

On Octonionic Nonassociative Ternary Gauge Field Theories

Carlos Castro

Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

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Abstract

A novel (to our knowledge) nonassociative Octonionic ternary gauge field theory is explicitly constructed based on a ternary-bracket structure involving the octonion algebra. The ternary bracket was defined earlier by Yamazaki. The antisymmetric rank-two field strength $F_{\mu\nu}$ is defined in terms of the ternary-bracket $[B_\mu, B_\nu, \mathbf{g}]$ involving an auxiliary octonionic-valued coupling $\mathbf{g} = g^a e_a$. The ternary bracket cannot be rewritten in terms of 2-brackets, $[A, B, C] \neq \frac{1}{4}[[A, B], C]$. It is found that gauge-invariant matter kinetic terms for an octonionic-valued scalar field can be introduced in the action if one starts instead with an octonionic-valued rank-three antisymmetric field strength $F_{\mu\nu\rho} = \partial_\rho A_{\mu\nu} + [A_{\mu\nu}, A_\rho, \mathbf{g}] +$ permutations, which is defined in terms of an antisymmetric tensor field of rank two $A_{\mu\nu} = A_{\mu\nu}^a e_a$ and $A_\mu = A_\mu^a e_a$. We conclude with some preliminary steps towards the construction of generalized ternary gauge field theories involving *both* 3-Lie algebras and octonions.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [7], [8], for references, among many others. A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [5], [9], [10]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative.

Recently, tremendous activity has been launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [12], [13] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$ -branes. The original BLG theory requires the algebraic structures of generalized

Lie 3-algebras and also of nonassociative algebras. Later developments by [14] provided a 3D Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [14] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [15], [16].

A Nonassociative Gauge theory based on the Moufang S^7 loop product (not a Lie algebra) has been constructed by [17]. Taking the algebra of octonions with a unit norm as the Moufang S^7 -loop, one reproduces a nonassociative octonionic gauge theory which is a generalization of the Maxwell and Yang-Mills gauge theories based on Lie algebras. *BPST*-like instantons solutions in $D = 8$ were also found. These solutions represented the physical degrees of freedom of the transverse 8-dimensions of superstring solitons in $D = 10$ preserving one and two of the 16 spacetime supersymmetries. Nonassociative deformations of Yang-Mills Gauge theories involving the left and right bimodules of the octonionic algebra were presented by [18].

The novel (to our knowledge) nonassociative octonionic ternary gauge theory developed in this work differs from the nonassociative gauge theories of [17] in many respects, mainly that it is based on a ternary bracket involving the octonion algebra that was proposed by Yamazaki [11]. It also differs from the work by [12], [13] in that our octonionic-valued gauge fields $B_\mu^a e_a$; $a = 0, 1, 2, \dots, 7$ are not, and cannot be represented, in terms of matrices $\mathbf{A}_\mu = A_\mu^{ab} f_{ab}^{cd} = (\tilde{A}_\mu)^{cd}$, defined in terms of f_{ab}^{cd} which are the structure constants of the 3-Lie algebra $[t_a, t_b, t^c] = f_{ab}^{cd} t_d$. This construction is not unlike writing the matrices $\mathbf{A}_\mu = A_\mu^a f_a^{bc} = (A_\mu)^{bc}$ of ordinary Yang-Mills gauge theory in terms of the adjoint representation of the gauge algebra : $[t_a, t_b] = f_{ab}^{cd} t_c$. Furthermore, our field strengths $F_{\mu\nu}$ are explicitly defined in terms of a 3-bracket $[B_\mu, B_\nu, \mathbf{g}]$ involving an auxiliary octonionic-valued scalar field $\mathbf{g} = g^a e_a$ which plays the role of a "coupling" function. Whereas the definition of $F_{\mu\nu}$ by [12], [13] was based on the standard commutator of the matrices $(\tilde{A}_\mu)_c^a (\tilde{A}_\nu)_b^c - (\tilde{A}_\nu)_c^a (\tilde{A}_\mu)_b^c$.

The nonassociative and noncommutative octonionic ternary gauge field theory is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket obeys the fundamental identity (generalized Jacobi identity) and was developed earlier by Yamazaki [11]. Given an octonion \mathbf{X} it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o e_o + x^m e_m, \quad m, n, p = 1, 2, 3, \dots, 7. \quad (1)$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o, \quad e_o e_i = e_i e_o = e_i, \quad e_i e_j = -\delta_{ij} e_o + c_{ijk} e_k, \quad i, j, k = 1, 2, 3, \dots, 7. \quad (2)$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations (124), (235), (346), (457), (561), (672), (713). The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_m = -e_m$

$$\bar{\mathbf{X}} = x^o e_o - x^m e_m. \quad (3)$$

and the norm is

$$N(\mathbf{X}) = | \langle \mathbf{X} \mathbf{X} \rangle |^{\frac{1}{2}} = | \text{Real} (\bar{\mathbf{X}} \mathbf{X}) |^{\frac{1}{2}} = | (x_o x_o + x_k x_k) |^{\frac{1}{2}}. \quad (4)$$

The inverse

$$\mathbf{X}^{-1} = \frac{\bar{\mathbf{X}}}{\langle \mathbf{X} \mathbf{X} \rangle}, \quad \mathbf{X}^{-1} \mathbf{X} = \mathbf{X} \mathbf{X}^{-1} = 1. \quad (5)$$

The non-vanishing associator is defined by

$$(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \quad (6)$$

In particular, the associator

$$\begin{aligned} (e_i, e_j, e_k) &= (e_i e_j) e_k - e_i (e_j e_k) = 2 d_{ijkl} e_l \\ d_{ijkl} &= \frac{1}{3!} \epsilon_{ijklmnp} c^{mnp}, \quad i, j, k, \dots = 1, 2, 3, \dots, 7 \end{aligned} \quad (7)$$

Yamazaki [11] defined the three-bracket as

$$[u, v, x] \equiv D_{u,v} x = \frac{1}{2} (u(vx) - v(ux) + (xv)u - (xu)v + u(xv) - (ux)v). \quad (8)$$

After a straightforward calculation when the indices span the imaginary elements $a, b, c, d = 1, 2, 3, \dots, 7$, and using the relationship [21]

$$c_{abd} c_{dcm} = - d_{abcm} + \delta_{ac} \delta_{bm} - \delta_{bc} \delta_{am} \quad (9a)$$

the ternary bracket becomes

$$[e_a, e_b, e_c] = f_{abcd} e_d = [d_{abcd} + 2 \delta_{ac} \delta_{bd} - 2 \delta_{bc} \delta_{ad}] e_d \quad (9b)$$

whereas e_0 has a vanishing ternary bracket

$$[e_a, e_b, e_0] = [e_a, e_0, e_b] = [e_0, e_a, e_b] = 0 \quad (9c)$$

It is important to note that $f_{abcd} \neq \pm c_{abd} c_{dcm}$ otherwise one would have been able to *rewrite* the ternary bracket in terms of ordinary 2-brackets as follows $[e_a, e_b, e_c] \sim \frac{1}{4} [[e_a, e_b], e_c]$.

The ternary bracket (8) obeys the fundamental identity

$$[[x, u, v], y, z] + [x, [y, u, v], z] + [x, y, [z, u, v]] = [[x, y, z], u, v] \quad (10)$$

A bilinear positive symmetric product $\langle u, v \rangle = \langle v, u \rangle$ is required such that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$\begin{aligned} \langle [u, v, x], y \rangle &= - \langle [u, v, y], x \rangle = - \langle x, [u, v, y] \rangle \Rightarrow \\ D_{u,v} \langle x, y \rangle &= 0 \end{aligned} \quad (11)$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [11]

$$\langle [u, v, x], y \rangle = \langle [x, y, u], y \rangle \quad (12)$$

The ternary product provided by Yamazaki (8) *obeys* the key fundamental identity (10) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices: $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however: $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -f_{dbca}$. The associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (10) as emphasized by [11]. For this reason we cannot use the associator to construct the 3-bracket.

We define the field strength in terms of the *ternary* bracket as

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu, \mathbf{g}] \quad (13)$$

where $\mathbf{g} = g^a e_a$ is an octonionic-valued "coupling" function which is not inert under octonionic gauge transformations. Only the scalar part of \mathbf{g} remains invariant. Under the local gauge transformations

$$\delta(B_\mu^m e_m) = - (\partial_\mu \Lambda^m(x)) e_m + \Lambda^{ab}(x) [e_a, e_b, B_\mu^c e_c] \quad (14)$$

$$\delta(g^m e_m) = \Lambda^{ab}(x) [e_a, e_b, g^c e_c] \quad (15)$$

and after some straightforward algebra involving an exchange of indices, one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets (8, 9b) transforms properly (homogeneously) under the ternary gauge transformations if, and only if, the gauge parameters (functions) $\Lambda^m(x); \Lambda^{ab}(x)$ are field-dependent and obey the following relationship

$$(\partial_{[\mu} \Lambda^{ab}) B_{\nu]}^c - (\partial_{[\mu} \Lambda^a) B_{\nu]}^b g^c = 0 \quad (16a)$$

involving the coupling function components g^c and the B_μ^c field. The antisymmetrization of indices $[\mu\nu]$ is performed with unit weight. Since the first term in (16a) is also antisymmetric in the internal ab indices, the second term in (16a) should be antisymmetric as well, and one can infer that $\partial_\mu \Lambda^a = B_\mu^a \Rightarrow \Lambda^a(x) = \int B_\mu^a dx^\mu$. Hence one has found the field-dependent expression for $\Lambda^a(x)$ that leads to the field-dependent relationship for $\Lambda^{ab}(x)$ in terms of $B_\mu^a(x)$ and $g^c(x)$ given by

$$(\partial_{[\mu} \Lambda^{ab}) B_{\nu]}^c - B_{[\mu}^a B_{\nu]}^b g^c = 0 \quad (16b)$$

The field dependence of the gauge parameter $\Lambda^a(x)$ does not occur in ordinary Yang-Mills theory because it is based on 2-brackets. Therefore, the gauge transformations themselves, involving ternary brackets, are now highly *nonlinear*

in the fields. This is one of the key differences. To sum up, if eq-(16a) is obeyed then $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F_{\mu\nu}^m e_m) = \Lambda^{ab} [e_a, e_b, F_{\mu\nu}^c e_c] = \Lambda^{ab} F_{\mu\nu}^c f_{abc}{}^m e_m \Rightarrow \delta F_{\mu\nu}^m = \Lambda^{ab} F_{\mu\nu}^c f_{abc}{}^m \quad (17)$$

The result (17) is a direct consequence of the fundamental identity (10) because the 3-bracket (8) is defined as a derivation

$$[[e_a, e_b, B_\mu], B_\nu, \mathbf{g}] + [B_\mu, [e_a, e_b, B_\nu], \mathbf{g}] + [B_\mu, B_\nu, [e_a, e_b, \mathbf{g}]] = [e_a, e_b, [B_\mu, B_\nu, \mathbf{g}]] \quad (18)$$

The parameter (function) $\Lambda^0(x)$ involved in the transformation $\delta B_\mu^0 = \partial_\mu \Lambda^0(x)$, corresponding to the real (identity) element e_0 of the octonion algebra leads to $\delta F_{\mu\nu}^0 = 0$, where the field strength component is Abelian-Maxwell-like $F_{\mu\nu}^0 = \partial_\mu B_\nu^0 - \partial_\nu B_\mu^0$. The finite ternary transformations can be obtained by "exponentiation" as follows

$$F' = F + \delta F + \frac{1}{2!} \delta(\delta F) + \frac{1}{3!} (\delta(\delta(\delta F))) + \dots \quad (19)$$

where $\delta(F_{\mu\nu}^m e_m) = \Lambda^{ab} [e_a, e_b, F_{\mu\nu}^c e_c]$; $\delta(\delta F) = \Lambda^{mn} [e_m, e_n, \Lambda^{ab} [e_a, e_b, F_{\mu\nu}^c e_c]]$; Given the octonionic valued field strength $F_{\mu\nu} = F_{\mu\nu}^a e_a$, with *real valued* components $F_{\mu\nu}^0, F_{\mu\nu}^i$; $i = 1, 2, 3, \dots, 7$, a gauge invariant action under ternary infinitesimal gauge transformations in D -dim is

$$S = - \frac{1}{4\kappa^2} \int d^D x \langle F_{\mu\nu} F^{\mu\nu} \rangle \quad (20)$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience. The $\langle \rangle$ operation extracting the e_0 part is defined as $\langle XY \rangle = \text{Real}(\bar{X}Y) = \langle YX \rangle = \text{Real}(YX)$. Under infinitesimal ternary gauge transformations of the action one has

$$\begin{aligned} \delta S &= - \frac{1}{4} \int d^D x \langle F_{\mu\nu} (\delta F^{\mu\nu}) + (\delta F_{\mu\nu}) F^{\mu\nu} \rangle = \\ &= - \frac{1}{4} \int d^D x \langle F_{\mu\nu}^c e_c \Lambda^{ab} [e_a, e_b, F^{\mu\nu n} e_n] \rangle + \\ &= - \frac{1}{4} \int d^D x \langle \Lambda^{ab} [e_a, e_b, F_{\mu\nu}^c e_c] F^{\mu\nu n} e_n \rangle = \\ &= - \frac{1}{4} \int d^D x \Lambda^{ab} F_{\mu\nu}^c F^{\mu\nu n} (\langle e_c f_{abnk} e_k \rangle + \langle f_{abck} e_k e_n \rangle) = 0. \end{aligned} \quad (21)$$

since

$$\langle e_c f_{abnk} e_k \rangle + \langle f_{abck} e_k e_n \rangle = f_{abnk} \delta_{ck} + f_{abck} \delta_{kn} = f_{abnc} + f_{abcn} =$$

$$[d_{abnc} + 2 \delta_{an} \delta_{bc} - 2 \delta_{bn} \delta_{ac}] + [d_{abcn} + 2 \delta_{ac} \delta_{bn} - 2 \delta_{bc} \delta_{an}] = 0 \quad (22)$$

because $d_{abnc} + d_{abcn} = 0$; $d_{nabc} + d_{cabn} = 0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. Invariance $\delta S = 0$, only occurs if, and only if, $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c] \neq \Lambda^{ab}[F^c e_c, e_a, e_b]$. The ordering inside the 3-bracket is crucial. One can check that if one sets $\delta F = \Lambda^{ab}[F^c e_c, e_a, e_b]$, the variation δS leads to a term in the integral which is *not* zero. However, under $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c]$, the variation δS is indeed zero as shown. This is a consequence of the fact that $[e_a, e_b, e_c] \neq [e_c, e_a, e_b]$ when the 3-bracket is given by eq-(8).

To show that the action is invariant under finite ternary gauge transformations requires to follow a few steps. Firstly, one defines

$$\langle x y \rangle \equiv \text{Real} [\bar{x} y] = \frac{1}{2} (\bar{x} y + \bar{y} x) \Rightarrow \langle x y \rangle = \langle y x \rangle \quad (23)$$

Despite nonassociativity, the *very special conditions*

$$x(\bar{x}u) = (x\bar{x})u; \quad x(u\bar{x}) = (xu)\bar{x}; \quad x(xu) = (xx)u; \quad x(ux) = (xu)x \quad (24)$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z \neq x(yz)$ one has that their real parts obey

$$\text{Real} [(x y) z] = \text{Real} [x (y z)] \quad (25)$$

Due to the nonassociativity of the algebra, in general one has that $(UF)U^{-1} \neq U(FU^{-1})$. However, if and only if $U^{-1} = \bar{U} \Rightarrow \bar{U}U = U\bar{U} = 1$, as a result of the the *very special conditions* (24) one has that $F' = (UF)U^{-1} = U(FU^{-1}) = UFU^{-1} = UF\bar{U}$ is *unambiguously* defined. One can equate the result of the exponentiation procedure in eq-(19) to the expression

$$F' = UFU^{-1} = UF\bar{U} = e^{\Sigma^k(\Lambda^{ab})e_k} (F^c t_c) e^{-\Sigma^k(\Lambda^{ab})e_k}; \quad k = 1, 2, 3, \dots, 7. \quad (26)$$

where $\Sigma^k(\Lambda^{ab})e_k$ is a complicated function of Λ^{ab} . It yields the finite gauge transformations which agree with the infinitesimal *ternary* ones when $\Lambda^{ab}(x)$ are *infinitesimals*. For instance, to lowest order in Λ^{ab} , one has that Σ^k satisfies $2\Sigma^k c_{kcd} = \Lambda^{ab} f_{abcd}$ and which follows by comparing the transformations in (19) to those in (26), to lowest order.

In ordinary associative Yang-Mills involving 2-brackets, it is well known that the finite gauge transformations are

$$(F_{\mu\nu}^n)' T_n = e^{i\Lambda^m T_m} F_{\mu\nu}^n T_n e^{-i\Lambda^m T_m}. \quad (27)$$

where T_m are the Hermitian Lie-algebra generators obeying the commutation relations $[T_m, T_n] = i f_{mnp} T_p$. It is a challenging work to derive the explicit functional dependence $\Sigma^k(\Lambda^{ab})e_k$ in eq-(26) that matches the transformation in eq-(19), to all orders in Λ^{ab} , for the ternary-brackets case.

Dropping the spacetime indices for convenience in the expressions for $F^{\mu\nu}, F_{\mu\nu}$, and by repeated use of eqs-(24-25), when $U^{-1} = \bar{U}$, the action density is also invariant under (unambiguously defined) gauge transformations of the form $F' = UFU^{-1} = U\bar{F}U$,

$$\begin{aligned} \langle F' F' \rangle &= \text{Re} [\bar{F}' F'] = \text{Re} [(U\bar{F}U^{-1})(UFU^{-1})] = \text{Re} [(U\bar{F})(U^{-1}(UFU^{-1}))] = \\ &\text{Re} [(U\bar{F})(U^{-1}U)(FU^{-1})] = \text{Re} [(U\bar{F})(FU^{-1})] = \text{Re} [(FU^{-1})(U\bar{F})] = \\ &\text{Re} [F(U^{-1}(U\bar{F}))] = \text{Re} [F(U^{-1}U)\bar{F}] = \text{Re} [F\bar{F}] = \text{Re} [\bar{F}F] = \langle FF \rangle. \end{aligned} \quad (28)$$

If the action (20) is invariant under finite ternary gauge transformations one can impose the condition $S[A_\mu^a; g^a] = S[(A_\mu^a)'; (g^a)' = C^a]$, where $\mathbf{C} = C^a e_a$ is a constant octonionic-valued coupling which can be obtained from gauging the octonionic-valued coupling function $\mathbf{g}(x)$ to a constant \mathbf{C} . The physical interpretation of the octonionic-valued coupling $\mathbf{g} = g^a e_a$ deserves further investigation. The real part of the coupling g^0 can be set to a constant, since g^0 is inert under gauge transformations, and it decouples from the definition of the field strength $F_{\mu\nu}$ because e_0 has a vanishing 3-bracket with other elements of the octonion algebra. The coupling $g^0 = \text{constant}$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g^0 [B_\mu, B_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the B_μ field as $F_{\mu\nu} = \frac{1}{g^0} (\partial_\mu (g^0 B_\nu) - \partial_\nu (g^0 B_\mu) + [g^0 B_\mu, g^0 B_\nu, \mathbf{g}])$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

Having formulated a gauge invariant action (20) the next step is to introduce gauge invariant matter terms like $(D_\mu \Phi)^2$ where $\Phi = \Phi^a e_a$ is an octonionic-valued scalar and $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi, \mathbf{g}]$. However, there is a caveat. If derivative $D_\mu \Phi$ transforms homogeneously, when $\delta \Phi = \Lambda^{ab} [e_a, e_b, \Phi]$, one arrives to the conditions $(\partial_\mu \Lambda^{ab}) \Phi^c - (\partial_\mu \Lambda^a) \Phi^b g^c = 0$ which would impose *additional* spurious constraints on the scalar field Φ . For this reason, gauge invariant matter terms in the action can be introduced if one starts instead with an octonionic-valued rank-three antisymmetric field strength

$$\begin{aligned} F_{\mu\nu\rho} &= \partial_\rho A_{\mu\nu} + \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \\ &[A_{\mu\nu}, A_\rho, \mathbf{g}] + [A_{\nu\rho}, A_\mu, \mathbf{g}] + [A_{\rho\mu}, A_\nu, \mathbf{g}] \end{aligned} \quad (29)$$

defined in terms of the antisymmetric tensor field of rank two $A_{\mu\nu} = A_{\mu\nu}^a e_a$, the field $A_\mu = A_\mu^a e_a$, and the auxiliary coupling function $\mathbf{g} = g^a e_a$. Under the local gauge transformations

$$\delta(A_{\mu\nu}^d e_d) = \partial_{[\mu} \Lambda_{\nu]}^d(x) e_d + \Lambda^{ab}(x) [e_a, e_b, A_{\mu\nu}^c e_c] \quad (30a)$$

$$\delta(A_\mu^d e_d) = -(\partial_\mu \Lambda^d(x)) e_d + \Lambda^{ab}(x) [e_a, e_b, A_\mu^c e_c]; \quad \delta(g^d e_d) = \Lambda^{ab}(x) [e_a, e_b, g^c e_c]; \quad (30b)$$

the antisymmetric field strength $F_{\mu\nu\rho}$ will transform homogeneously

$$\delta(F_{\mu\nu\rho}^d e_d) = \Lambda^{ab} [e_a, e_b, F_{\mu\nu\rho}^c e_c] \quad (30c)$$

if, and only if, the following conditions are met

$$\partial_{[\mu}\Lambda_{\nu]}^a A_\rho^b g^c - A_{\mu\nu}^a (\partial_\rho\Lambda^b) g^c + (\partial_\rho\Lambda^{ab}) A_{\mu\nu}^c = 0 \quad (31)$$

in conjunction with similar equations obtained by a permutation of the space-time indices. A particular solution to the field-dependent conditions on the gauge parameters (31) is

$$\partial_\rho\Lambda^b = A_\rho^b \Rightarrow \Lambda^b(x) = \int_0^x A_\rho^b(x') dx'^\rho \quad (32a)$$

$$\partial_{[\mu}\Lambda_{\nu]}^a(x) = A_{\mu\nu}^a(x); \quad \Lambda^{ab} = constant \quad (32b)$$

Therefore, eqs-(32) determine the field-dependent behavior of $\Lambda^a(x), \Lambda_\nu^a(x)$ in terms of the gauge fields $A_\rho, A_{\mu\nu}$. One must emphasize that despite that $\Lambda^{ab} = constant$ in (32b) this does *not* mean that one has rigid *global* transformations for the gauge fields $A_\mu, A_{\mu\nu}$ in eqs-(30a, 30b), due to the fact that the gauge parameters $\Lambda^a(x), \Lambda_\nu^a(x)$ are explicitly x -dependent !. Therefore, one has truly local (nonlinear) gauge transformations for the gauge fields. It is true, however, that the homogeneous transformation for the field strength $\delta(F_{\mu\nu\rho}^d e_d)$ given by eq-(30c) does exhibit a rigid global behavior when $\Lambda^{ab} = constant$. There are other solutions to eq-(31) besides those in eqs-(32) that do not involve setting $\Lambda^{ab} = constant$. In this case there is also a field dependence on the coupling function $g^c(x)$. For simplicity, we shall focus only in the solutions in eqs-(32).

Omitting internal indices, now one can introduce gauge invariant scalar matter by defining the covariant derivative in an explicit *nonlinear* manner as

$$D_\mu\Phi = \partial_\mu\Phi + l^2 [A_{\mu\nu}, A^\nu, \Phi,] \quad (33)$$

the above nonlinear covariant derivative is defined both in terms of A_μ and $A_{\mu\nu}$. l is a parameter of length dimensions that must be introduced because $A_{\mu\nu}$ has dimensions of $length^{-2}$. One may verify that $D_\mu\Phi$ transforms homogeneously when $\delta\Phi = \Lambda^{ab}[e_a, e_b, \Phi]$ if, and only if, the *same* conditions in eqs-(31) are satisfied. In this case no additional spurious constraints on the fields are introduced. Hence, the same field-dependent conditions on the gauge parameters given by eqs-(32a,32b) are still valid. Furthermore, an action

$$S = \int d^Dx < -\frac{1}{2} \frac{1}{(3!\kappa^2)} F_{\mu\nu\rho} F^{\mu\nu\rho} + \frac{1}{2} (D_\mu\Phi)^2 > \quad (34)$$

is invariant under the gauge transformations given by eqs-(30); κ is a parameter of suitable dimensions introduced in order to render the action dimensionless. The motivation in constructing an octonionic-valued field strength in terms of ternary brackets is because the ordinary 2-bracket does *not* obey the Jacobi identity

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 3 d_{ijkl} e_l \neq 0 \quad (35)$$

If one has the ordinary Yang-Mills expression for the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (36)$$

because the 2-bracket does *not* obey the Jacobi identity, one has an extra (spurious) term in the expression for

$$[D_\mu, D_\nu] \Phi = [F_{\mu\nu}, \Phi] + (A_\mu, A_\nu, \Phi) \quad (37)$$

given by the crucial contribution of the non-vanishing associator $(A_\mu, A_\nu, \Phi) = (A_\mu A_\nu) \Phi - A_\mu (A_\nu \Phi) \neq 0$. For this reason, due to the non-vanishing condition (35), the ordinary Yang-Mills field strength does *not* transform homogeneously under ordinary gauge transformations involving the parameters $\Lambda = \Lambda^a e_a$

$$\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda] \quad (38)$$

and it yields an extra contribution of the form

$$\delta F_{\mu\nu} = [F_{\mu\nu}, \Lambda] + (\Lambda, A_\mu, A_\nu) \quad (39)$$

As a result of the additional contribution (Λ, A_μ, A_ν) in eq-(39), the ordinary Yang-Mills action $S = \int \langle F_{\mu\nu} F^{\mu\nu} \rangle$ will *no* longer be gauge invariant. Under infinitesimal variations eqs-(38), the variation of the action is *no* longer zero but receives spurious contributions of the form $\delta S = -4 F_{\mu\nu}^l \Lambda^i A^{\mu j} A^{\nu k} d_{ijkl} \neq 0$ due to the non-associativity of the octonion algebra.

To finalize we discuss further constructions, like having an octonionic-valued and $SU(N)$ -valued gauge field $\mathbf{A}_\mu = A_\mu^{am} (e_a \otimes T_m)$ involving the $SU(N)$ algebra generators $T_m, m = 1, 2, 3, \dots, N^2 - 1$ and the octonion algebra generators $e_a, a = 0, 1, 2, 3, \dots, 7$; i.e. one has octonionic-valued components for the $SU(N)$ gauge fields. The commutator is

$$[\mathbf{A}_\mu, \mathbf{A}_\nu] = [A_\mu^{am} (e_a \otimes T_m), A_\nu^{bn} (e_b \otimes T_n)] = \frac{1}{2} A_\mu^{am} A_\nu^{bn} \{e_a, e_b\} \otimes [T_m, T_n] + \frac{1}{2} A_\mu^{am} A_\nu^{bn} [e_a, e_b] \otimes \{T_m, T_n\} \quad (40)$$

where

$$\{e_a, e_b\} = -2 \delta_{ab} e_o, \quad [e_a, e_b] = 2 c_{abc} e_c \quad (41)$$

and for the $SU(N)$ Hermitian generators one has

$$\{T_m, T_n\} = \frac{1}{N} \delta_{mn} + d_{mnp} T_p, \quad [T_m, T_n] = i f_{mnp} T_p \quad (42)$$

One may note that the r.h.s of (40) involves both commutators and anti-commutators. Due to the fact that the octonion algebra does not obey the

Jacobi identities this will spoil the gauge invariance of typical Yang-Mills actions as described before. Let us have instead a ternary Lie algebra (3-Lie algebra) obeying the ternary commutation relations

$$[T_m, T_n, T_p] = f_{mnpq} T_q \quad (43)$$

and such that the ternary-bracket structure-constants f_{mnpq} obey the fundamental identity. A 3-Lie-algebra and octonionic-valued field is defined by $\mathbf{A}_\mu \equiv A_\mu^{ma} (T_m \otimes e_a)$. However, the triple commutator

$$[\mathbf{A}_\mu, \mathbf{A}_\nu, \mathbf{A}_\rho] = [A_\mu^{mi} (T_m \otimes e_i), A_\nu^{nj} (T_n \otimes e_j), A_\rho^{pk} (T_p \otimes e_k)] \quad (44)$$

would furnish a very *complicated* expression for the r.h.s of eq-(44). To simplify matters one could define the ternary bracket as

$$[\mathbf{A}_\mu, \mathbf{A}_\nu, \mathbf{A}_\rho] \equiv A_\mu^{mi} A_\nu^{nj} A_\rho^{pk} [T_m, T_n, T_p] \otimes [e_i, e_j, e_k] = \\ A_\mu^{mi} A_\nu^{nj} A_\rho^{pk} f_{mnpq} f_{ijkl} (T_q \otimes e_l) \quad (45)$$

so that one has closure in the r.h.s of eq-(45). It is warranted to explore further these generalized ternary gauge field theories involving 3-Lie algebras and octonions. In particular, the plausible physical applications of the action (34) and the role of these octonionic ternary gauge theories in M-theory.

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