0(n, k, \alpha, \epsilon)$ *such that for any* $M > M_0(n, k, \alpha, \epsilon)$*, we have*

$$\int_{S_3} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \leq c \epsilon \log \epsilon^{-1} \frac{\log^{n-\alpha/2-1} M}{M}$$

for some constant $c(n, k, \alpha)$.

Proof. We decompose

$$\int_{S_3} E[d_{1,k}^\alpha | X_1 = x] p(x) dx = \sum_{i=0}^{i^*} \int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx + \int_{\tilde{S}_{3,C}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx.$$

By Lemma 17,

$$\sum_{i=0}^{i^*} \int_{\tilde{S}_{3,i}} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \leq c\epsilon \frac{\log^{n-\alpha/2-1} M}{M} \sum_{i=0}^{\infty} 2^{-i} (\log \epsilon^{-1} + i + 1).$$

Lemma 18 finalizes the proof. □

9 Proof of Theorem 2

Previously we have examined the regions S_1, S_2 and S_3, which were defined in terms of ϵ and M. We decompose

$$\begin{aligned}(M \log^{\alpha/2+1-n} M) E[d_{1,k}^\alpha] &= M \log^{\alpha/2+1-n} M (\int_{S_1} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \\ &+ \int_{S_2} E[d_{1,k}^\alpha | X_1 = x] p(x) dx + \int_{S_3} E[d_{1,k}^\alpha | X_1 = x] p(x) dx) \\ &= I_{1,\epsilon,M} + I_{2,\epsilon,M} + I_{3,\epsilon,M}.\end{aligned}$$

with

$$\begin{aligned} I_{1,\epsilon,M} &= M \log^{\alpha/2+1-n} M \int_{S_1} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \\ I_{2,\epsilon,M} &= M \log^{\alpha/2+1-n} M \int_{S_2} E[d_{1,k}^\alpha | X_1 = x] p(x) dx \\ I_{3,\epsilon,M} &= M \log^{\alpha/2+1-n} M \int_{S_3} E[d_{1,k}^\alpha | X_1 = x] p(x) dx. \end{aligned}$$

Lemmas 9 and 19 show that

$$\lim_{\epsilon \to 0} \limsup_{M \to \infty} I_{1,\epsilon,M} + I_{3,\epsilon,M} = 0.$$

Also by Lemma 15,

$$\lim_{\epsilon \to 0} \limsup_{M \to \infty} I_{2,\epsilon,M} = \frac{2^{n-\alpha/2-1} n V_n}{(k-1)!} \int_0^\infty g\left(\frac{1}{y}\right) dy.$$

We conclude that

$$\begin{aligned}\lim_{M \to \infty} (M \log^{\alpha/2+1-n} M) E[d_{1,k}^\alpha] &= \lim_{\epsilon \to 0} \lim_{M \to \infty} (I_{1,\epsilon,M} + I_{2,\epsilon,M} + I_{3,\epsilon,M}) \\ &= \frac{2^{n-\alpha/2-1} n V_n}{(k-1)!} \int_0^\infty g\left(\frac{1}{y}\right) dy.\end{aligned}$$

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