On Algebraic Mųlti-Group Spaces

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Abstract: A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining classical of a group with Smarandache multi-spaces, the conception of a multi-group space is introduced in this paper, which is a generalization of the classical algebraic structures, such as the group, filed, body, $\cdots$, etc.. Similar to groups, some characteristics of a multi-group space are obtained in this paper.

Key words: multi-space, group, multi-group space, theorem.

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1.Introduction

The notion of multi-spaces is introduced by Smarandache in [5] under his idea of hybrid mathematics: *combining different fields into a unifying field*([6]). Today, this idea is widely accepted by the world of sciences. For mathematics, definite or exact solution under a given condition is not the only object for mathematician. New creation power has emerged and new era for the mathematics has come now.

A Smarandache multi-space is defined by

Definition 1.1 *For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

The conception of multi-group space is a generalization of the classical algebraic structures, such as the group, filed, body, $\cdots$, etc., which is defined as follows.

Definition 1.2 *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a complete multi-space with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations $\times$ and $\circ$, $\times \neq \circ$, there is one operation,* *for example the operation $\times$ satisfying the distribution law to the operation $\circ$ if their operation results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group space.

Remark: The following special cases convince us that multi-group spaces are generalization of group, field and body, $\cdots$, etc..

(i) If $n = 1$, then $\widetilde{G} = (G_1; \times_1)$ is just a group.

(ii) If $n = 2, G_1 = G_2 = \widetilde{G}$, Then $\widetilde{G}$ is a body. If $(G_1; \times_1)$ and $(G_2; \times_2)$ are commutative groups, then $\widetilde{G}$ is a field.

Notice that in [7][8] various bispaces, such as bigroup, bisemigroup, biquasigroup, biloop, bigroupoid, biring, bisemiring, bivector, bisemivector, binear-ring, $\cdots$, etc., consider two operation on two different sets are introduced.

2. **Characteristics of multi-group spaces**

For a multi-group space $\widetilde{G}$ and a subset $\widetilde{G_1} \subset \widetilde{G}$, if $\widetilde{G_1}$ is also a multi-group space under a subset $O(\widetilde{G_1}), O(\widetilde{G_1}) \subset O(\widetilde{G})$, then $\widetilde{G}$ is called a *multi-group subspace*, denoted by $\widetilde{G_1} \preceq \widetilde{G}$. We have the following criterion for the multi-group subspaces.

Theorem 2.1 *For a multi-group space $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$, a subset $\widetilde{G_1} \subset \widetilde{G}$ is a multi-group subspace if and only if for any integer $k, 1 \leq k \leq n$, $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$.*

Proof If $\widetilde{G_1}$ is a multi-group space with the operation set $O(\widetilde{G_1}) = \{\times_{i_j} | 1 \leq j \leq s\} \subset O(\widetilde{G})$, then

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i) = \bigcup_{j=1}^{s} G'_{i_j}$$

where $G'_{i_j} \preceq G_{i_j}$ and $(G_{i_j}; \times_{i_j})$ is a group. Whence, if $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then there exist an integer $l, k = i_l$ such that $\widetilde{G_1} \bigcap G_k = G'_{i_l}$, i.e., $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$.

Now if for any integer k, $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$, let N denote the index set k with $\widetilde{G_1} \bigcap G_k \neq \emptyset$. Then

$$\widetilde{G_1} = \bigcup_{j \in N} (\widetilde{G_1} \bigcap G_j)$$

and $(\widetilde{G_1}\bigcap G_j, \times_j)$ is a group. Since $\widetilde{G_1}\subset\widetilde{G}$ and $O(\widetilde{G_1})\subset O(\widetilde{G})$, the associative law and distribute law are true for the $\widetilde{G_1}$. Therefore, $\widetilde{G_1}$ is a multi-group subspace of $\widetilde{G}$. ♮

For a finite multi-group subspace, we have the following criterion.

Theorem 2.2 *Let $\widetilde{G}$ be a finite multi-group space with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. A subset $\widetilde{G_1}$ of $\widetilde{G}$ is a multi-group subspace under an operation subset $O(\widetilde{G_1}) \subset O(\widetilde{G})$ if and only if for each operation $\times$ in $O(\widetilde{G_1})$, $(\widetilde{G_1}; \times)$ is complete.*

Proof Notice that for a multi-group space $\widetilde{G}$, its each multi-group subspace $\widetilde{G_1}$ is complete.

Now if $\widetilde{G_1}$ is a complete set under each operation $\times_i$ in $O(\widetilde{G_1})$, we know that $(\widetilde{G_1}\bigcap G_i; \times_i)$ is a group (see also [9]) or an empty set. Whence, we get that

$$\widetilde{G_1} = \bigcup_{i=1}^{n}(\widetilde{G_1}\bigcap G_i).$$

Therefore, $\widetilde{G_1}$ is a multi-group subspace of $\widetilde{G}$ under the operation set $O(\widetilde{G_1})$. ♮

For a multi-group subspace $\widetilde{H}$ of the multi-group space $\widetilde{G}$, $g \in \widetilde{G}$, define

$$g\widetilde{H} = \{g \times h | h \in \widetilde{H}, \times \in O(\widetilde{H})\}.$$

Then for $\forall x, y \in \widetilde{G}$,

$$x\widetilde{H}\bigcap y\widetilde{H} = \emptyset \ \text{ or } \ x\widetilde{H} = y\widetilde{H}.$$

In fact, if $x\widetilde{H}\bigcap y\widetilde{H} \neq \emptyset$, let $z \in x\widetilde{H}\bigcap y\widetilde{H}$, then there exist elements $h_1, h_2 \in \widetilde{H}$ and operations $\times_i$ and $\times_j$ such that

$$z = x \times_i h_1 = y \times_j h_2.$$

Since $\widetilde{H}$ is a multi-group subspace, $(\widetilde{H}\bigcap G_i; \times_i)$ is a subgroup. Whence, there exists an inverse element h_1^{-1} in $(\widetilde{H}\bigcap G_i; \times_i)$. We have that

$$x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.$$

That is,

$$x = y \times_j h_2 \times_i h_1^{-1}.$$

Whence,

$$x\widetilde{H} \subseteq y\widetilde{H}.$$

Similarly, we can also get that

$$x\widetilde{H} \supseteq y\widetilde{H}.$$

Therefore, we get that

$$x\widetilde{H} = y\widetilde{H}.$$

Denote the union of two set A and B by $A \bigoplus B$ if $A \bigcap B = \emptyset$. Then we get the following result by the previous proof.

Theorem 2.3 *For any multi-group subspace $\widetilde{H}$ of a multi-group space $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x \in T} x\widetilde{H}.$$

For the case of finite groups, since there is only one binary operation $\times$ and $|x\widetilde{H}| = |y\widetilde{H}|$ for any $x, y \in \widetilde{G}$, We get the following corollary, which is just Lagrange theorem for finite groups.

Corollary 2.1(Lagrange theorem) *For any finite group G, if H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

For a multi-group space $\widetilde{G}$ and $g \in \widetilde{G}$, denote by $\overrightarrow{O(g)}$ all the binary operations associative with g and by $\widetilde{G}(\times)$ the elements associative with the binary operation $\times$. For a multi-group subspace $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\times)$, if $\forall h \in \widetilde{H}$,

$$g \times h \times g^{-1} \in \widetilde{H},$$

then call $\widetilde{H}$ a *normal multi-group subspace* of $\widetilde{G}$, denoted by $\widetilde{H} \lhd \widetilde{G}$. If $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$, similar to the normal subgroup of a group, it can be shown that $g \times \widetilde{H} = \widetilde{H} \times g$, where $g \in \widetilde{G}(\times)$. We have the following result.

Theorem 2.4 *Let $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ be a multi-group space with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. Then a multi-group subspace $\widetilde{H}$ of $\widetilde{G}$ is normal if and only if for any integer $i, 1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or $\widetilde{H} \bigcap G_i = \emptyset$.*

Proof We have known that

$$\widetilde{H} = \bigcup_{i=1}^{n} (\widetilde{H} \bigcap G_i).$$

If for any integer $i, 1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$, then we know that for $\forall g \in G_i, 1 \leq i \leq n$,

$$g \times_i (\widetilde{H} \bigcap G_i) \times_i g^{-1} = \widetilde{H} \bigcap G_i.$$

Whence, for $\forall\circ \in O(\widetilde{H})$ and $\forall g \in \overrightarrow{\widetilde{G}(\circ)}$,

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}.$$

That is, $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$.

Now if $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$, then by definition, we know that for $\forall\circ \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\circ)$,

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}.$$

Not loss of generality, we assume that $\circ = \times_k$, then we get that

$$g \times_k (\widetilde{H} \bigcap G_k) \times_k g^{-1} = \widetilde{H} \bigcap G_k.$$

Therefore, $(\widetilde{H} \bigcap G_k; \times_k)$ is a normal subgroup of $(G_k, \times_k)$. For operation $\circ$ is chosen arbitrarily, we know that for any integer i, $1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or an empty set. $\natural$

For a multi-group space $\widetilde{G}$ with an operation set $O(\widetilde{G}) = \{\times_i | \ 1 \leq i \leq n\}$, an order of operations in $O(\widetilde{G})$ is said an *oriented operation sequence*, denoted by $\overrightarrow{O}(\widetilde{G})$. For example, if $O(\widetilde{G}) = \{\times_1, \times_2 \times_3\}$, then $\times_1 \succ \times_2 \succ \times_3$ is an oriented operation sequence and $\times_2 \succ \times_1 \succ \times_3$ is also an oriented operation sequence.

For an oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, we construct a series of normal multi-group subspaces

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_m = \{1_{\times_n}\}$$

by the following programming.

STEP 1: *Construct a series*

$$\widetilde{G} \rhd \widetilde{G}_{11} \rhd \widetilde{G}_{12} \rhd \cdots \rhd \widetilde{G}_{1l_1}$$

under the operation $\times_1$*.*

STEP 2: *If a series*

$$\widetilde{G}_{(k-1)l_1} \rhd \widetilde{G}_{k1} \rhd \widetilde{G}_{k2} \rhd \cdots \rhd \widetilde{G}_{kl_k}$$

has be constructed under the operation $\times_k$ *and* $\widetilde{G}_{kl_k} \neq \{1_{\times_n}\}$, *then construct a series*

$$\widetilde{G}_{kl_1} \rhd \widetilde{G}_{(k+1)1} \rhd \widetilde{G}_{(k+1)2} \rhd \cdots \rhd \widetilde{G}_{(k+1)l_{k+1}}$$

under the operation $\times_{k+1}$*.*

This programming is terminated until the series

$$\widetilde{G}_{(n-1)l_1} \rhd \widetilde{G}_{n1} \rhd \widetilde{G}_{n2} \rhd \cdots \rhd \widetilde{G}_{nl_n} = \{1_{\times_n}\}$$

has be constructed under the operation $\times_n$*.*

The number m is called the length of the series of normal multi-group subspaces. For a series

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_n = \{1_{\times_n}\}$$

of normal multi-group subspaces, if for any integer $k, s, 1 \leq k \leq n, 1 \leq s \leq l_k$, there exists a normal multi-group subspace $\widetilde{H}$ such that

$$\widetilde{G}_{ks} \rhd \widetilde{H} \rhd \widetilde{G}_{k(s+1)},$$

then $\widetilde{H} = \widetilde{G}_{ks}$ or $\widetilde{H} = \widetilde{G}_{k(s+1)}$, we call this series is *maximal*. For a maximal series of finite normal multi-group subspaces, we have the following result.

Theorem 2.5 *For a finite multi-group space* $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ *and an oriented operation sequence* $\overrightarrow{O}(\widetilde{G})$, *the length of maximal series of normal multi-group subspaces is a constant, only dependent on* $\widetilde{G}$ *itself.*

Proof The proof is by induction on the integer n.

For $n = 1$, the maximal series of normal multi-group subspaces is just a composition series of a finite group. By Jordan-Hölder theorem (see [1] or [3]), we know the length of a composition series is a constant, only dependent on $\widetilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume the assertion is true for cases of $n \leq k$. We prove it is true in the case of $n = k + 1$. Not loss of generality, assume the order of binary operations in $\overrightarrow{O}(\widetilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \rhd G_2 \rhd \cdots \rhd G_s = \{1_{\times_1}\}.$$

By Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Theorem 3.6, we know a maximal series of normal multi-group subspace of $\widetilde{G}$ gotten by the STEP 1 under the operation $\times_1$ is

$$\widetilde{G} \rhd \widetilde{G} \setminus (G_1 \setminus G_2) \rhd \widetilde{G} \setminus (G_1 \setminus G_3) \rhd \cdots \rhd \widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\}).$$

Notice that $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is still a multi-group space with less or equal to k operations. By the induction assumption, we know the length of its maximal series of normal multi-group subspaces is only dependent on $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$, is a constant. Therefore, the length of a maximal series of normal multi-group subspaces is also a constant, only dependent on $\widetilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal multi-group subspaces of $\widetilde{G}$ is a constant under an oriented operations $\overrightarrow{O}(\widetilde{G})$, only dependent on $\widetilde{G}$ itself. ♮

As a special case, we get the following corollary.

Corollary 2.2(Jordan-Hölder theorem) *For a finite group G, the length of the composition series is a constant, only dependent on G.*

3. Open Problems on Multi-group Spaces

Problem 3.1 *Establish a decomposition theory for multi-group spaces.*

In group theory, we know the following decomposition results([1][3]) for a group.

Let G be a finite Ω-group. Then G can be uniquely decomposed as a direct product of finite non-decomposition Ω-subgroups.

Each finite Abelian group is a direct product of its Sylow p-subgroups.

Then Problem 3.1 can be restated as follows.

Whether can we establish a decomposition theory for multi-group spaces similar to above two results in group theory, especially, for finite multi-group spaces?

Problem 3.2 *Define the conception of simple multi-group spaces for multi-group spaces. For finite multi-group spaces, whether can we find all simple multi-group spaces?*

For finite groups, we know that there are four simple group classes ([9]):

Class 1: the cyclic groups of prime order;

Class 2:the alternating groups $A_n, n \geq 5$;

Class 3: the 16 groups of Lie types;

Class 4: the 26 sporadic simple groups.

Problem 2.3 *Determine the structure properties of a multi-group space generated by finite elements.*

For a subset A of a multi-group space $\widetilde{G}$, define its spanning set by

$$\langle A \rangle = \{a \circ b | a, b \in A \text{ and } \circ \in O(\widetilde{G})\}.$$

If there exists a subset $A \subset \widetilde{G}$ such that $\widetilde{G} = \langle A \rangle$, then call $\widetilde{G}$ is generated by A. Call $\widetilde{G}$ is *finitely generated* if there exist a finite set A such that $\widetilde{G} = \langle A \rangle$. Then Problem 2.3 can be restated by

Can we establish a finite generated multi-group theory similar to the finite generated group theory?

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