Ternary Octonionic Gauge Field Theories

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Abstract

A ternary gauge field theory is explicitly constructed based on a totally antisymmetric ternary-bracket structure associated with a 3-Lie algebra. It is shown that the ternary infinitesimal gauge transformations do obey the key *closure* relations $[\delta_1, \delta_2] = \delta_3$. Invariant actions for the 3-Lie algebra-valued gauge fields and scalar fields are displayed. We analyze and point out the difficulties in formulating a nonassociative Octonionic ternary gauge field theory based on a ternary-bracket associated with the octonion algebra and defined earlier by Yamazaki. It is shown that a Yang-Mills-like quadratic action is invariant under global (rigid) transformations involving the Yamazaki ternary octonionic bracket, and that there is *closure* of these global (rigid) transformations based on constant antisymmetric parameters $\Lambda^{ab} = -\Lambda^{ba}$. Promoting the latter parameters to spacetime dependent ones $\Lambda^{ab}(x^\mu)$ allows to build an octonionic ternary gauge field theory when one imposes gauge *covariant* constraints on the latter gauge parameters leading to field-dependent gauge parameters and nonlinear gauge transformations. In this fashion one does not spoil the gauge invariance of the quadratic action under this restricted set of gauge transformations and which are tantamount to spacetime-dependent scalings (homothecy) of the gauge fields.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [7], [8], for references, among many others. A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [5], [9], [10]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative.

Recently, tremendous activity has been launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [15], [16] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$-branes. The original BLG theory requires the algebraic structures of generalized Lie 3-algebras and also of nonassociative algebras. Later developments by [17] provided a $3D$ Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [17] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [18], [19].

A Nonassociative Gauge theory based on the Moufang S^7 loop product (not a Lie algebra) has been constructed by [20]. Taking the algebra of octonions with a unit norm as the Moufang S^7-loop, one reproduces a nonassociative octonionic gauge theory which is a generalization of the Maxwell and Yang-Mills gauge theories based on Lie algebras. $BPST$-like instantons solutions in $D = 8$ were also found. These solutions represented the physical degrees of freedom of the transverse 8-dimensions of superstring solitons in $D = 10$ preserving one and two of the 16 spacetime supersymmetries. Nonassociative deformations of Yang-Mills Gauge theories involving the left and right bimodules of the octonionic algebra were presented by [21].

In section **2** we develop a ternary gauge field theory formulation associated to a 3-Lie algebra and whose structure constants are totally antisymmetric in all their indices. An invariant action involving the 3-Lie algebra-valued gauge field and scalar field is provided. It is shown that there is *closure* of the gauge variations on the fields. In section **3** the Nonassociative Octonionic ternary gauge field theory is presented and it differs mainly from the prior formulation (besides nonassociativity) due to the fact that the structure constants are not totally antisymmetric in all their indices. As a result one encounters difficulties in formulating a gauge invariant quadratic action and having closure of the gauge variations on the fields.

It is shown, nevertheless, that the Yang-Mills like action is invariant under $different$ global (rigid) transformations involving ternary octonionic brackets and antisymmetric constant parameters $\Lambda^{ab} = -\Lambda^{ba}, a, b = 1, 2, 3, ...7$. It is shown that a Yang-Mills-like quadratic action is invariant under global (rigid) transformations involving the Yamazaki ternary octonionic bracket, and that there is *closure* of these global (rigid) transformations based on constant antisymmetric parameters $\Lambda^{ab} = -\Lambda^{ba}$. Promoting the latter parameters to spacetime dependent ones $\Lambda^{ab}(x^\mu)$ allows to build an octonionic ternary gauge field theory when one imposes gauge *covariant* constraints on the latter gauge parameters leading to field-dependent gauge parameters and nonlinear gauge transformations. In this fashion one does not spoil the gauge invariance of the quadratic action under this restricted set of gauge transformations and which are tantamount to spacetime-dependent scalings of the gauge fields.

The ternary gauge theory developed in this work differs from the work by [15], [16] in that our 3-Lie algebra-valued gauge field strengths $F_{\mu\nu}$ are explicitly defined in terms of a 3-bracket $[A_\mu, A_\nu, \mathbf{g}]$ involving a 3-Lie algebra-valued coupling $\mathbf{g} = g^a t_a$. Whereas the definition of $F_{\mu\nu}$ by [15], [16] was based on the standard commutator of the matrices $(\tilde{A}_\mu)^a_c (\tilde{A}_\nu)^c_b \; - \; (\tilde{A}_\nu)^a_c (\tilde{A}_\mu)^c_b$. These matrices were defined as $\mathbf{A}_\mu = A^{ab}_\mu f_{ab}{}^{cd} = (\tilde{A}_\mu)^{cd}$ and given in terms of the structure constants $f_{ab}{}^{cd}$ of the 3-Lie algebra $[t_a, t_b, t^c] = f_{ab}{}^{cd} t_d$.

2 3-Lie-Algebra-valued Gauge Field Theories

In this section we will construct a gauge field theory based on a 3-Lie algebra-valued gauge fields. As outlined by [15], one introduces a basis T^a for the 3-Lie algebra and one expands the gauge field $A_\mu = A^a_\mu T_a, a = 1, ..., N$, where N is the dimension of the 3-Lie algebra. The structure constants are introduced as

$$[\ T_a,\ T_b,\ T_c\ ] \ = \ f_{abc}{}^d\ T_d \tag{2.1}$$

such that $f_{abc}{}^d = f_{[abc]}{}^d$. The trace-form provides a metric $h^{ab} = Tr(T^a, T^b)$. that we can use to raise indices: $f^{abcd} = h^{de} f^{abc}{}_e$. On physical grounds one assume that h^{ab} is positive definite.

A bilinear positive symmetric product written as $< X, Y > = < Y, X > = Tr(X, Y) = Tr(Y, X)$ is required and such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$< \ [u,\ v,\ x],\ y\ > \ = \ - \ < \ [u,\ v,\ y],\ x\ > \ = \ - \ < \ x,\ [u,\ v,\ y]\ > \ \Rightarrow$$

$$D_{u,v}\ <\ x,\ y\ > \ = \ <\ [u,\ v,\ x],\ y\ > \ + \ <\ x,\ [u,\ v,\ y]\ > \ = \ 0 \tag{2.2}$$

The symmetric product remains invariant under derivations. Since the ternay bracket is totally antisymmetric one can rewrite $< [u, v, x], y > = < [x, u, v], y >$, and from (2.2) one infers that the 3-Lie algebra admits a totally antisymmetric ternary product which satisfies

$$Tr(\ [A,\ B,\ C\ ],\ D\ ) \ = \ - \ Tr(\ A,\ [\ B,\ C,\ D\ ]\ ) \tag{2.3}$$

The condition (2.3) on the trace-form implies that $f^{abcd} = -f^{dabc}$ and this further implies that the structure constants are totally *antisymmetric* in all their indices $f^{abcd} = f^{[abcd]}$, in analogy with the familiar result in Lie algebras.

If gauge symmetries act as a derivation

$$\delta(\ [\ X,\ Y,\ Z]\ ) \ = \ [\ \delta X,\ Y,\ Z\ ] \ + \ [\ X,\ \delta Y,\ Z] \ + \ [\ X,\ Y,\ \delta Z\ ] \tag{2.4}$$

this leads to the fundamental identity

$$[\ U, V, [X, Y, Z]\ ] \ = \ [\ [U, V, X],\ Y,\ Z\ ] + [\ X,\ [U, V, Y],\ Z\ ] + [\ X,\ Y,\ [U, V, Z]\ ] \tag{2.5}$$

which plays a role analogous to the Jacobi identity in ordinary Lie algebras. Among the key properties we shall use are the fundamental identity (2.5) and the fact that the structure constants $f_{[abcd]}$ are totally *antisymmetric* in all of their indices.

The simplest nontrivial Lie 3-algebra is $\mathcal{A}_4$. It has 4 generators T_a, $a = 1,2,3,4$. The ternary bracket is defined by $[T_a, T_b, T_c] = \epsilon_{abcd} T_d$. The invariant metric of $\mathcal{A}_4$ is δ_{ab}. The 3-Lie algebra $\mathcal{A}_4$ [11] is a natural generalization of the Lie algebra $su(2)$. It was conjectured in [12] and later proved in [13] that the only finite dimensional Lie 3-algebras with a positive-definite metric are the trivial algebra, $\mathcal{A}_4$, and their direct sums. On the other hand, it is possible to define many infinite dimensional Lie 3-algebras with positive-definite metrics. All the Nambu-Poisson algebras are of this kind [12].

We define the field strength in terms of the *ternary* bracket as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; [\; A_\mu,\; A_\nu,\; \mathbf{g}\;] \tag{2.6}$$

where $\mathbf{g} = g^a T_a$ is a 3-Lie-algebra valued "coupling" which is inert under gauge transformations. Under the local gauge transformations the field transforms as

$$\delta(A^d_\mu\; T_d) \;=\; -\; (\partial_\mu \Lambda^d(x))\; T_d \;+\; [\; \Lambda^a(x)\; T_a,\; A^b_\mu\; T_b,\; g^c\; T_c\;] \tag{2.7}$$

and the 3-Lie-algebra-valued coupling is gauge invariant

$$\delta(g^d\; T_d) \;=\; [\Lambda^a(x)\; T_a,\; \mathbf{g},\; \mathbf{g}] = 0 \tag{2.8}$$

since the ternary brackets $[X, Y, Y] = 0$. After some straightforward algebra one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the ternary-brackets (2.6) transforms properly (homogeneously) under the ternary gauge transformations because $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda^d = 0$, and

$$\big(\; (\partial_\mu\Lambda^a) A^b_\nu g^c - (\partial_\nu\Lambda^a) A^b_\mu g^c - (\partial_\mu\Lambda^a) A^b_\nu g^c - (\partial_\nu\Lambda^b) A^a_\mu g^c \;\big)\; f_{abc}{}^d \;=\; 0 \tag{2.9}$$

is *identically* zero. The second and fourth terms in (2.9) are symmetric under the exchange of indices $a \leftrightarrow b$ so they will cancel out due to the antisymmetry $f_{abcd} = -f_{bacd}$. This simply can be seen by rewriting the fourth term as

$$-\; (\partial_\nu\Lambda^b)\; A^a_\mu\; g^c\; f_{abc}{}^d \;=\; (\partial_\nu\Lambda^b)\; A^a_\mu\; g^c\; f_{bac}{}^d \;=\; (\partial_\nu\Lambda^a)\; A^b_\mu\; g^c\; f_{abc}{}^d \tag{2.10}$$

and relabeling the $a \leftrightarrow b$ indices in the last line of (2.10). Therefore $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F^d_{\mu\nu}\; T_d) \;=\; [\; \Lambda^a(x)\; T_a,\; F^b_{\mu\nu}\; T_b,\; g^c\; T_c\;] \;=\; \Lambda^a(x)\; F^b_{\mu\nu}\; g^c f_{abc}{}^d\; T_d \;\Rightarrow$$

$$\delta F^d_{\mu\nu} \;=\; \Lambda^a(x)\; F^b_{\mu\nu}\; g^c\; f_{abc}{}^d \tag{2.11}$$

The result (2.11) is a direct consequence of the fundamental identity; i.e the ternary bracket is a derivation with respect to the first two entries

$$[\; \Lambda,\; \mathbf{g},\; [A_\mu,\; A_\nu,\; \mathbf{g}]\;] \;=$$

$$[\ [\Lambda,\ \mathbf{g},\ A_\mu],\ A_\nu,\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ [\Lambda,\ \mathbf{g},\ A_\nu],\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ A_\nu,\ [\Lambda,\ \mathbf{g},\ \mathbf{g}]\ ] \quad (2.12)$$

Because the ternary bracket is totally antisymmetric under the exchange of any pair of indices, one may exchange the entries in (2.12) as follows

$$[\ \Lambda,\ \mathbf{g},\ [A_\mu,\ A_\nu,\ \mathbf{g}]\ ]\ =\ -\ [\ \Lambda,\ [A_\mu,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ]$$

$$[\Lambda,\ \mathbf{g},\ A_\mu]\ =\ -\ [\Lambda,\ A_\mu,\ \mathbf{g}];\quad [\Lambda,\ \mathbf{g},\ A_\nu]\ =\ -\ [\Lambda,\ A_\nu,\ \mathbf{g}];.... \quad (2.13a)$$

leading to the relation

$$[\ [\Lambda,\ A_\mu,\ \mathbf{g}],\ A_\nu,\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ [\Lambda,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ A_\nu,\ [\Lambda,\ \mathbf{g},\ \mathbf{g}]\ ]\ =$$

$$[\ \Lambda,\ [A_\mu,\ A_\nu,\ \mathbf{g}],\ \mathbf{g}\ ] \quad (2.13b)$$

which is precisely the relation required in order to show that $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations.

Furthermore, by writing $A_\mu = A_\mu^a T_a; \Lambda_1 = \Lambda_1^a T_a; \Lambda_2 = \Lambda_2^b T_b$, ..., after some straightforward algebra, one can verify that the infinitesimal gauge transformations (2.7, 2.8) obey the *closure* conditions

$$(\delta_{\Lambda_2}\ \delta_{\Lambda_1}\ -\ \delta_{\Lambda_1}\ \delta_{\Lambda_2})\ A_\mu\ =\ \delta_{[\Lambda_1,\Lambda_2,\mathbf{g}]}\ A_\mu\ =$$

$$-\ \partial_\mu(\ [\Lambda_1,\ \Lambda_2,\ \mathbf{g}]\ )\ +\ [\ [\Lambda_1,\ \Lambda_2,\ \mathbf{g}],\ A_\mu,\ g\ ]\ =$$

$$\delta_{\Lambda_3}\ A_\mu\ =\ -\ (\partial_\mu \Lambda_3)\ +\ [\ \Lambda_3,\ A_\mu,\ \mathbf{g}\ ];\quad \Lambda_3\ =\ [\ \Lambda_1,\ \Lambda_2,\ \mathbf{g}\ ] \quad (2.14)$$

if, and only if, the 3-Lie algebra-valued coupling $\mathbf{g} = g^a T_a$ is *constant*. The result in (2.14) is a consequence of the fundamental identity, in the particular case that f_{abcd} is totally antisymmetric in all of its indices, leading to a full antisymmetry of the ternary bracket.

The finite ternary gauge transformations can be obtained by "exponentiation" as follows

$$F'\ =\ F\ +\ \delta F\ +\ \frac{1}{2!}\delta(\delta F)\ +\frac{1}{3!}(\delta(\delta(\delta F)))\ +\ .... \quad (2.15a)$$

where

$$\delta F\ =\ [\ \Lambda^a\ T_a,\ F^b_{\mu\nu}\ T_b,\ g^c\ T_c\ ];\ \ \delta(\delta F)\ =\ [\ \Lambda^m\ T_m,\ [\ \Lambda^a\ T_a,\ F^b_{\mu\nu}\ T_b,\ g^c\ T_c\ ],\ g^n\ T_n\ ];\ .... \quad (2.15b)$$

A gauge invariant action under ternary infinitesimal gauge transformations in D-dim is given

$$S\ =\ -\ \frac{1}{4\kappa^2}\ \int\ d^Dx\ <\ F_{\mu\nu}\ F^{\mu\nu}\ > \quad (2.16)$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience.

Under infinitesimal ternary gauge transformations of the ordinary quadratic action one has

$$\delta\ S\ =\ -\frac{1}{4}\ \int\ d^Dx\ <\ F_{\mu\nu}\ (\delta F^{\mu\nu})\ +\ (\delta F_{\mu\nu})\ F^{\mu\nu}\ >\ =$$

$$-\frac{1}{4} \int d^D x \ < \ F^c_{\mu\nu} \ T_c \ [\ \Lambda^a T_a, \ F^{\mu\nu \ b} \ T_b, \ g^n \ T_n] \ > \ +$$

$$-\frac{1}{4} \int d^D x \ < \ [\Lambda^a \ T_a, \ F^b_{\mu\nu} \ T_b, \ g^n \ T_n] \ F^{\mu\nu \ c} \ T_c \ > \ =$$

$$-\frac{1}{4} \int d^D x \ \Lambda^a \ g^n \ F^b_{\mu\nu} \ F^{\mu\nu \ c} \ \left(\ < \ T_c \ f_{abn}{}^k \ T_k \ > \ + \ < \ f_{abn}{}^k \ T_k \ T_c \ > \ \right) \ =$$

$$-\frac{1}{2} \int d^D x \ \Lambda^a \ g^n \ F^b_{\mu\nu} \ F^{\mu\nu \ c} \ f_{abnc} \ = \ 0 \tag{2.17}$$

the last terms terms inside the integrand in eq-(2.17) vanish identically due to the full antisymmetry of $f_{abnc} = -f_{acnb}$, and the symmetry of $F^b_{\mu\nu} \ F^{\mu\nu \ c}$ under the exchange of indices $b \leftrightarrow c$. Therefore, the action is invariant under infinitesimal ternary gauge transformations $\delta S = 0$.

The physical interpretation of the 3-Lie algebra-valued coupling $\mathbf{g} = g^a T_a$ deserves further investigation. As described by [15] one can augment the 3-Lie algebra by including an element T^0 that associates with everything, or more precisely, that satisfies $f^{0ab}{}_d = 0$. If we assume that $h^{0b} = 0$ if $b \neq 0$, one finds that $f_{abc0} = 0$. Since T_0 decouples from the ternary brackets $[A, B, g^0 T_0] = 0$, the physical coupling $g^0 = \textit{constant}$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g^0 [A_\mu, A_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the A_μ field as $F_{\mu\nu} = \frac{1}{g^0} \left(\partial_\mu (g^0 A_\nu) - \partial_\nu (g^0 A_\mu) + [g^0 A_\mu, g^0 A_\nu, \mathbf{g}] \right)$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

Having formulated a gauge invariant action (2.16) the next step is to introduce gauge invariant matter terms like $(D_\mu \Phi)^2$ where $\Phi = \Phi^a T_a$ is 3-Lie algebra-valued scalar and $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \ \Phi, \ \mathbf{g}]$. The derivative $D_\mu \Phi$ transforms homogeneously, when

$$\delta(\Phi^d T_d) \ = \ [\ \Lambda^a T_a, \ \Phi^b \ T_b, \ g^c \ T_c \ ] \ \Rightarrow \ \delta(D_\mu \Phi) \ = \ [\ \Lambda, \ D_\mu \Phi, \ \mathbf{g} \ ] \tag{2.18}$$

The action

$$S \ = \ \int d^4 x \ < \ - \frac{1}{2 \ (g^0)^2} \ F_{\mu\nu} \ F^{\mu\nu} \ + \ \frac{1}{2} \ (D_\mu \Phi)^2 \ > \tag{2.19}$$

is invariant under the infinitesimal gauge transformations given by eqs-(2.11,2.18). To show invariance under finite gauge transformations via the "exponentiation" procedure in eqs-(2.15) is much more cumbersome.

In the next section we shall analyze Nonassociative Octonionic Ternary Gauge Field Theories based on a ternary octonionic product with the fundamental difference, besides the nonassociativity, that the structure constants f_{abcd} are *no* longer totally antisymmetric in their indices. Thus the bracket in the octonion case $[[A, B]] \equiv [A, B, \mathbf{g}]$ is *not* effectively a Lie bracket (as it was in the

3-Lie algebra case in this section) because the bracket $[[A, B]]$ in the octonion case does *not* obey the Jacobi identity because the structure constants f_{abcd} are no longer totally antisymmetric in their indices.

3 Nonassociative Octonionic Ternary Gauge Field Theories

The nonassociative and noncommutative octonionic ternary gauge field theory is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket obeys the fundamental identity (generalized Jacobi identity) and was developed earlier by Yamazaki [14]. Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o\ e_o\ +\ x^m\ e_m, \quad m, n, p = 1, 2, 3, ...7. \tag{3.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_i = e_i e_o = e_i,\ e_i e_j = -\delta_{ij} e_o\ + c_{ijk} e_k,\ i, j, k = 1, 2, 3, ...7. \tag{3.2}$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations $(124), (235), (346), (457), (561), (672), (713)$. The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_m = -e_m$

$$\bar{\mathbf{X}} = x^o\ e_o\ -\ x^m\ e_m. \tag{3.3}$$

and the norm is

$$N(\mathbf{X})\ =\ |\ <\ \mathbf{X}\ \mathbf{X}\ >\ |^{\frac{1}{2}}\ = |\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ |^{\frac{1}{2}}\ = |\ (x_o\ x_o\ + x_k\ x_k)\ |^{\frac{1}{2}}. \tag{3.4}$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{<\ \mathbf{X}\ \mathbf{X}\ >}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{3.5}$$

The non-vanishing associator is defined by

$$(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{3.6}$$

In particular, the associator

$$\begin{aligned} (e_i, e_j, e_k)\ &=\ (e_i e_j) e_k - e_i (e_j e_k)\ =\ 2\ d_{ijkl}\ e_l \\ d_{ijkl}\ &=\ \frac{1}{3!}\ \epsilon_{ijklmnp}\ c^{mnp},\ i, j, k.... = 1, 2, 3, ...7 \end{aligned} \tag{3.7}$$

Yamazaki [14] defined the three-bracket as

$$[\, u,\ v,\ x\,] \ \equiv\ D_{u,v}\, x \ =\ \frac{1}{2}\ (\ u(vx) - v(ux)\ +\ (xv)u\ -\ (xu)v\ +\ u(xv)\ -\ (ux)v\ )\,. \tag{3.8}$$

After a straightforward calculation when the indices span the imaginary elements $a, b, c, d = 1, 2, 3, ..., 7$, and using the relationship [22]

$$c_{abd}\ c_{dcm}\ =\ -\ d_{abcm}\ +\ \delta_{ac}\ \delta_{bm}\ -\ \delta_{bc}\ \delta_{am} \tag{3.9a}$$

the ternary bracket becomes

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcd}\ e_d\ =\ [\ d_{abcd}\ + 2\ \delta_{ac}\ \delta_{bd}\ - 2\ \delta_{bc}\ \delta_{ad}\ ]\ e_d \tag{3.9b}$$

whereas e_0 has a vanishing ternary bracket

$$[\ e_a,\ e_b,\ e_0\ ]\ =\ [\ e_a,\ e_0,\ e_b\ ]\ =\ [\ e_0,\ e_a,\ e_b\ ]\ =\ 0 \tag{3.9c}$$

It is important to note that $f_{abcd} \neq \pm\ c_{abd}\ c_{dcm}$ otherwise one would have been able to *rewrite* the ternary bracket in terms of ordinary 2-brackets as follows $[e_a,\ e_b,\ e_c] \sim \frac{1}{4}[\ [e_a,\ e_b],\ e_c\ ]$.

The ternary bracket (3.8) obeys the fundamental identity

$$[\ [x,\ u,\ v],\ y,\ z\ ] + [\ x,\ [y,\ u,\ v],\ z\ ] + [\ x,\ y,\ [z,\ u,\ v]\ ]\ =\ [\ [x,\ y,\ z],\ u,\ v\ ] \tag{3.10}$$

A bilinear positive symmetric product $< u, v >=< v, u >$ is required such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$<\ [u,\ v,\ x],\ y\ >\ =\ -\ <\ [u,\ v,\ y],\ x\ >\ =\ -\ <\ x,\ [u,\ v,\ y]\ >\ \Rightarrow$$

$$D_{u,v}\ < x, y >\ =\ 0 \tag{3.11}$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [14]

$$<\ [u,\ v,\ x],\ y\ >\ =\ <\ [x,\ y,\ u],\ v\ > \tag{3.12}$$

The ternary product provided by Yamazaki (3.8) *obeys* the key fundamental identity (3.10) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices : $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however : $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -\ f_{dbca}$. The associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (3.10) as emphasized by [14]. For this reason we cannot use the associator to construct the 3-bracket.

We define the field strength in terms of the *ternary* bracket as before

$$F_{\mu\nu}\ =\ \partial_\mu A_\nu\ -\ \partial_\nu A_\mu\ +\ [\ A_\mu,\ A_\nu,\ \mathbf{g}\ ] \tag{3.13}$$

where $\mathbf{g} = g^a e_a$ is an octonionic-valued "coupling" function. Under the infinitesimal ternary gauge transformations $\delta A_\mu = -\partial_\mu \Lambda + [\Lambda, A_\mu, \mathbf{g}] \Rightarrow \delta(F^d_{\mu\nu}\ e_d) = [\Lambda^a(x) e_a, F^b_{\mu\nu} e_b, g^c e_c]$, the ordinary quadratic action

$$S = -\frac{1}{4\kappa^2} \int d^D x < F_{\mu\nu}\ F^{\mu\nu} > \tag{3.14}$$

is *not* invariant under ternary infinitesimal gauge transformations as we shall see next. κ is a suitable dimensionful constant introduced to render the action dimensionless. The octonionic valued field strength is $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, and has *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ i = 1, 2, 3, ..., 7$. The $<\ \ >$ operation extracting the e_0 part is defined as $< XY >= Real(\bar{X} Y) =< YX >= Real(\bar{Y} X)$. Under infinitesimal ternary gauge transformations of the ordinary quadratic action one has

$$\delta\ S = -\frac{1}{4\kappa^2} \int d^D x < F_{\mu\nu}\ (\delta F^{\mu\nu}) + (\delta F_{\mu\nu})\ F^{\mu\nu} > =$$

$$-\frac{1}{4\kappa^2} \int d^D x < F^c_{\mu\nu}\ e_c\ [\ \Lambda^a e_a,\ F^{\mu\nu\ b}\ e_b,\ g^n e_n] > +$$

$$-\frac{1}{4\kappa^2} \int d^D x < [\Lambda^a e_a,\ F^b_{\mu\nu}\ e_b,\ g^n e_n]\ F^{\mu\nu\ c}\ e_c > =$$

$$-\frac{1}{4\kappa^2} \int d^D x\ \Lambda^a\ F^c_{\mu\nu}\ F^{\mu\nu\ b}\ (\ < e_c\ f_{abnk}\ e_k > + < f_{abnk}\ e_k\ e_c > \ ) =$$

$$-\frac{1}{2\kappa^2} \int d^D x\ \Lambda^a\ g^n\ F^c_{\mu\nu}\ F^{\mu\nu\ b}\ f_{abnc} =$$

$$-\frac{1}{\kappa^2} \int d^D x\ \big(\ (\Lambda^a g_a)\ (F^b_{\mu\nu} F^{\mu\nu}_b) - (\Lambda^a F^{\mu\nu}_a)\ (g_c F^c_{\mu\nu})\ \big) \neq 0 \tag{3.15}$$

Hence, because

$$f_{abnc} = (\ d_{abnc} + 2\ \delta_{an}\ \delta_{bc} - 2\ \delta_{bn}\ \delta_{ac}\ ) \tag{3.16}$$

is *not* antisymmetric under the exchange of indices $b \leftrightarrow c:\ f_{abnc} \neq -f_{acnb}$, eq-(3.15) is *not* zero like it was in section **2**. Had f_{abnc} been fully antisymmetric then the variation δS would have been zero due to the fact that $F^c_{\mu\nu}\ F^{\mu\nu\ b}$ is symmetric under $b \leftrightarrow c$. Concluding, in the octonionic ternary algebra case, one has that $\delta S \neq 0$.

Another problem due to the fact that f_{abcd} is *not* totally antisymmetric in all of its indices is that there *is no closure* of the infinitesimal octonionic ternary gauge transformations $\delta A_\mu = -\partial_\mu \Lambda + [\Lambda, A_\mu, \mathbf{g}]$ like it occurs in eq-(2.14). Furthermore, as mentioned earlier, the bracket in the octonion case $[[A, B]] \equiv [A, B, \mathbf{g}]$ is *not* effectively a Lie bracket (as it was in the 3-Lie algebra case) because the bracket $[[A, B]]$ in the octonion case does *not* obey the Jacobi identity because the structure constants f_{abcd} are no longer totally antisymmetric in their indices. Because the associator ternary operation for octonions

$(x,y,z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (3.10) one cannot use the associator to construct the 3-bracket and this rules out the use of the totally antisymmetric d_{abcd}.

Nevertheless, the quadratic Yang-Mills-like action (3.14) is invariant under the global (rigid) transformations defined by

$$\delta(A_\mu^d\ e_d)\ =\ \Lambda^{ab}[\ e_a,\ e_b,\ A_\mu^c\ e_c\ ] \tag{3.17a}$$

and

$$\delta(g^d\ e_d)\ =\ \Lambda^{ab}[\ e_a,\ e_b,\ g^c\ e_c] \tag{3.17b}$$

when one introduces the constant antisymmetric parameters $\Lambda^{ab} = -\Lambda^{ba}$. One may note now that the coupling $g^c e_c$ is *not* inert under the transformations (3.17b). Only the real part g^0 is inert. After some straightforward algebra one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets transforms properly (homogeneously) under the global (rigid) transformations (3.17)

$$\delta(F^m_{\mu\nu}\ e_m)\ =\ \Lambda^{ab}\ [\ e_a,\ e_b,\ F^c_{\mu\nu}\ e_c\ ]\ =\ \Lambda^{ab}\ F^c_{\mu\nu}\ f_{abc}{}^m\ e_m\ \Rightarrow\ \delta F^m_{\mu\nu}\ =\ \Lambda^{ab}\ F^c_{\mu\nu}\ f_{abc}{}^m \tag{3.18}$$

The result (3.18) is a direct consequence of the fundamental identity (3.10) because the 3-bracket (3.8) is defined as a derivation

$$[\ [e_a,\ e_b,\ A_\mu],\ A_\nu,\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ [e_a,\ e_b,\ A_\nu],\ \mathbf{g}\ ]\ +\ [\ A_\mu,\ A_\nu,\ [e_a,\ e_b,\ \mathbf{g}]\ ]\ =$$

$$[\ e_a,\ e_b,\ [A_\mu,\ A_\nu,\ \mathbf{g}]\ ] \tag{3.19}$$

The finite ternary global transformations can be obtained by "exponentiation" as follows

$$F'\ =\ F\ +\ \delta F\ +\ \frac{1}{2!}\delta(\delta F)\ +\frac{1}{3!}(\delta(\delta(\delta F)))\ +\ .... \tag{3.20}$$

where $\delta(F^m_{\mu\nu}\ e_m) = \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]$; $\delta(\delta F) = \Lambda^{mn}[\ e_m, e_n, \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]\ ]$; ...... Given the octonionic valued field strength $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, with *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ \ i = 1,2,3,.....,7$, one can verify that the quadratic action (3.14) is indeed *invariant* under the ternary infinitesimal global (rigid) transformations (3.17)

$$\delta\ S\ =\ -\ \frac{1}{4\kappa^2}\ \int\ d^Dx\ <\ F_{\mu\nu}\ (\delta F^{\mu\nu})\ +\ (\delta F_{\mu\nu})\ F^{\mu\nu}\ >\ =$$

$$-\ \frac{1}{4\kappa^2}\ \int\ d^Dx\ <\ F^c_{\mu\nu}\ e_c\ \Lambda^{ab}\ [e_a,\ e_b,\ F^{\mu\nu\ n}\ e_n]\ >\ +$$

$$-\ \frac{1}{4\kappa^2}\ \int\ d^Dx\ <\ \Lambda^{ab}\ [e_a,\ e_b,\ F^c_{\mu\nu}\ e_c]\ F^{\mu\nu\ n}\ e_n\ >\ =$$

$$-\frac{1}{4\kappa^2}\ \int\ d^Dx\ \Lambda^{ab}\ F^c_{\mu\nu}\ F^{\mu\nu\ n}\ (\ <\ e_c\ f_{abnk}\ e_k\ >\ +\ <\ f_{abck}\ e_k\ e_n\ >\ )\ =\ 0. \tag{3.21}$$

as a result of

$$< e_c \, f_{abnk} \, e_k \; > + < \; f_{abck} \, e_k \, e_n \; > \; = \; f_{abnk} \, \delta_{ck} + \; f_{abck} \, \delta_{kn} \; = \; f_{abnc} + f_{abcn} \; =$$

$$[\; d_{abnc} \; + 2 \; \delta_{an} \; \delta_{bc} \; - 2 \; \delta_{bn} \; \delta_{ac} \;] \; + \; [\; d_{abcn} \; + 2 \; \delta_{ac} \; \delta_{bn} \; - 2 \; \delta_{bc} \; \delta_{an} \;] \; = \; 0 \quad (3.22)$$

because $d_{abnc}+d_{abcn}=0$; $d_{nabc}+d_{cabn}=0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. Invariance $\delta S=0$, only occurs if, and only if, $\delta F=\Lambda^{ab}[e_a,e_b,F^c e_c] \neq \Lambda^{ab}[F^c e_c,e_a,e_b]$. The ordering inside the 3-bracket is crucial. One can check that if one sets $\delta F=\Lambda^{ab}[F^c e_c,e_a,e_b]$, the variation δS leads to a term in the integral which is *not* zero. However, under $\delta F=\Lambda^{ab}[e_a,e_b,F^c e_c]$, the variation δS is indeed zero as shown. This is a consequence of the fact that $[e_a,e_b,e_c]\neq[e_c,e_a,e_b]$ when the 3-bracket is given by eq-(3.8, 3.9b).

To show closure of the global transformations (3.17) one needs to recast them in terms of *derivations* as

$$\delta_1 A_\mu \; = \; \delta_1 (A^k_\mu \; e_k) \; = \; \Lambda^{ab}_1 [\; e_a, \; e_b, \; A^c_\mu \; e_c \;] \; = \; \Lambda^{ab}_1 \; D_{e_a,e_b} \; A_\mu \qquad (3.23)$$

$$\delta_2 A_\mu \; = \; \delta_2 (A^k_\mu \; e_k) \; = \; \Lambda^{cd}_2 [\; e_c, \; e_d, \; A^l_\mu \; e_l \;] \; = \; \Lambda^{cd}_2 \; D_{e_c,e_d} \; A_\mu \qquad (3.24)$$

so that, by recurring to the fundamental identity (3.10) in order to evaluate the commutator of two derivations and after relabeling indices, one arrives at

$$[\delta_1, \; \delta_2] \, A_\mu \; = \; \Lambda^{cd}_2 \, \Lambda^{ab}_1 \, [\, D_{e_c,e_d}, \; D_{e_a,e_b} \,] \, A_\mu \; = \; \Lambda^{cd}_2 \, \Lambda^{ab}_1 \; \left(\; D_{[e_c,e_d,e_a],e_b} \; + \; D_{e_a,[e_c,e_d,e_b]} \; \right) \; A_\mu \; =$$

$$\Lambda^{cd}_2 \; \Lambda^{ab}_1 \; (\; [\; [e_c,e_d,e_a], \; e_b, \; A_\mu \;] \; + \; [\; e_a, \; [e_c,e_d,e_b], \; A_\mu \;] \;) \; =$$

$$\Lambda^{cd}_2 \; \Lambda^{ab}_1 \; f_{cdak} \; [\; e_k, \; e_b, \; A_\mu \;] \; + \; \Lambda^{cd}_2 \; \Lambda^{ab}_1 \; f_{cdbk} \; [\; e_a, \; e_k, \; A_\mu \;] \; =$$

$$- \; (\; \Lambda^{cd}_2 \; \Lambda^{ak}_1 \; f_{cda}{}^b \; - \; \Lambda^{cd}_2 \; \Lambda^{ab}_1 \; f_{cda}{}^k \;) \, [\, e_k, \; e_b, \; A_\mu \;] \; = \; \Lambda^{kb}_3 \; [\; e_k, \; e_b, \; A_\mu \;] \; = \; \delta_3 A_\mu \qquad (3.25)$$

Therefore the (constant) antisymmetric parameter resulting from the closure of two global transformations is given by

$$\Lambda^{kb}_3 \; = \; - \; (\; \Lambda^{cd}_2 \; \Lambda^{ak}_1 \; f_{cda}{}^b \; - \; \Lambda^{cd}_2 \; \Lambda^{ab}_1 \; f_{cda}{}^k \;) \qquad (3.26)$$

To show that the action is invariant under finite ternary global transformations requires to follow a few steps. Firstly, one defines

$$< \; x \, y \; > \; \equiv \; Real \, [\, \bar{x} \, y \,] \; = \; \frac{1}{2} \, (\, \bar{x} \, y + \bar{y} \, x \,) \; \Rightarrow \; < \; x \, y \; > \; = \; < \; y \, x \; > \quad (3.27)$$

Despite nonassociativity, the *very special conditions*

$$x(\bar{x}u) \; = \; (x\bar{x})u; \quad x(u\bar{x}) \; = \; (xu)\bar{x}; \quad x(xu) \; = \; (xx)u; \quad x(ux) \; = \; (xu)x \quad (3.28)$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z\neq x(yz)$ one has that their real parts obey

$$Real \; [\; (x \; y) \; z \;] \; = \; Real \; [x \; (y \; z) \;] \qquad (3.29)$$

Due to the nonassociativity of the algebra, in general one has that $(UF)U^{-1} \neq U(FU^{-1})$. However, if and only if $U^{-1} = \bar{U} \Rightarrow \bar{U}U = U\bar{U} = 1$, as a result of the the *very special conditions* (3.28) one has that $F' = (UF)U^{-1} = U(FU^{-1}) = UFU^{-1} = UF\bar{U}$ is *unambiguously* defined. One can equate the result of the exponentiation procedure in eq-(3.20) to the expression

$$F' \ = \ UFU^{-1} \ = \ UF\bar{U} \ = \ e^{\Sigma^k(\Lambda^{ab})e_k} \ (F^c \ t_c) \ e^{-\Sigma^k(\Lambda^{ab})e_k}; \quad k = 1,2,3,....,7. \tag{3.30}$$

where $\Sigma^k(\Lambda^{ab})e_k$ is a complicated function of Λ^{ab}. It yields the finite global transformations which agree with the infinitesimal *ternary* ones when Λ^{ab} are *infinitesimals*. For instance, to lowest order in Λ^{ab}, one has that Σ^k satisfies $2\Sigma^k c_{kcd} = \Lambda^{ab} f_{abcd}$ and which follows by comparing the transformations in (3.17) to those in (3.20), to lowest order.

In ordinary associative Yang-Mills involving 2-brackets, it is well known that the finite gauge transformations are

$$(F^n_{\mu\nu})' \ T_n \ = \ e^{i\Lambda^m T_m} \ F^n_{\mu\nu} T_n \ e^{-i\Lambda^m T_m}. \tag{3.31}$$

where T_m are the Hermitian Lie-algebra generators obeying the commutation relations $[T_m, T_n] = i f_{mnp} T_p$. It is a challenging work to derive the explicit functional dependence $\Sigma^k(\Lambda^{ab})e_k$ in eq-(3.30) that matches the transformation in eq-(3.20), to all orders in Λ^{ab}, for the ternary-brackets case.

Dropping the spacetime indices for convenience in the expressions for $F^{\mu\nu}, F_{\mu\nu}$, and by repeated use of eqs-(3.28, 3.29), when $U^{-1} = \bar{U}$, the action density is also invariant under (unambiguously defined) transformations of the form $F' = UFU^{-1} = UF\bar{U}$,

$$< \ F' \ F' \ > \ = \ Re\,[\bar{F}' \ F'] \ = \ Re\,[(U\bar{F}U^{-1})\,(UFU^{-1})] \ = \ Re\,[(U\bar{F})\,(\ U^{-1}\,(UF\,U^{-1})\ )] \ =$$

$$Re\,[(U \ \bar{F}) \ (U^{-1} \ U) \ (FU^{-1})] \ = \ Re\,[(U\bar{F}) \ (FU^{-1})] \ = \ Re\,[(FU^{-1}) \ (U\bar{F})] \ =$$

$$Re\,[F\,(\ U^{-1}\,(U\,\bar{F})\ )] \ = \ Re\,[F\,(U^{-1}U)\,\bar{F}] \ = \ Re\,[F\,\bar{F}] \ = \ Re\,[\bar{F}\,F] \ = \ < \ F\,F \ > . \tag{3.32}$$

The real part of the coupling g^0 is inert under global transformations (3.17b) and it decouples from the definition of the field strength $F_{\mu\nu}$ because e_0 has a vanishing 3-bracket with other elements of the octonion algebra. The coupling $g^0 = \textit{constant}$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g^0[A_\mu, A_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the A_μ field as $F_{\mu\nu} = \frac{1}{g^0}\left(\partial_\mu(g^0 A_\nu) - \partial_\nu(g^0 A_\mu) + [g^0 A_\mu, g^0 A_\nu, \mathbf{g}]\right)$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

The motivation in constructing an octonionic-valued field strength in terms of ternary brackets is because the ordinary 2-bracket does *not* obey the Jacobi identity

$$[\ e_i,\ [\ e_j,\ e_k\ ]\ ] + [\ e_j,\ [\ e_k,\ e_i\ ]\ ] + [\ e_k,\ [\ e_i,\ e_j\ ]\ ] \ = \ 3\ d_{ijkl}\ e_l \ \neq \ 0 \quad (3.33)$$

If one has the ordinary Yang-Mills expression for the field strength

$$F_{\mu\nu} \ = \ \partial_\mu A_\nu \ - \ \partial_\nu A_\mu \ + \ [\ A_\mu,\ A_\nu\ ] \quad (3.34)$$

because the 2-bracket does *not* obey the Jacobi identity, one has an extra (spurious) term in the expression for

$$[\ D_\mu,\ D_\nu\ ]\ \Phi \ = \ [\ F_{\mu\nu},\ \Phi\ ] \ + \ (\ A_\mu,\ A_\nu,\ \Phi\ ) \quad (3.35)$$

given by the crucial contribution of the non-vanishing associator $(A_\mu, A_\nu, \Phi) = (A_\mu A_\nu)\Phi - A_\mu(A_\nu\Phi) \neq 0$. For this reason, due to the non-vanishing condition (3.33), the ordinary Yang-Mills field strength does *not* transform homogeneously under ordinary gauge transformations involving the parameters $\Lambda = \Lambda^a e_a$

$$\delta A_\mu = \ \partial_\mu \Lambda \ + \ [A_\mu, \Lambda] \quad (3.36)$$

and it yields an extra contribution of the form

$$\delta F_{\mu\nu} \ = \ [F_{\mu\nu}, \Lambda] \ + \ (\ \Lambda,\ A_\mu,\ A_\nu) \quad (3.37)$$

As a result of the additional contribution (Λ, A_μ, A_ν) in eq-(3.38), the ordinary Yang-Mills action $S = \int < F_{\mu\nu} F^{\mu\nu} >$ will *no* longer be gauge invariant. Under infinitesimal variations eqs-(3.37), the variation of the action is *no* longer zero but receives spurious contributions of the form $\delta S = -4 F^l_{\mu\nu} \Lambda^i A^{\mu j} A^{\nu k} d_{ijkl} \neq 0$ due to the non-associativity of the octonion algebra.

On the other hand, by promoting the constant antisymmetric parameters Λ^{ab} in the transformations (3.17) to x^μ-dependent ones $\Lambda^{ab}(x^\mu)$ leads to *inhomogeneous* terms $(\partial_{[\mu} \Lambda^{ab}(x)) A^c_{\nu]} f_{abcd} e_d$ in the transformations laws for the field strength $F^d_{\mu\nu} e_d$ and which will spoil the gauge invariance of the quadratic action (3.14). One may impose the gauge *covariant* constraints

$$(\partial_{[\mu} \Lambda^{ab}(x))\ A^c_{\nu]}\ f_{abcd} \ = \ 0, \quad a, b, c, d = 1, 2, 3, ..., 7 \quad (3.39)$$

which lead to field-dependent gauge parameters $\Lambda^{ab}(A_\mu)$ and nonlinear gauge transformations. Performing a second gauge variation on the gauge covariant constraints, by varying only the gauge fields and not the parameters in eq-(3.39), yields

$$(\partial_{[\mu} \Lambda^{ab}(x))\ \delta A^c_{\nu]}\ f_{abcd} \ = \ (\partial_{[\mu} \Lambda^{ab}(x))\ (\Lambda^{ij}(x)\ A^k_{\nu]}\ f_{ijk}{}^c)\ f_{abcd} \ = \ 0 \quad (3.40)$$

eqs-(3.40) are compatible with eqs-(3.39) when

$$\delta A^c_\nu \ = \ \Lambda^{ij}(x)\ A^k_\nu\ f_{ijk}{}^c \ = \ \Lambda(x)\ A^c_\nu \ \Rightarrow$$

$$A_\nu^{c'} \ - \ A_\nu^c \ = \ \Lambda(x) \ A_\nu^c \ \Rightarrow \ A_\nu^{c'} \ = \ (1 \ + \ \Lambda(x)) \ A_\nu^c; \quad c = 1,2,3,....,7 \tag{3.41}$$

where $\Lambda(x)$ is another spacetime-dependent parameter. Thus, the conditions (3.41) yield a homothecy (spacetime dependent) scaling of the gauge fields. Eqs-(3.39, 3.41) determine the functional dependence of the parameters $\Lambda^{ab}(x), \Lambda(x)$ in terms of the gauge fields A_μ. One still maintains *closure* of two gauge transformations (3.17), by varying only the gauge fields and not the parameters, given by eqs-(3.25, 3.26), when the gauge parameters $\Lambda_1^{ab}(x), \Lambda_2^{cd}(x)$ are spacetime-dependent. The scaling transformations can be exponentiated furnishing $A_\nu^{c'} = e^{\Lambda(x)} A_\nu^c$. The field-dependent parameters can be turned into infinitesimal ones by multiplying them by an infinitesimal ϵ as $\Lambda^{ab}[A_\mu] \rightarrow \epsilon\Lambda^{ab}[A_\mu]$; $\Lambda[A_\mu] \rightarrow \epsilon\Lambda[A_\mu]$.

To sum up, one can promote the constant antisymmetric parameters Λ^{ab} in the transformations (3.17) to x^μ-dependent ones $\Lambda^{ab}(x^\mu)$ such that the *inhomogeneous* terms $(\partial_{[\mu}\Lambda^{ab}(x))A^c_{\nu]} f_{abcd} e_d$ in the transformations laws for the field strength $F^d_{\mu\nu} e_d$ can be set to *zero* if, and only if, one imposes the gauge *covariant* constraints (3.39) that lead to field-dependent gauge parameters $\Lambda^{ab}(A_\mu)$; i.e. one has nonlinear gauge transformations. In this fashion one does not spoil the gauge invariance of the quadratic action (3.14) under the *restricted* set of gauge transformations resulting in the homothecy/scaling of the fields (3.41). One should note that the scaling parameter $\Lambda(x)$ is not arbitrary but field-dependent due to the gauge covariant constraints (3.39) and that $\delta F^d_{\mu\nu} = \Lambda^{ab} F^c_{\mu\nu} f_{abc}{}^d \neq \Lambda(x) F^d_{\mu\nu}$ despite that $\delta A^d_\mu = \Lambda(x) \ A^d_\mu$.

To finalize we discuss further constructions, like having an octonionic-valued and $SU(N)$-valued gauge field $\mathbf{A}_\mu = A^{am}_\mu (e_a \otimes T_m)$ involving the $SU(N)$ algebra generators $T_m, m = 1,2,3,...., N^2 - 1$ and the octonion algebra generators $e_a, a = 0,1,2,3,....,7$; i.e. one has octonionic-valued components for the $SU(N)$ gauge fields. The commutator is

$$[\ \mathbf{A}_\mu, \ \mathbf{A}_\nu \ ] \ = \ [\ A^{am}_\mu \ (e_a \otimes T_m), \ A^{bn}_\nu \ (e_b \otimes T_n) \ ] \ =$$

$$\frac{1}{2} \ A^{am}_\mu \ A^{bn}_\nu \ \{e_a, \ e_b\} \otimes [T_m, \ T_n] \ + \frac{1}{2} \ A^{am}_\mu \ A^{bn}_\nu \ [e_a, \ e_b] \otimes \{T_m, \ T_n\} \tag{3.42}$$

where

$$\{e_a, \ e_b\} \ = \ -2 \ \delta_{ab} \ e_o, \quad [e_a, \ e_b] \ = \ 2 \ c_{abc} \ e_c \tag{3.43}$$

and for the $SU(N)$ Hermitian generators one has

$$\{T_m, \ T_n\} \ = \ \frac{1}{N} \ \delta_{mn} \ + \ d_{mnp} \ T_p, \quad [T_m, \ T_n] \ = \ i \ f_{mnp} \ T_p \tag{3.44}$$

One may note that the r.h.s of (3.42) involves both commutators and anticommutators as it occurs in Supersymmetry. Due to the fact that the octonion algebra does not obey the Jacobi identities this will spoil the gauge invariance of typical Yang-Mills actions as described above. Let us have instead a ternary Lie algebra (3-Lie algebra) obeying the ternary commutation relations

$$[\ T_m, \ T_n, \ T_p \ ] \ = \ f_{mnpq} \ T_q \tag{3.45}$$

and such that the ternary-bracket structure-constants f_{mnpq} obey the fundamental identity. A 3-Lie-algebra and octonionic-valued field is defined by $\mathbf{A}_\mu \equiv A_\mu^{ma}\ (T_m \otimes e_a)$. However, the triple commutator

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ =\ [\ A_\mu^{mi}\ (T_m \otimes e_i),\ A_\nu^{nj}\ (T_n \otimes e_j),\ A_\rho^{pk}\ (T_p \otimes e_k)\ ] \quad (3.46)$$

would furnish a very *complicated* expression for the r.h.s of eq-(3.46). To simplify matters one could define the ternary bracket as

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ \equiv\ A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ [T_m,\ T_n,\ T_p] \otimes [e_i, e_j, e_k]\ =$$

$$A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ f_{mnpq}\ f_{ijkl}\ (T_q \otimes e_l) \quad (3.47)$$

so that one has closure in the r.h.s of eq-(3.47).

To conclude, if one were able to find an octonionic ternary bracket that is totally antisymmetric in all of its indices, and which satisfies the fundamental identity, one would be able to construct a nonassociative octonionic ternary gauge field theory and build an invariant quadratic action. However in this case one would effectively recover a Lie bracket structure $[[A, B]] \equiv [A, B, \mathbf{g}]$, obeying the Jacobi identity when the structure constants f_{abcd} are totally antisymmetric in their indices. The ternary bracket [4] is totally antisymmetric but it does not obey the fundamental identity. It is warranted to explore further these generalized ternary gauge field theories involving 3-Lie algebras and octonions in M-theory.

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