Gravity as a Manifestation of de Sitter Invariance over a Galois Field

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Abstract:

We consider a system of two free bodies in de Sitter invariant quantum mechanics. De Sitter invariance is understood such that a set of operators describing a system satisfies commutation relations of the de Sitter algebra. Our approach does not involve quantum field theory, de Sitter space and its geometry (metric and connection). At very large distances the standard relative distance operator describes a well known cosmological acceleration. In particular, the cosmological constant problem does not exist and there is no need to involve dark energy or other fields for solving this problem. At the same time, for systems of macroscopic bodies this operator does not have correct properties at lesser distances and should be modified. We propose a modification which has correct properties, reproduces Newton's gravity and the precession of Mercury's perihelion if the width of the de Sitter momentum distribution δ for a macroscopic body is inversely proportional to its mass m. We argue that fundamental quantum theory should be based on a Galois field with a large characteristic p which is a fundamental constant characterizing laws of physics in our Universe. Then one can give a natural explanation that $\delta = const\, R/(mG)$ where R is the radius of the Universe (such that $\Lambda = 3/R^2$ is the cosmological constant) and G is a quantity defining Newton's gravity. A very rough estimation gives $G \approx R/(m_N lnp)$ where m_N is the nucleon mass. If R is of order $10^{26}m$ then lnp is of order 10^{80} and therefore p is of order $exp(10^{80})$. In the formal limit $p \to \infty$ gravity disappears, i.e. in our approach gravity is a consequence of finiteness of nature.

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Contents

Chapter 1

Introduction

1.1 The main idea of this work

Let us consider an isolated system of two particles and pose a question whether they interact or not. In theoretical physics there is no unambiguous criterion for answering this question. For example, in classical (i.e. non quantum) nonrelativistic and relativistic mechanics the criterion is clear and simple: if the relative acceleration of the particles is zero they do not interact, otherwise they interact. However, those theories are based on Galilei and Poincare symmetries, respectively and there is no reason to believe that such symmetries are exact symmetries of nature.

In quantum mechanics the criterion can be as follows. If E is the energy operator of the two-particle system and E_i $(i = 1, 2)$ is the energy operator of particle i then one can formally define the interaction operator U such that

$$E = E_1 + E_2 + U \tag{1.1}$$

Therefore the criterion can be such that the particles do not interact if $U = 0$, i.e. $E = E_1 + E_2$.

In local quantum field theory (QFT) the criterion is also clear and simple: the particles interact if they can exchange by virtual quanta of some fields. For example, the electromagnetic interaction between the particles means that they can exchange by virtual photons, the gravitational interaction - that they can exchange by virtual gravitons etc. In that case U in Eq. (1.1) is an effective operator obtained in the approximation when all degrees of freedom except those corresponding to the given particles can be integrated out.

A problem with approaches based on Eq. (1.1) is that the answer should be given in terms of invariant quantities while energies are reference frame dependent. Therefore one should consider the two-particle mass operator. In standard Poincare invariant theory the free mass operator is given by $M = M_0(\mathbf{q}) = (m_1^2+\mathbf{q}^2)^{1/2}+(m_2^2+\mathbf{q}^2)^{1/2}$ where the m_i are the particle masses and $\mathbf{q}$ is the relative momentum operator. In classical approximation $\mathbf{q}$ becomes the relative momentum and M_0 becomes a function of $\mathbf{q}$ not depending on the relative distance r between the particles. Therefore the relative acceleration is zero and this case can be treated as noninteracting.

Consider now a two-particle system in de Sitter (dS) invariant theory. A question arises how dS invariance should be understood on quantum level. Typically it is understood such that QFT should be considered on dS space. However, as argued e.g. in Ref. [1] (see also Sect. 1.3 of the present paper), the notion of spacetime background is not physical. On quantum level the only consistent definition of dS invariance is that the operators describing the system satisfy commutation relations of the dS algebra. This definition does not involve General Relativity (GR), QFT, dS space and its geometry (metric, connection etc.). Then the only consistent definition of an elementary particle is that it is described by an irreducible representation (IR) of the dS algebra. Therefore a possible definition of the free two-particle system can be such that the system is described by a representation where not only the energy but all other operators are given by sums of the corresponding single-particle operators. In representation theory such a representation is called the tensor products of IRs.

In other words, we consider only quantum mechanics of two free particles in dS invariant theory. In that case, as shown in Refs. [2, 3, 4] and others (see also Sect. 2.3 of the present paper), the two-particle mass operator can be explicitly calculated. It can be written as $M = M_0(\mathbf{q}) + V$ where V is an operator depending not only on $\mathbf{q}$. In classical approximation V becomes a function depending on r. As a consequence, the relative acceleration is not zero and the result for the relative acceleration describes a well known cosmological repulsion (sometimes called dS antigravity). From a formal point of view this result coincides with that obtained in GR on dS spacetime. However, our result has been obtained without involving Riemannian geometry, metric, connection and dS spacetime.

One might argue that the above situation contradicts the law of inertia according to which if particles do not interact then their relative acceleration must be zero. However, this law has been postulated in Galilei and Poincare invariant theories and there is no reason to believe that it will be valid for other symmetries. Another argument might be such that dS invariance implicitly implies existence of other particles which interact with the two particles under consideration. Therefore the above situation resembles a case when two particles not interacting with each other are moving with different accelerations in a nonhomogeneous field and therefore their relative acceleration is not zero. This argument has much in common with a well known discussion of whether empty spacetime can have a curvature and whether a nonzero curvature implies the existence of dark energy or other fields. However, as already noted, in Sect. 1.3 we argue that fundamental quantum theory should not involve spacetime at all. Therefore our result demonstrates that the cosmological constant problem does not exist and the cosmological acceleration can be easily (and naturally) explained without involving dark energy or other fields.

In QFT interactions can be only local and there are no interactions at a distance (sometimes called direct interactions), when particles interact without an intermediate field. In particular, a potential interaction (when the force of the interaction depends only on the distance between the particles) can be only a good approximation in situations when the particle velocities are much less than the speed of light c. The explanation is such that if the force of the interaction depends only on the distance between the particles and the distance is slightly changed then the particles will feel the change immediately, but this contradicts the statement that no interaction can be transmitted with the speed greater than the speed of light. Although standard QFT is based on Poincare symmetry, physicists typically believe that the notion of interaction adopted in QFT is valid for any symmetry. However, the above discussion shows that the dS antigravity is not caused by exchange of any virtual particles. In particular a question about the speed of propagation of dS antigravity in not physical. In other words, the dS antigravity is an example of a true direct interaction. It is also possible to say that the dS antigravity is not an interaction at all but simply an inherent property of dS invariance.

On quantum level, de Sitter and anti de Sitter (AdS) symmetries are widely used for investigating QFT in curved spacetime. However, it seems rather paradoxical that such a simple case as a free two-body system in dS invariant theory has not been widely discussed. According to our observations, such a situation is a manifestation of the fact that even physicists working on dS QFT are not familiar with basic facts about IRs of the dS algebra. It is difficult to imagine how standard Poincare invariant quantum theory can be constructed without involving well known results on IRs of the Poincare algebra. Therefore it is reasonable to think that when Poincare invariance is replaced by dS one, IRs of the Poincare algebra should be replaced by IRs of the dS algebra. However, physicists working on QFT in curved spacetime argue that fields are more fundamental than particles and therefore there is no need to involve IRs.

Our discussion shows that the notion of interaction depends on symmetry. For example, when we consider a system of two particles which from the point of view of dS symmetry are free (since they are described by a tensor product of IRs), from the point of view of our experience based on Galilei or Poincare symmetries they are not free since their relative acceleration is not zero. This poses a question whether not only dS antigravity but other interactions are in fact not interactions but effective interactions emerging when a higher symmetry is treated in terms of a lower one.

In particular, is it possible that quantum symmetry is such that on classical level the relative acceleration of two free particles is described by the same expression as that given by the Newton gravitational law and corrections to it? This possibility has been first discussed in Ref. [2]. It is clear that this possibility is not in mainstream according to which gravity is a manifestation of the graviton exchange. We will not discuss whether or not the results on binary pulsars can be treated as a strong indirect indication of the existence of gravitons and why gravitons have not been experimentally detected yet. We believe that until the nature of gravity has been unambiguously understood, different possibilities should be investigated. We believe that a very strong argument in favor of our approach is as follows. In contrast to theories based on Poincare and AdS symmetries, in the dS case the spectrum of the free mass operator is not bounded below by $(m_1 + m_2)$. As a consequence, it is not a problem to indicate states where the mean value of the mass operator has an additional contribution $-Gm_1m_2/r$ with possible corrections (here G is the gravitational constant). A problem is to understand reasons why macroscopic bodies have such wave functions.

If we accept dS symmetry then the first step is to investigate the structure of dS invariant theory from the point of view of IRs of the dS algebra. This problem is discussed in Refs. [3, 4, 1]. In Ref. [2] we discussed a possibility that gravity is simply a manifestation of the fact that fundamental quantum theory should be based not on complex numbers but on a Galois field with a large characteristic p which is a fundamental constant defining the laws of physics in our Universe. This approach to quantum theory, which we call GFQT, has been discussed in Refs. [5, 6, 7] and other publications. In Refs. [8, 9] we discussed additional arguments in favor of our hypothesis about gravity. We believe that the results of the present paper give strong indications that our hypothesis is correct. Before proceeding to the derivation of the results, we would like to discuss a general structure of fundamental quantum theory.

1.2 Remarks on the cosmological constant problem

The discovery of the cosmological repulsion (see e.g. Refs. [10, 11]) has ignited a vast discussion on how this phenomenon should be interpreted. The majority of authors treat this phenomenon as an indication that the cosmological constant (CC) Λ in GR is positive and therefore the spacetime background has a positive curvature. According to Refs. [12, 13], the observational data on the value of Λ define it with the accuracy better than 5%. Therefore the possibilities that $\Lambda = 0$ or $\Lambda < 0$ are practically excluded. To discuss the CC problem in greater details, we first discuss the following well-known problem: How many independent dimensionful constants are needed for a complete description of nature? A paper [14] represents a trialogue between three well known scientists: M.J. Duff, L.B. Okun and G. Veneziano (see also Ref. [15] and references therein). The results of their discussions are summarized as follows: *LBO develops the traditional approach with three constants, GV argues in favor of at most two (within superstring theory), while MJD advocates zero.* According to Ref. [16], a possible definition of a fundamental constant might be such that it cannot be calculated in the existing theory. We would like to give arguments in favor of the opinion of the first author in Ref. [14]. One of our goals is to argue that the cosmological and gravitational constants cannot be fundamental physical quantities.

Consider a measurement of a component of angular momentum. The result depends on the system of units. As shown in quantum theory, in units $\hbar/2 = 1$ the result is given by an integer $0, \pm 1, \pm 2, ...$ But we can reverse the order of units and say that in units where the angular momentum is an integer l, its value in $kg \cdot m^2/sec$ is $(1.05457162 \cdot 10^{-34} \cdot l/2) kg \cdot m^2/sec$. Which of those two values has more physical significance? In units where the angular momentum components are integers, the commutation relations between the components are

$$[M_x, M_y] = 2iM_z \quad [M_z, M_x] = 2iM_y \quad [M_y, M_z] = 2iM_x$$

and they do not depend on any parameters. Then the meaning of l is clear: it shows how big the angular momentum is in comparison with the minimum nonzero value 1. At the same time, the measurement of the angular momentum in units $kg \cdot m^2/sec$ reflects only a historic fact that at macroscopic conditions on the Earth in the period between the 18th and 21st centuries people measured the angular momentum in such units.

The fact that quantum theory can be written without the quantity $\hbar$ at all is usually treated as a choice of units where $\hbar = 1/2$ (or $\hbar = 1$). We believe that a better interpretation of this fact is simply that quantum theory tells us that physical results for measurements of the components of angular momentum should be given in integers. Then the question why $\hbar$ is as it is, is not a matter of fundamental physics since the answer is: because we want to measure components of angular momentum in $kg \cdot m^2/sec$.

Our next example is the measurement of velocity v. Let $(E, \mathbf{p})$ be a particle four-momentum defined by its energy and momentum. Then in special relativity the quantity $I_{2P} = E^2 - \mathbf{p}^2c^2$ is an invariant which is denoted as m^2c^4. The reason is that in usual situations $I_{2P} \geq 0$ and m coincides with the standard particle mass. However, the mathematical structure of special relativity does not impose any restrictions on the values of observable quantities E and $\mathbf{p}$; in particular it does not prohibit the case $I_{2P} < 0$. Particles for which this case takes place are called tachyons and their possible existence is widely discussed in the literature. The velocity vector $\mathbf{v}$ is defined as $\mathbf{v} = \mathbf{p}c^2/E$. The fact that any relativistic theory can be written without involving c is usually described as a choice of units where $c = 1$. Then for known particles the quantity $v = |\mathbf{v}|$ can take only values in the range [0,1] while for tachyons it can take values in the range $(1, \infty)$. However, we can again reverse the order of units and say that relativistic theory tells us that for known particles the results for measurements of velocity should be given by values in [0,1] while in general they should be given by values in $[0, \infty)$. Then the question why c is as it is, is again not a matter of physics since the answer is: because we want to measure velocity in m/sec.

One might pose a question whether or not the values of $\hbar$ and c may change with time. As far as $\hbar$ is concerned, this is a question that if the angular momentum equals one then its value in $kg \cdot m^2/sec$ will always be $1.05457162 \cdot 10^{-34}/2$ or not. It is obvious that this is not a problem of fundamental physics but a problem how the units (kg, m, sec) are defined. In other words, this is a problem of metrology and cosmology. At the same time, the value of c will always be the same since the modern *definition* of meter is the length which light passes during $(1/(3 \cdot 10^8))sec$.

It is often believed that the most fundamental constants of nature are $\hbar$, c and G. The units where $\hbar = c = G = 1$ are called Planck units. Another well known notion is the $c\hbar G$ cube of physical theories. The meaning is that any relativistic theory should contain c, any quantum theory should contain $\hbar$ and any gravitational theory should contain G. However, the above remarks indicates that the meaning should be the opposite. In particular, relativistic theory *should not* contain c and quantum theory *should not contain* $\hbar$. The problem of treating G is a key problem of this paper and will be discussed below.

A standard phrase that relativistic theory becomes non-relativistic one when $c \to \infty$ should be understood such that if relativistic theory is rewritten in conventional (but not physical!) units then c will appear and one can take the limit $c \to \infty$. A more physical description of the transition is that all the velocities in question are much less than unity. We will see in Section 2.3 that those definitions are not equivalent. Analogously, a more physical description of the transition from quantum to classical theory should be that all angular momenta in question are very large rather than $\hbar \to 0$.

Consider now what happens if we assume that dS symmetry is fundamental. We will see that in our approach dS symmetry has nothing to do with dS space but now we consider standard notion of this symmetry. The dS space is a four-dimensional manifold in the five-dimensional space defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2 \tag{1.2}$$

In the formal limit $R \to \infty$ the action of the dS group in a vicinity of the point $(0, 0, 0, 0, x_4 = R)$ becomes the action of the Poincare group on Minkowski space. In the literature, instead of R, the cosmological constant (CC) $\Lambda = 3/R^2$ is often used. Then $\Lambda > 0$ in the dS case, $\Lambda < 0$ in the AdS one and $\Lambda = 0$ for Poincare symmetry. The dS space can be parameterized without using the quantity R at all if instead of x_a ($a = 0, 1, 2, 3, 4$) we define dimensionless variables $\xi_a = x_a/R$. It is also clear that the elements of the SO(1,4) group do not depend on R since they are products of conventional and hyperbolic rotations. So the dimensionful value of R appears only if one wishes to measure coordinates on the dS space in terms of coordinates of the flat five-dimensional space where the dS space is embedded in. This requirement does not have a fundamental physical meaning. Therefore the value of R defines only a scale factor for measuring coordinates in the dS space. By analogy with c and $\hbar$, the question why R is as it is, is not a matter of fundamental physics since the answer is: because we want to measure distances in meters. In particular, there is no guaranty that the CC is really a constant, *i.e.*, does not change with time. It is also obvious that if dS symmetry is assumed from the beginning then the value of Λ has no relation to the value of G.

If one assumes that spacetime background is fundamental then in the spirit of GR it is natural to think that the empty spacetime is flat, *i.e.*, that $\Lambda = 0$ and this was the subject of the well-known dispute between Einstein and de Sitter. However, as noted above, it is now accepted that $\Lambda \neq 0$ and, although it is very small, it is positive rather than negative. If we accept parameterization of the dS space as in Eq. (1.2) then the metric tensor on the dS space is

$$g_{\mu\nu} = \eta_{\mu\nu} - x_\mu x_\nu/(R^2 + x_\rho x^\rho) \tag{1.3}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, $\eta_{\mu\nu}$ is the diagonal tensor with the components $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$ and a summation over repeated indices is assumed. It is easy to calculate the Christoffel symbols in the approximation where all the components of the vector x are much less than R: $\Gamma_{\mu,\nu\rho} = -x_\mu \eta_{\nu\rho}/R^2$. Then a direct calculation shows that in the nonrelativistic approximation the equation of motion for a single particle is

$$\mathbf{a} = \mathbf{r}c^2/R^2 \tag{1.4}$$

where $\mathbf{a}$ and $\mathbf{r}$ are the acceleration and the radius vector of the particle, respectively.

Suppose now that we have a system of two noninteracting particles and $(\mathbf{r}_i, \mathbf{a}_i)$ $(i = 1, 2)$ are their radius vectors and accelerations, respectively. Then Eq. (1.4) is valid for each particle if $(\mathbf{r}, \mathbf{a})$ is replaced by $(\mathbf{r}_i, \mathbf{a}_i)$, respectively. Now if we define the relative radius vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ then they will satisfy the same Eq. (1.4) which shows that the dS antigravity is repulsive. It terms of Λ it reads $\mathbf{a} = \Lambda \mathbf{r} c^2/3$ and therefore in the AdS case we have attraction rather than repulsion.

The fact that even a single particle in the Universe has a nonzero acceleration might be treated as contradicting the law of inertia but, as already noted, this law has been postulated only for Galilean or Poincare symmetries and we have $\mathbf{a} = 0$ in the limit $R \to \infty$. A more serious problem is that, according to standard experience, any particle moving with acceleration necessarily emits gravitational waves, any charged particle emits electromagnetic waves *etc.* Does this experience work in the dS world? This problem is intensively discussed in the literature (see e.g., Ref. [17] and references therein). Suppose we accept that, according to GR, the loss of energy in gravitational emission is proportional to the gravitational constant. Then one might say that in the given case it is not legitimate to apply GR since the constant G characterizes interaction between different particles and cannot be used if only one particle exists in the world. However, the majority of authors proceed from the assumption that the empty dS space cannot be literally empty. If the Einstein equations are written in the form $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ where $T_{\mu\nu}$ is the stress-energy tensor of matter then the case of empty space is often treated as a vacuum state of a field with the stress-energy tensor $T^{vac}_{\mu\nu}$ such that $(8\pi G/c^4)T^{vac}_{\mu\nu} = -\Lambda g_{\mu\nu}$. This field is often called dark energy. With such an approach one implicitly returns to Einstein's point of view that a curved space cannot be empty. Then the fact that $\Lambda \neq 0$ is treated as a dark energy on the flat background. In other words, this is an assumption that Poincare symmetry is fundamental while dS one is emergent.

However, in this case a new problem arises. The corresponding quantum theory is not renormalizable and with reasonable cutoffs, the quantity Λ in units $\hbar = c = 1$ appears to be of order $1/l_P^2 = 1/G$ where l_P is the Planck length. It is obvious that since in the above theory the only dimensionful quantities in units $\hbar = c = 1$ are G and Λ, and the theory does not have other parameters, the result that $G\Lambda$ is of order unity seems to be natural. However, this value of Λ is at least by 120 orders of magnitude greater than the experimental one. In supergravity the disagreement can be reduced but even in best scenarios it exceeds 40 orders of magnitude. Numerous efforts to solve this CC problem have not been successful so far although many explanations have been proposed.

Many physicists argue that in the spirit of GR, the theory should not depend on the choice of the spacetime background (a principle of background independence) and there should not be a situation when the flat background is preferable. Moreover, although GR has been confirmed in several experiments in Solar system, it is not clear whether it can be extrapolated to cosmological distances. In other words, our intuition based on GR with $\Lambda = 0$ cannot be always correct if $\Lambda \neq 0$. In Ref. [18] this point of view is discussed in details. The authors argue that a general case of Einstein's equation is when Λ is present and there is no reason to believe that a special case $\Lambda = 0$ is preferable.

In summary, numerous attempts to resolve the CC problem have not converged to any universally accepted theory. All those attempts are based on the notion of spacetime background and in the next section we discuss whether this notion is physical.

1.3 Should physical theories involve spacetime background?

From the point of view of quantum theory, any physical quantity can be discussed only in conjunction with the operator defining this quantity. For example, in standard quantum mechanics the quantity t is a parameter, which has the meaning of time only in classical limit since there is no operator corresponding to this quantity. The problem of how time should be defined on quantum level is very difficult and is discussed in a vast literature (see e.g., Refs. [19] and references therein). It has been also well known since the 1930s [20] that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator. In other words, the coordinates cannot be exactly measured even in situations when exact measurements are allowed by the non-relativistic uncertainty principle. In the introductory section of the well-known textbook [21] simple arguments are given that for a particle with mass m, the coordinates cannot be measured with the accuracy better than the Compton wave length $\hbar/mc$. This fact is mentioned in practically every textbook on quantum field theory (see e.g., Ref. [22]). Hence, the exact measurement is possible only either in the non-relativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$). Another well known example discussed in standard textbooks on Quantum Electrodynamics (see e.g. Ref. [23]) is that in this theory there is no way to define a photon wave function in coordinate representation.

We accept a principle that any definition of a physical quantity is a description how this quantity should be measured. In quantum theory this principle has been already implemented but we believe that it should be valid in classical theory as well. From this point of view, one can discuss if *coordinates of particles* can be measured with a sufficient accuracy, while the notion of spacetime background, regardless of whether it is flat or curved, does not have a physical meaning. Indeed, this notion implies that spacetime coordinates are meaningful even if they refer not to real particles but to points of a manifold which exists only in our imagination. However, such coordinates are not measurable. To avoid this problem one might try to treat spacetime background as a reference frame. Note that even in GR, which is a pure classical (*i.e.*, non-quantum) theory, the meaning of reference frame is not clear. In standard textbooks (see e.g., Ref. [24]) the reference frame in GR is defined as a collection of weightless bodies, each of which is characterized by three numbers (coordinates) and is supplied by a clock. Such a notion (which resembles ether) is not physical even on classical level and for sure it is meaningless on quantum level. There is no doubt that GR is a great achievement of theoretical physics and has achieved great successes in describing experimental data. At the same time, it is based on the notions of spacetime background or reference frame, which do not have a clear physical meaning.

In classical field theories (e.g. in classical electrodynamics), spatial coordinates are meaningful only as the coordinates of test particles. However, in GR spacetime is described not only by coordinates but also by a curvature. The philosophy of GR is that matter creates spacetime curvature and in the absence of matter spacetime should be flat. Therefore $\Lambda \neq 0$ implicitly implies that spacetime is not empty. However, the notion of spacetime without matter is fully unphysical and, in our opinion, it is a nonphysical feature of GR that there are solutions when matter disappears but spacetime still exists and has a curvature (a zero curvature for Minkowski spacetime and a nonzero curvature if $\Lambda \neq 0$). This feature cannot be justified even taking into account the fact that GR is a pure classical theory. In some approaches (see e.g. Ref. [25]), when matter disappears, the metric tensor becomes not the Minkowskian one but zero, i.e. spacetime disappears too. Also, as argued in Ref. [26], the metric tensor should be dimensionful since $g_{\mu\nu}dx^\mu dx^\nu$ should be scale independent. Then the absolute value of the metric tensor is proportional to the number of particles in the Universe.

In Loop Quantum Gravity (LQG), spacetime is treated on quantum level as a special state of quantum gravitational field. This construction is rather complicated and one of its main goals is to have a quantum generalization of spacetime such that GR should be recovered as a classical limit of quantum theory. However, so far

LQG has not succeeded in proving that GR is a special case of LQG in classical limit.

Another approach where spacetime is not fundamental but emergent is based on holographic principle and the recent work by Verlinde [27]. As noted in this paper, "*Space is in the first place a device introduced to describe the positions and movements of particles. Space is therefore literally just a storage space for information...*". This implies that the emergent spacetime is meaningful only if matter is present. The author of Ref. [27] states that in his approach one can recover Einstein equations where the coordinates and curvature refer to the emergent spacetime. However, it is not clear how to treat the fact that the formal limit when matter disappears is possible and spacetime formally remains although, if it is emergent, it cannot exist without matter.

In quantum theory, if we have a system of particles, its wave function (represented as a Fock state or in other forms) gives the maximum possible information about this system and there is no other way of obtaining any information about the system except from its wave function. So the information encoded in the emergent space should be somehow extracted from the system wave function. However, to the best of our knowledge, there is no theory relating the emergent space with the system wave function. Typically the emergent space is described in the same way as the "fundamental" space, i.e. as a manifold and it is not clear how the points of this manifold are related to the wave function. The above arguments showing that the "fundamental" space is not physical can be applied to the emergent space as well. In particular, the coordinates of the emergent space are not measurable and it is not clear what is the meaning of those coordinates where there are no particles at all. It is also known that at present the holographic principle is only a hypothesis which has not been experimentally verified. At the same time, since the nature of gravity is a very difficult fundamental problem, we believe that different approaches for solving this problem should be welcome.

We believe that in view of this discussion, it is unrealistic to expect that successful quantum theory of gravity will be based on quantization of GR or on emergent spacetime. The results of GR might follow from quantum theory of gravity only in situations when spacetime coordinates of *real bodies* is a good approximation while in general the formulation of quantum theory should not involve spacetime background at all. One might take objection that coordinates of spacetime background in GR can be treated only as parameters defining possible gauge transformations while final physical results do not depend on these coordinates. Analogously, although the quantity x in the Lagrangian density $L(x)$ is not measurable, it is only an auxiliary tool for deriving equations of motion in classical theory and constructing Hilbert spaces and operators in quantum theory. After this construction has been done, one can safely forget about background coordinates and Lagrangian. In other words, a problem is whether nonphysical quantities can be present at intermediate stages of physical theories. This problem has a long history discussed in a vast literature. Probably Newton was the first who introduced the notion of spacetime background but, as noted in a paper in Wikipedia, "Leibniz thought instead that space was a collection of relations between objects, given by their distance and direction from one another". As noted above, the assumption that spacerime exists and has a curvature even when matter is absent is not physical. We believe that at the fundamental level unphysical notions should not be present even at intermediate stages. So Lagrangian can be at best treated as a hint for constructing a fundamental theory. As stated in Ref. [21], local quantum fields and Lagrangians are rudimentary notion, which will disappear in the ultimate quantum theory. Those ideas have much in common with the Heisenberg S-matrix program and were rather popular till the beginning of the 1970's. In view of successes of gauge theories they have become almost forgotten.

In summary, although the most famous successes of theoretical physics have been obtained in theories involving spacetime background, this notion does not have a physical meaning. Therefore a problem arises how to explain the fact that physics seems to be local with a good approximation. In Section 2.3 it is shown that the result of GR on the dS space given by Eq. (1.4) is simply a consequence of dS symmetry on quantum level when semiclassical approximation works with a good accuracy. For deriving this result there is no need to involve dS space, metric, connection, dS QFT and other sophisticated methods. The first step in our approach is discussed in the next section.

1.4 Symmetry on quantum level

If we accept that quantum theory should not proceed from spacetime background, a problem arises how symmetry should be defined on quantum level. Note that each system is described by a set of independent operators and they somehow commute with each other. We accept that by definition, the rules how they commute define a Lie algebra which is treated as a symmetry algebra.

Such a definition of symmetry on quantum level is in the spirit of Dirac's paper [28]. We believe that for understanding this Dirac's idea the following example might be useful. If we define how the energy should be measured (e.g., the energy of bound states, kinetic energy *etc.*), we have a full knowledge about the Hamiltonian of our system. In particular, we know how the Hamiltonian should commute with other operators. In standard theory the Hamiltonian is also interpreted as an operator responsible for evolution in time, which is considered as a classical macroscopic parameter. In situations when this parameter is a good approximate parameter, macroscopic transformations from the symmetry group corresponding to the evolution in time have a meaning of evolution transformations. However, there is no guaranty that such an interpretation is always valid (e.g., at the very early stage of the Universe). In general, according to principles of quantum theory, self-adjoint operators in Hilbert spaces represent observables but there is no requirement that parameters defining a family of unitary transformations generated by a self-adjoint operator are eigenvalues of another self-adjoint operator. A well known example from standard quantum mechanics is that if P_x is the x component of the momentum operator then the family of unitary transformations generated by P_x is $exp(iP_x x/\hbar)$ where $x \in (-\infty, \infty)$ and such parameters can be identified with the spectrum of the position operator. At the same time, the family of unitary transformations generated by the Hamiltonian H is $exp(-iHt/\hbar)$ where $t \in (-\infty, \infty)$ and those parameters cannot be identified with a spectrum of a self-adjoint operator on the Hilbert space of our system. In the relativistic case the parameters x can be formally identified with the spectrum of the Newton-Wigner position operator [20] but it is well known that this operator does not have all the required properties for the position operator. So, although the operators $exp(iP_x x/\hbar)$ and $exp(-iHt/\hbar)$ are well defined, their physical interpretation as translations in space and time is not always valid.

The *definition* of the dS symmetry on quantum level is that the operators M^{ab} $(a, b = 0, 1, 2, 3, 4,\ M^{ab} = -M^{ba})$ describing the system under consideration satisfy the commutation relations *of the dS Lie algebra* so(1,4), *i.e.*,

$$[M^{ab}, M^{cd}] = -i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{1.5}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$. These relations do not depend on any free parameters. One might say that this is a consequence of the choice of units where $\hbar = c = 1$. However, as noted above, any fundamental theory should not involve the quantities $\hbar$ and c.

With such a definition of symmetry on quantum level, dS symmetry looks more natural than Poincare symmetry. In the dS case all the ten representation operators of the symmetry algebra are angular momenta while in the Poincare case only six of them are angular momenta and the remaining four operators represent standard energy and momentum. If we define the operators P^μ as $P^\mu = M^{4\mu}/R$ then in the formal limit when $R \to \infty$, $M^{4\mu} \to \infty$ but the quantities P^μ are finite, the relations (1.5) become the commutation relations for representation operators of the Poincare algebra such that the dimensionful operators P^μ are the four-momentum operators. Note also that the above definition of the dS symmetry has nothing to do with dS space and its curvature.

In view of the above remarks, one might think that the dS analog of the energy operator is M^{40}. However, in dS theory all the operators M^{a0} $(a = 1, 2, 3, 4)$ are on equal footing. This poses a problem whether a parameter describing the evolution defined by the Hamiltonian is a fundamental quantity even on classical level (see Sect. 3.7).

A theory based on the above definition of the dS symmetry on quantum level cannot involve quantities which are dimensionful in units $\hbar = c = 1$. In particular, we inevitably come to conclusion that the dS space, the gravitational constant and the cosmological constant cannot be fundamental. The latter appears only as a parameter replacing the dimensionless operators $M^{4\mu}$ by the dimensionful operators P^μ which have the meaning of momentum operators only if R is rather large. Therefore the cosmological constant problem does not arise at all but instead we have a problem why nowadays Poincare symmetry is so good approximate symmetry. This is rather a problem of cosmology but not quantum physics.

1.5 Remarks on semiclassical approximation in quantum mechanics

In quantum theory, states of a system are represented by elements of a projective Hilbert space. The fact that a Hilbert space H is projective means that if $\psi \in H$ is a state then $const\,\psi$ is the same state. The matter is that not the probability itself but only relative probabilities of different measurement outcomes have a physical meaning. In particular, normalization of states to one is only a matter of convention. This observation will be important in Chap. 4 while in this and the next chapters we will always work with states ψ such that $||\psi|| = 1$ where $||...||$ is a norm. It is defined such that if $(...,...)$ is a scalar product in H then $||\psi|| = (\psi,\psi)^{1/2}$.

In quantum theory every physical quantity is described by a selfadjoint operator. Each selfadjoint operator is Hermitian i.e. satisfies the property $(\psi_2, A\psi_1) = (A\psi_2, \psi_1)$ for any states belonging to the domain of A. If A is an operator of some quantity then the mean value of the quantity and its uncertainty in state ψ are given by $\bar{A} = (\psi, A\psi)$ and $\Delta A = ||(A-\bar{A})\psi||$, respectively. The condition that a quantity corresponding to the operator A is semiclassical in state ψ can be defined such that $|\Delta A| \ll |\bar{A}|$. This implies that the quantity can be semiclassical only if $|\bar{A}|$ is rather large. In particular, if $\bar{A} = 0$ then the quantity cannot be semiclassical.

Let B be an operator corresponding to another physical quantity and $\bar{B}$ and ΔB be the mean value and the uncertainty of this quantity, respectively. We can write $AB = \{A,B\}/2 + [A,B]/2$ where the commutator $[A,B] = AB - BA$ is anti-Hermitian and the anticommutator $\{A,B\} = AB + BA$ is Hermitian. Let $[A,B] = -iC$ and $\bar{C}$ be the mean value of the operator C.

A question arises whether two physical quantities corresponding to the operators A and B can be simultaneously semiclassical in state ψ. Since $||\psi_1||||\psi_2|| \geq |(\psi_1,\psi_2)|$, we have that

$$\Delta A \Delta B \geq \frac{1}{2}|(\psi, (\{A-\bar{A}, B-\bar{B}\} + [A,B])\psi)| \tag{1.6}$$

Since $(\psi, \{A-\bar{A}, B-\bar{B}\}\psi)$ is real and $(\psi, [A,B]\psi)$ is imaginary, we get

$$\Delta A \Delta B \geq \frac{1}{2}|\bar{C}| \tag{1.7}$$

This condition is known as a general uncertainty relation between two quantities. A well known special case is that if P is the x component of the momentum operator and X is the operator of multiplication by x then $[P,X] = -i\hbar$ and $\Delta p \Delta x \geq \hbar/2$. The states where $\Delta p \Delta = \hbar/2$ are called coherent ones. They are treated such that the momentum and the coordinate are simultaneously semiclassical in a maximal possible way. A well known example is that if

$$\psi(x) = \frac{1}{a\sqrt{\pi}} exp[\frac{i}{\hbar}p_0 x - \frac{1}{2a^2}(x - x_0)^2]$$

then $\bar{X} = x_0$, $\bar{P} = p_0$, $\Delta x = a/\sqrt{2}$ and $\Delta p = \hbar/(a\sqrt{2})$.

Consider first a one dimensional motion. In standard textbooks on quantum mechanics, the presentation starts with a wave function $\psi(x)$ in coordinate space since it is implicitly assumed that the meaning of space coordinates is known. Then a question arises why $P = -i\hbar d/dx$ should be treated as the momentum operator. The explanation is as follows.

Consider wave functions having the form $\psi(x) = exp(ip_0 x/\hbar)a(x)$ where the amplitude $a(x)$ has a sharp maximum near $x = x_0 \in [x_1, x_2]$ such that $a(x)$ is not small only when $x \in [x_1, x_2]$. Then Δx is of order $x_2 - x_1$ and the condition that the coordinate is semiclassical is $\Delta x \ll |x_0|$. Since $-i\hbar d\psi(x)/dx = p_0\psi(x) - i\hbar exp(ip_0 x/\hbar)da(x)/dx$, we see that $\psi(x)$ will be approximately the eigenfunction of $-i\hbar d/dx$ with the eigenvalue p_0 if $|p_0 a(x)| \gg \hbar|da(x)/dx|$. Since $|da(x)/dx|$ is of order $|a(x)/\Delta x|$, we have a condition $|p_0\Delta x| \gg \hbar$. Therefore if the momentum operator is $-i\hbar d/dx$, the uncertainty of momentum Δp is of order $\hbar/\Delta x$, $|p_0| \gg \Delta p$ and this implies that the momentum is also semiclassical. At the same time, $|p_0\Delta x|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[x_1, x_2]$. Therefore the number of oscillations should be much greater than unity. In particular, semiclassical approximation cannot be valid if Δx is very small, but on the other hand, Δx cannot be very large since it should be much less than x_0. Another justification of the fact that $-i\hbar d/dx$ is the momentum operator is that in the formal limit $\hbar \to 0$ the Schroedinger equation becomes the Hamilton-Jacobi equation. This discussion resembles a well known discussion on the validity of geometrical optics: it is valid when the wave length is much less than characteristic dimensions of the problem.

We conclude that the choice of $-i\hbar d/dx$ as the momentum operator is justified from the requirement that in semiclassical approximation this operator becomes the classical momentum. However, it is obvious that this requirement does not define the operator uniquely: any operator $\tilde{P}$ such that $\tilde{P} - P$ disappears in semiclassical limit, also can be called the momentum operator.

One might say that the choice $P = -i\hbar d/dx$ can also be justified from the following considerations. In nonrelativistic quantum mechanics we assume that the theory should be invariant under the action of the Galilei group, which is a group of transformations of Galilei spacetime. The x component of the momentum operator should be the generator corresponding to spatial translations along the x axis and $-i\hbar d/dx$ is precisely the required operator. In this consideration one assumes that spacetime has a physical meaning while, as noted in Sect. 1.3, this is not the case.

As noted in Sect. 1.4, one should start not from spacetime but from a symmetry algebra. Therefore in nonrelativistic quantum mechanics we should start from the Galilei algebra and consider its IRs. For simplicity we again consider a one dimensional case. Let $P_x = P$ be one of representation operators in an IR of the Galilei algebra. We can implement this IR in a Hilbert space of functions $\psi(p)$ such that $\int_{-\infty}^{\infty} |\psi(p)|^2 dp < \infty$ and P is the operator of multiplication by p, i.e. $P\psi(p) = p\psi(p)$. Then a question arises how the operator of the x coordinate should be defined. In contrast with the momentum operator, the coordinate one is not defined by the representation and so it should be defined from additional assumptions. Probably a future quantum theory of measurements will make it possible to construct operators of physical quantities from the rules how these quantities should be measured. However, at present we can construct necessary operators only from rather intuitive considerations.

By analogy with the above discussion, one can say that semiclassical wave functions should be of the form $\psi(p) = exp(-ix_0p/\hbar)a(p)$ where the amplitude $a(p)$ has a sharp maximum near $p = p_0 \in [p_1, p_2]$ such that $a(p)$ is not small only when $p \in [p_1, p_2]$. Then Δp is of order $p_2 - p_1$ and the condition that the momentum is semiclassical is $\Delta p \ll |p_0|$. Since $i\hbar d\psi(p)/dp = x_0\psi(p) + i\hbar exp(-ix_0p/\hbar)da(p)/dp$, we see that $\psi(p)$ will be approximately the eigenfunction of $i\hbar d/dp$ with the eigenvalue x_0 if $|x_0 a(p)| \gg \hbar |da(p)/dp|$. Since $|da(p)/dp|$ is of order $|a(p)/\Delta p|$, we have a condition $|x_0 \Delta p| \gg \hbar$. Therefore if the coordinate operator is $X = i\hbar d/dp$, the uncertainty of coordinate Δx is of order $\hbar/\Delta p$, $|x_0| \gg \Delta x$ and this implies that the coordinate defined in such a way is also semiclassical. We can also note that $|x_0\Delta p|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[p_1, p_2]$ and therefore the number of oscillations should be much greater than unity. It is also clear that semiclassical approximation cannot be valid if Δp is very small, but on the other hand, Δp cannot be very large since it should be much less than p_0.

Although this definition of the coordinate operator has much in common with standard definition of the momentum operators, several questions arise. First of all, by analogy with the discussion about the momentum operator, one can say that the condition that in classical limit the coordinate operator should become the classical coordinate does not define the operator uniquely. One might require that the coordinate operator should correspond to translations in momentum space or be the operator of multiplication by x where the x representation is defined as a Fourier transform of the p representation but these requirements are not justified. The condition $|x_0| \gg \Delta x$ might seem to be unphysical since x_0 depends on the choice of the origin in the x space while Δx does not depend on this choice. Therefore a conclusion whether the coordinate is semiclassical or not depends on the choice of the reference frame. However, one can notice that not the coordinate itself has a physical meaning but only a relative coordinate between two particles.

Nevertheless, the above definition of the coordinate operator is not fully in line with what we think is a physical coordinate operator. To illustrate this point, consider, for example a measurement of the distance between some particle and the electron in a hydrogen atom. We expect that Δx cannot be less than the Bohr radius. Therefore if x_0 is of order of the Bohr radius, the coordinate cannot be semiclassical. One might think that the accuracy of the coordinate measurement can be defined as $|\Delta x/x_0|$ and therefore if we succeed in keeping Δx to be of order of the Bohr radius when we increase $|x_0|$ then the coordinate will be measured with a better and better accuracy when $|x_0|$ becomes greater. This intuitive understanding might be correct if the distance to the electron is measured in a laboratory where a distance is of order of centimeters or meters. However, is this intuition correct when we measure distances between macroscopic bodies? In the spirit of GR, the distance between two bodies which are far from each other should be measured by sending a light signal and waiting when it returns back. However, when a reflected signal is obtained, some time has passed and we don't know what happened to the body of interest (e.g. if the body is moving with a high speed, if the Universe is expanding etc.). For such experiments the logic is opposite to what we have with the standard definition of the coordinate operator in quantum mechanics: the accuracy of measurements is better not when the distance is greater but when it is less. One might think that if we consider not very long time intervals then for nonrelativistic particles such a measurement defines the coordinate with a good accuracy. However, it is a problem how to define the distance operator between a macroscopic body and a photon. This observation is in line with the remarks in Sect. 1.3 that it is not possible to define the photon wave function in coordinate representation.

Consider now the nonrelativistic Schroedinger equation for a free particle with the mass m and energy E. A separation of variables in spherical coordinates makes it possible to consider this equation separately in states with the orbital angular momentum l:

$$\frac{d^2\chi}{dr^2}+[\frac{2mE}{\hbar^2}-\frac{l(l+1)}{r^2}]\chi=0 \tag{1.8}$$

where $\chi(r)$ is a radial wave function such that $\int_0^\infty |\chi(r)|^2dr<\infty$. A solution satisfying the condition $\chi(0)=0$ is proportional to $j_l(kr)$ where $k=(2mE)^{1/2}/\hbar$ and

$$j_l(x)=(-x)^l(\frac{1}{x}\frac{d}{dx})^l\frac{sinx}{x} \tag{1.9}$$

is a spherical Bessel function of the lth order. One might think that at very large distances this solution is semiclassical. The asymptotic expression for $j_l(x)$ at very large distances can be obtained from Eq. (1.9) by differentiating only $sinx$ and the result is $j_l(x)\approx sin(x-l\pi/2)$. Therefore the result is a linear combination of two rapidly oscillating exponents and semiclassical approximation is valid. A question arises at what distances this approximation is correct. The asymptotic decomposition of the spherical Bessel function at large distances is (see e.g. Ref. [29])

$$j_l(x) = sin(x - l\pi/2)[\sum_{m=0}^{M-1}(-1)^m(l+1/2,2m)(2x)^{-2m} + O(|x|)^{-2M}] +$$
$$cos(x - l\pi/2)[\sum_{m=0}^{M-1}(-1)^m(l+1/2,2m+1)(2x)^{-2m-1} + O(|x|)^{-2M-1}] \quad (1.10)$$

where $(a,b) = 2^{-2b}(4a^2-1)(4a^2-3^2)\cdots[4a^2-(2b-1)^2]/b!$ is the Hankel symbol. Therefore when l is large, for the validity of the approximation $j_l(x) \approx sin(x-l\pi/2)$, the quantity $x = kr$ should be not only much greater than l but even much greater than l^2. Nevertheless, as argued in standard textbooks (see e.g. Ref. [30]), for the validity of semiclassical approximation it suffices to require that $l \gg 1$ regardless whether $j_l(x) \approx sin(x - l\pi/2)$ is valid or not. We will investigate the validity of semiclassical approximation in Sect. 3.4. In macroscopic experiments the condition $l \gg 1$ is always satisfied with a very high accuracy. For illustration, consider a photon moving in approximately radial direction away from the the Earth surface. Suppose that the photon energy equals the bound energy of the ground state of the hydrogen atom (27.2ev). Then in units $c = \hbar = 1$ this energy is of order $10^7/cm$. Therefore even if the lever arm of the photon trajectory with respect to the Earth center is $1cm$, the value of l is of order 10^7. In other experiments considered in GR this value is greater by many orders of magnitude. On the other hand, it is usually tacitly assumed that the width of the momentum distribution is sufficiently large. However, in Chaps. 3-5 we argue that in GFQT this is not always the case.

1.6 The content of this paper

In Chap. 2 we construct IRs of the dS algebra following the book by Mensky [31]. This construction makes it possible to show that the well known cosmological repulsion is simply a kinematical effect in dS quantum mechanics. The derivation involves only standard quantum mechanical notions. It does not require dealing with dS space, metric tensor, connection and other notions of Riemannian geometry. As argued in the preceding sections, fundamental quantum theory should not involve spacetime at all. In our approach the cosmological constant problem does not exist and there is no need to involve dark energy or other fields for explaining this problem.

In Chap. 3 we construct IRs in the basis where all quantum numbers are discrete. This makes it possible to investigate for which two-body wave functions one can get standard Newton's law of gravity and the precession of Mercury's perihelion. The explicit construction is given in Sects. 3.5 and 3.6. In Sects. 3.7 and 3.8 we discuss the problem of evolution in dS theory and well known gravitational experiments with light.

In Chap. 4 we argue that fundamental quantum theory should be based on Galois fields rather than complex numbers. In our approach, standard theory is a special case of a quantum theory over a Galois field (GFQT) in a formal limit when the characteristic of the field p becomes infinitely large. We tried to make the presentation as self-contained as possible without assuming that the reader is familiar with Galois fields.

In Chap. 5 we construct semiclassical states in GFQT and discuss the problem of calculating the gravitational constant. Finally, Chap. 6 is the discussion.

Chapter 2

Basic properties of de Sitter invariant quantum theories

2.1 dS invariance vs. AdS and Poincare invariance

As already mentioned, the motivation for this work is to investigate whether standard gravity can be obtained in the framework of a free theory. In standard nonrelativistic approximation, gravity is characterized by the term $-Gm_1m_2/r$ in the mean value of the mass operator. Here G is the gravitational constant, m_1 and m_2 are the particle masses and r is the distance between the particles. Since the kinetic energy is always positive, the free nonrelativistic mass operator is positive definite and therefore there is no way to obtain gravity in the framework of the free theory. Analogously, in Poincare invariant theory the spectrum of the free two-body mass operator belongs to the interval $[m_1 + m_2, \infty)$ while the existence of gravity necessarily requires that the spectrum should contain values less than $m_1 + m_2$.

In theories where the symmetry algebra is the AdS algebra so(2,3), the structure of IRs is well known (see e.g. Ref. [32]). In particular, for positive energy IRs the AdS Hamiltonian has the spectrum in the interval $[m, \infty)$ and m has the meaning of the mass. Therefore the situation is pretty much analogous to that in Poincare invariant theories. In particular, the free two-body mass operator again has the spectrum in the interval $[m_1+m_2, \infty)$ and therefore there is no way to reproduce gravitational effects in the free AdS invariant theory.

As noted in Sect. 1.2, the existing experimental data practically exclude the possibility that $\Lambda \leq 0$ since the cosmological acceleration is not zero and is a consequence of repulsion, not attraction. This is a strong argument in favor of dS symmetry vs. Poincare and AdS ones. As argued in Sect. 1.4, quantum theory should start not from spacetime but from a symmetry algebra. Therefore the choice of dS symmetry is natural and the cosmological constant problem does not exist. However, the majority of physicists prefer to start from a flat spacetime and treat Poincare symmetry as fundamental while dS one as emergent.

In contrast to the situation in Poincare and AdS invariant theories, the free mass operator in dS theory is not bounded below by the value of $m_1 + m_2$. The discussion in Sect. 2.3 shows that this property by no means implies that the theory is unphysical. Therefore if one has a choice between Poincare, AdS and dS symmetries then the only chance to describe gravity in a free theory is to choose dS symmetry.

2.2 IRs of the dS Algebra

If we accept dS symmetry on quantum level as described in Sect. 1.4, a question arises how elementary particles in quantum theory should be defined. A discussion of numerous controversial approaches can be found, for example in the recent paper [33]. Although particles are observables and fields are not, in the spirit of QFT, fields are more fundamental than particles, and a possible definition is as follows [34]: *It is simply a particle whose field appears in the Lagrangian. It does not matter if it's stable, unstable, heavy, light—if its field appears in the Lagrangian then it's elementary, otherwise it's composite.* Another approach has been developed by Wigner in his investigations of unitary irreducible representations (UIRs) of the Poincare group [35]. In view of this approach, one might postulate that a particle is elementary if the set of its wave functions is the space of an IR of the symmetry group or Lie algebra in the given theory. Since we do not accept approaches based on spacetime background then by analogy with the Wigner approach we accept that, *by definition*, elementary particles in dS invariant theory are described by IRs of the dS algebra by Hermitian operators. For different reasons, there exists a vast literature not on such IRs but on UIRs of the dS group. References to this literature can be found e.g., in our papers [4, 1] where we used the results on UIRs of the dS group for constructing IRs of the dS algebra by Hermitian operators. In this section we will describe the construction proceeding from an excellent description of UIRs of the dS group in a book by Mensky [31]. The final result is given by explicit expressions for the operators M^{ab} in Eq. (2.15). The readers who are not interested in technical details can skip the derivation.

The elements of the SO(1,4) group will be described in the block form

$$g = \left\| \begin{array}{ccc} g_0^0 & \mathbf{a}^T & g_4^0 \\ \mathbf{b} & r & \mathbf{c} \\ g_0^4 & \mathbf{d}^T & g_4^4 \end{array} \right\| \tag{2.1}$$

where

$$\mathbf{a} = \left\| \begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array} \right\| \quad \mathbf{b}^T = \| \; b_1 \quad b_2 \quad b_3 \; \| \quad r \in SO(3) \tag{2.2}$$

and the subscript ^{T} means a transposed vector.

UIRs of the SO(1,4) group belonging to the principle series of UIRs are induced from UIRs of the subgroup H (sometimes called "little group") defined as follows [31]. Each element of H can be uniquely represented as a product of elements of the subgroups SO(3), A and $\mathbf{T}$: $h = r\tau_A \mathbf{a_T}$ where

$$\tau_A = \left\| \begin{array}{ccc} cosh(\tau) & 0 & sinh(\tau) \\ 0 & 1 & 0 \\ sinh(\tau) & 0 & cosh(\tau) \end{array} \right\| \qquad \mathbf{a_T} = \left\| \begin{array}{ccc} 1+\mathbf{a}^2/2 & -\mathbf{a}^T & \mathbf{a}^2/2 \\ -\mathbf{a} & 1 & -\mathbf{a} \\ -\mathbf{a}^2/2 & \mathbf{a}^T & 1-\mathbf{a}^2/2 \end{array} \right\| \tag{2.3}$$

The subgroup A is one-dimensional and the three-dimensional group $\mathbf{T}$ is the dS analog of the conventional translation group (see e.g., Ref. [31]). We believe it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the SO(3) group. It should also be clear when r is a true element of the SO(3) group or belongs to the SO(3) subgroup of the SO(1,4) group. Note that standard UIRs of the Poincare group are induced from the little group, which is a semidirect product of SO(3) and four-dimensional translations and so the analogy between UIRs of the Poincare and dS groups is clear.

Let $r \to \Delta(r;\mathbf{s})$ be an UIR of the group SO(3) with the spin $\mathbf{s}$ and $\tau_A \to exp(im_{dS}\tau)$ be a one-dimensional UIR of the group A, where m_{dS} is a real parameter. Then UIRs of the group H used for inducing to the SO(1,4) group, have the form

$$\Delta(r\tau_A\mathbf{a_T}; m_{dS}, \mathbf{s}) = exp(im_{dS}\tau)\Delta(r;\mathbf{s}) \tag{2.4}$$

We will see below that m_{dS} has the meaning of the dS mass and therefore UIRs of the SO(1,4) group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let G=SO(1,4) and $X = G/H$ be the factor space (or coset space) of G over H. The notion of the factor space is well known (see e.g., Ref. [31]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class x. The choice of representatives is not unique since if x_G is a representative of the class $x \in G/H$ then $x_G h_0$, where h_0 is an arbitrary element from H, also is a representative of the same class. It is well known that X can be treated as a left G space. This means that if $x \in X$ then the action of the group G on X can be defined as follows: if $g \in G$ then gx is a class containing gx_G (it is easy to verify that such an action is correctly defined). Suppose that the choice of representatives is somehow fixed. Then $gx_G = (gx)_G(g,x)_H$ where $(g,x)_H$ is an element of H. This element is called a factor.

The explicit form of the operators M^{ab} depends on the choice of representatives in the space G/H. As explained in papers on UIRs of the SO(1,4) group (see e.g., Ref. [31]), to obtain the possible closest analogy between UIRs of the SO(1,4) and Poincare groups, one should proceed as follows. Let $\mathbf{v}_L$ be a representative of the Lorentz group in the factor space SO(1,3)/SO(3) (strictly speaking, we should consider $SL(2,C)/SU(2)$). This space can be represented as the velocity hyperboloid with the Lorentz invariant measure

$$d\rho(\mathbf{v}) = d^3\mathbf{v}/v_0 \tag{2.5}$$

where $v_0 = (1+\mathbf{v}^2)^{1/2}$. Let $I \in SO(1,4)$ be a matrix which formally has the same form as the metric tensor η. One can show (see e.g., Ref. [31] for details) that $X = G/H$ can be represented as a union of three spaces, X_+, X_- and X_0 such that X_+ contains classes $\mathbf{v}_L h$, X_- contains classes $\mathbf{v}_L I h$ and X_0 has measure zero relative to the spaces X_+ and X_-.

As a consequence, the space of UIR of the SO(1,4) group can be implemented as follows. If s is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of UIR of the group SU(2) with the spin s. Then the space of UIR is the space of functions $\{f_1(\mathbf{v}), f_2(\mathbf{v})\}$ on two Lorentz hyperboloids with the range in the space of UIR of the group SU(2) with the spin s and such that

$$\int [||f_1(\mathbf{v})||^2 + ||f_2(\mathbf{v})||^2] d\rho(\mathbf{v}) < \infty \tag{2.6}$$

It is well-known that positive energy UIRs of the Poincare and AdS groups (associated with elementary particles) are implemented on an analog of X_+ while negative energy UIRs (associated with antiparticles) are implemented on an analog of X_-. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy UIRs are fully independent. At the same time, the dS group contains such elements (e.g., I [31]) and for this reason its UIRs can be implemented only on the union of X_+ and X_-. Even this fact is a strong indication that UIRs of the dS group cannot be interpreted in the same way as UIRs of the Poincare and AdS groups.

A general construction of the operators M^{ab} is as follows. We first define right invariant measures on $G = SO(1,4)$ and H. It is well known that for semisimple Lie groups (which is the case for the dS group), the right invariant measure is simultaneously the left invariant one. At the same time, the right invariant measure $d_R(h)$ on H is not the left invariant one, but has the property $d_R(h_0 h) = \Delta(h_0) d_R(h)$, where the number function $h \to \Delta(h)$ on H is called the module of the group H. It is easy to show [31] that

$$\Delta(r\tau_A \mathbf{a_T}) = exp(-3\tau) \tag{2.7}$$

Let $d\rho(x)$ be a measure on $X = G/H$ compatible with the measures on G and H. This implies that the measure on G can be represented as $d\rho(x) d_R(h)$. Then one can show [31] that if X is a union of X_+ and X_- then the measure $d\rho(x)$ on each Lorentz hyperboloid coincides with that given by Eq. (2.5). Let the representation space be implemented as the space of functions $\varphi(x)$ on X with the range in the space of UIR of the SU(2) group such that

$$\int ||\varphi(x)||^2 d\rho(x) < \infty \tag{2.8}$$

Then the action of the representation operator $U(g)$ corresponding to $g \in G$ is defined as

$$U(g)\varphi(x) = [\Delta((g^{-1}, x)_H)]^{-1/2} \Delta((g^{-1}, x)_H; m_{dS}, \mathbf{s})^{-1} \varphi(g^{-1}x) \tag{2.9}$$

One can directly verify that this expression defines a unitary representation. Its irreducibility can be proved in several ways (see e.g., Ref. [31]).

As noted above, if X is the union of X_+ and X_-, then the representation space can be implemented as in Eq. (2.4). Since we are interested in calculating only the explicit form of the operators M^{ab}, it suffices to consider only elements of $g \in G$ in an infinitely small vicinity of the unit element of the dS group. In that case one can calculate the action of representation operators on functions having the carrier in X_+ and X_- separately. Namely, as follows from Eq. (2.7), for such $g \in G$, one has to find the decompositions

$$g^{-1}\mathbf{v}_L = \mathbf{v}'_L r'(\tau')_A(\mathbf{a}')_{\mathbf{T}} \quad g^{-1}\mathbf{v}_L I = \mathbf{v}"_L I r"(\tau")_A(\mathbf{a}")_{\mathbf{T}} \tag{2.10}$$

where $r', r" \in SO(3)$. In these expressions it suffices to consider only elements of H belonging to an infinitely small vicinity of the unit element.

The problem of choosing representatives in the spaces SO(1,3)/SO(3) or SL(2.C)/SU(2) is well known in standard theory. The most usual choice is such that $\mathbf{v}_L$ as an element of SL(2,C) is given by

$$\mathbf{v}_L = \frac{v_0 + 1 + \mathbf{v}\sigma}{\sqrt{2(1+v_0)}} \tag{2.11}$$

Then by using a well known relation between elements of SL(2,C) and SO(1,3) we obtain that $\mathbf{v}_L \in SO(1,4)$ is represented by the matrix

$$\mathbf{v}_L = \left\| \begin{array}{ccc} v_0 & \mathbf{v}^T & 0 \\ \mathbf{v} & 1 + \mathbf{v}\mathbf{v}^T/(v_0+1) & 0 \\ 0 & 0 & 1 \end{array} \right\| \tag{2.12}$$

As follows from Eqs. (2.4) and (2.9), there is no need to know the expressions for $(\mathbf{a}')_{\mathbf{T}}$ and $(\mathbf{a}")_{\mathbf{T}}$ in Eq. (2.10). We can use the fact [31] that if e is the five-dimensional vector with the components $(e^0 = 1, 0, 0, 0, e^4 = -1)$ and $h = r\tau_A\mathbf{a}_{\mathbf{T}}$, then $he = exp(-\tau)e$ regardless of the elements $r \in SO(3)$ and $\mathbf{a}_{\mathbf{T}}$. This makes it possible to easily calculate $(\mathbf{v}'_L, \mathbf{v}"_L, (\tau')_A, (\tau")_A)$ in Eq. (2.10). Then one can calculate $(r', r")$ in these expressions by using the fact that the SO(3) parts of the matrices $(\mathbf{v}'_L)^{-1}g^{-1}\mathbf{v}_L$ and $(\mathbf{v}"_L)^{-1}g^{-1}\mathbf{v}_L$ are equal to r' and $r"$, respectively.

The relation between the operators $U(g)$ and M^{ab} is as follows. Let L_{ab} be the basis elements of the Lie algebra of the dS group. These are the matrices with the elements

$$(L_{ab})^c_d = \delta^c_d\eta_{bd} - \delta^c_b\eta_{ad} \tag{2.13}$$

They satisfy the commutation relations

$$[L_{ab}, L_{cd}] = \eta_{ac}L_{bd} - \eta_{bc}L_{ad} - \eta_{ad}L_{bc} + \eta_{bd}L_{ac} \tag{2.14}$$

Comparing Eqs. (1.5) and (2.14) it is easy to conclude that the M^{ab} should be the representation operators of $-iL^{ab}$. Therefore if $g = 1 + \omega_{ab}L^{ab}$, where a sum over repeated indices is assumed and the ω_{ab} are such infinitely small parameters that $\omega_{ab} = -\omega_{ba}$ then $U(g) = 1 + i\omega_{ab}M^{ab}$.

We are now in position to write down the final expressions for the operators M^{ab}. Their action on functions with the carrier in X_+ has the form

$$
\begin{aligned}
&\mathbf{J} = l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N} == -iv_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&\mathbf{B} = m_{dS}\mathbf{v} + i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},\\
&\mathcal{E} = m_{dS}v_0 + iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2})
\end{aligned}
\tag{2.15}
$$

where $\mathbf{J} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = \{M^{41}, M^{42}, M^{43}\}$, $\mathbf{s}$ is the spin operator, $\mathbf{l}(\mathbf{v}) = -i\mathbf{v}\times\partial/\partial\mathbf{v}$ and $\mathcal{E} = M^{40}$. The action of the generators on functions with the carrier in X_- is analogous [4, 1] but the corresponding expressions will not be needed in this paper.

2.2.1 Discussion

In deriving Eq. (2.15) we used only the commutation relations (1.5), no approximations have been made and the results are exact. In particular, the dS space, the cosmological constant and the Riemannian geometry have not been involved at all. Nevertheless, the expressions for the representation operators is all we need to have the maximum possible information in quantum theory.

As shown in the literature (see e.g. Ref. [31]), the above construction of IRs applies to IRs of the principle series where m_{dS} is a real parameter such that $|m_{dS}| > 0$. Therefore such IRs are called massive.

A problem arises how m_{dS} is related to the standard particle mass m in Poincare invariant theory. A general notion of contraction has been developed in Ref. [36]. In our case it can be performed as follows. Let us assume that $m_{dS} > 0$ and define $m = m_{dS}/R$, $\mathbf{P} = \mathbf{B}/R$ and $E = \mathcal{E}/R$. The set of operators $(E, \mathbf{P})$ is the Lorentz vector since its components can be written as $M^{4\nu}$ ($\nu = 0, 1, 2, 3$). Then, as follows from Eq. (1.5), in the limit when $R \to \infty$, $m_{dS} \to \infty$ but m_{ds}/R is finite, one obtains a standard representation of the Poincare algebra for a particle with the mass m such that $\mathbf{P} = m\mathbf{v}$ is the particle momentum and $E = mv_0$ is the particle energy. Therefore m is the standard mass in Poincare invariant theory and the operators of the Lorentz algebra $(\mathbf{N}, \mathbf{J})$ have the same form for the Poincare and dS algebras.

In Sect. 1.2 we argued that fundamental physical theory should not contain dimensional parameters at all. In this connection it is interesting to note that the de Sitter mass m_{dS} is a ratio of the radius of the Universe R to the Compton wave length of the particle under consideration. Therefore even for elementary particles the de Sitter masses are very large. For example, if R is of order $10^{26}m$ then the de

Sitter masses of the electron, the Earth and the Sun are of order 10^{39}, 10^{93} and 10^{99}, respectively.

In Standard Model (based on Poicare invariance) only massless Weyl particles are treated as fundamental. Such particles are characterized by helicity rather than spin. In view of this fact one can pose a problem whether the above IRs have a physical meaning. First of all, we note that the results on IRs can be applied not only to elementary particles but even to macroscopic bodies when it suffices to consider their motion as a whole. This is the case when the distances between the bodies are much greater that their sizes. In Poincare invariant theory, IRs describing massless Weyl particles can be obtained as a limit of massive IRs when $m \to 0$ with a special case of representatives in the factor space $SL(2,C)/SU(2)$. However, in dS theory such a limit does not exist and therefore there are no Weyl particles in dS theory (see Sect. 6 in Ref. [1]). In his book [31] Mensky notes that as dS analogs of massless IRs one might consider either IRs with $m_{dS} = 0$ belonging to the complementary series or IRs with $-im_{dS} = 1/2$ belonging to the discrete series but these possibilities have not been investigated in details. In standard theory it is believed that the photon is a true massless particle. The present upper level for its mass is $10^{-18}ev$ which seems to be an extremely tiny quantity. However, the corresponding dS mass is of order 10^{15} and so even the mass which is treated as extremely small in Poincare invariant theory might be very large in dS invariant theory. In the present paper we assume that the photon can be described by IRs of the principle series discussed above.

The operator $\mathbf{N}$ contains $i\partial/\partial\mathbf{v}$ which is proportional to the standard coordinate operator $i\partial/\partial\mathbf{p}$. The factor v_0 in $\mathbf{N}$ is needed for Hermiticity since the volume element is given by Eq. (2.5). Such a construction can be treated as a relativistic generalization of standard coordinate operator and then the orbital part of $\mathbf{N}$ is proportional to the Newton-Wigner position operator [20]. However, it is well known that this operator does not satisfy all the requirements for the coordinate operator. First of all, as noted in Sect. 1.3, in relativistic theory the coordinate cannot be measured with the accuracy better than $\hbar/mc$. Another argument is as follows. If we find eigenfunctions of the x component of the Newton-Wigner position operator with eigenvalues x and construct a wave function which at $t = 0$ has a finite carrier in x then, as follows from the Schroedinger equation with the relativistic Hamiltonian, at any $t > 0$ this function will have an infinite carrier. In other words, the wave function will be instantly spread over the whole space while the speed of propagation should not exceed c. These remarks show that the construction of the physical coordinate operator is far from being obvious.

As noted in Sect. 1.3, in experiments with the photon discussed in GR, the orbital angular momentum is very large. Therefore in this case the spin term in $\mathbf{J}$ can be neglected. The same is true for macroscopic bodies if their internal rotation is not extremely fast. Since $|v_0 > |\mathbf{v}|$, the orbital part of the operator $\mathbf{N}$ is also much greater than its spin part. The orbital part of the operator $\mathbf{B}$ is typically much greater than its spin part; this is clear even from the fact that in Poincare limit this part is proportional to R while the spin does not depend on R. In view of these remarks, in the present paper we will not consider spin effects.

It is well known that in Poincare invariant theory the operator $I_{2P} = E^2 - \mathbf{P}^2$ is the Casimir operator, *i.e.*, it commutes with all the representation operators. According to the well known Schur lemma in representation theory, all elements in the space of IR are eigenvectors of the Casimir operators with the same eigenvalue. In particular, they are the eigenvectors of the operator I_{2P} with the eigenvalue m^2. As follows from Eq. (1.5), in the dS case the Casimir operator of the second order is

$$I_2 = -\frac{1}{2}\sum_{ab} M_{ab}M^{ab} = \mathcal{E}^2 + \mathbf{N}^2 - \mathbf{B}^2 - \mathbf{J}^2 \qquad (2.16)$$

and a direct calculation shows that for operators (2.15) the numerical value of I_2 is $m_{dS}^2 - s(s+1) + 9/4$. In Poincare invariant theory the value of the spin is related to the Casimir operator of the fourth order which can be constructed from the Pauli-Lubanski vector. An analogous construction exists in dS invariant theory but we will not dwell on this.

2.3 dS quantum mechanics and cosmological repulsion

Consider the nonrelativistic approximation when $|\mathbf{v}| \ll 1$. If we wish to work with units where the dimension of velocity is m/sec, we should replace $\mathbf{v}$ by $\mathbf{v}/c$. If $\mathbf{p} = m\mathbf{v}$ then it is clear from the expressions for $\mathbf{B}$ in Eq. (2.15) that $\mathbf{p}$ becomes the real momentum $\mathbf{P}$ only in the limit $R \to \infty$. Now by analogy with nonrelativistic quantum mechanics (see Sect. 1.5), we *define* the position operator $\mathbf{r}$ as $i\partial/\partial\mathbf{p}$ and in that case the operator $\mathbf{N}$ in Eq. (2.15) becomes $-E\mathbf{r}$. At this stage we do not have any coordinate space yet. However, the consideration in Sect. 1.5 shows that there exist states where both, $\mathbf{p}$ and $\mathbf{r}$ are semiclassical. In this approximation we can treat them as usual vectors and neglect their commutators. Then as follows from Eq. (2.15)

$$\mathbf{P} = \mathbf{p} + mc\mathbf{r}/R \quad H = \mathbf{p}^2/2m + c\mathbf{p}\mathbf{r}/R \qquad (2.17)$$

where $H = E - mc^2$ is the classical nonrelativistic Hamiltonian. As follows from these expressions,

$$H(\mathbf{P}, \mathbf{r}) = \frac{\mathbf{P}^2}{2m} - \frac{mc^2\mathbf{r}^2}{2R^2} \qquad (2.18)$$

The last term in this expression is the dS correction to the nonrelativistic Hamiltonian. It is interesting to note that the nonrelativistic Hamiltonian depends on c although it is usually believed that c can be present only in relativistic theory. This illustrates the fact mentioned in Section 1.2 that the transition to nonrelativistic theory understood as $|\mathbf{v}| \ll 1$ is more physical than that understood as $c \to \infty$. The presence of c in Eq. (2.18) is a consequence of the fact that this expression is written in standard units. In nonrelativistic theory c is usually treated as a very large quantity. Nevertheless, the last term in Eq. (2.18) is not large since we assume that R is very large. The result for one particle given by Eq. (1.4) is now a consequence of the equations of motion for the Hamiltonian given by Eq. (2.18).

Another way to show that our results are compatible with GR is as follows. The well known result of GR is that if the metric is stationary and differs slightly from the Minkowskian one then in the nonrelativistic approximation the curved spacetime can be effectively described by a gravitational potential $\varphi(\mathbf{r}) = (g_{00}(\mathbf{r}) - 1)/2c^2$. We now express x_0 in Eq. (1.2) in terms of a new variable t as $x_0 = t + t^3/6R^2 - t\mathbf{x}^2/2R^2$. Then the expression for the interval becomes

$$ds^2 = dt^2(1 - \mathbf{r}^2/R^2) - d\mathbf{r}^2 - (\mathbf{r}d\mathbf{r}/R)^2 \tag{2.19}$$

Therefore, the metric becomes stationary and $\varphi(\mathbf{r}) = -\mathbf{r}^2/2R^2$ in agreement with Eq. (2.18).

Consider now a system of two free particles described by the variables $\mathbf{p}_j$ and $\mathbf{r}_j$ $(j = 1, 2)$. Define the standard nonrelativistic variables

$$\begin{aligned} &\mathbf{P}_{12} = \mathbf{p}_1 + \mathbf{p}_2 \quad \mathbf{q} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2) \\ &\mathbf{R}_{12} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \end{aligned} \tag{2.20}$$

where now we use $\mathbf{r}$ to denote the relative radius vector. Then if the particles are described by Eq. (2.15), the two-particle operators $\mathbf{P}$ and $\mathbf{E}$ in the non-relativistic approximation are given by

$$\mathbf{P} = \mathbf{P}_{12} + \frac{iM_0(\mathbf{q})}{R}\frac{\partial}{\partial \mathbf{P}_{12}} \quad E = M_0(\mathbf{q}) + \frac{\mathbf{P}_{12}^2}{2M_0(\mathbf{q})} + \frac{i}{R}(\mathbf{P}_{12}\frac{\partial}{\partial \mathbf{P}_{12}} + \mathbf{q}\frac{\partial}{\partial \mathbf{q}} + 3) \tag{2.21}$$

where $M_0(\mathbf{q}) = m_1 + m_2 + \mathbf{q}^2/2m_{12}$ and m_{12} is the reduced two-particle mass. As a consequence, the nonrelativistic mass operator $(E^2 - \mathbf{P}^2)^{1/2}$ in first order in $1/R$ is given by

$$M = m_1 + m_2 + \frac{\mathbf{q}^2}{2m_{12}} + \frac{i}{R}(\mathbf{q}\frac{\partial}{\partial \mathbf{q}} + \frac{3}{2}) \tag{2.22}$$

Therefore the classical internal nonrelativistic two-body Hamiltonian is

$$H_{nr}(\mathbf{q}, \mathbf{r}) = \frac{\mathbf{q}^2}{2m_{12}} + \frac{\mathbf{qr}}{R} \tag{2.23}$$

where $\mathbf{q}$ and $\mathbf{r}$ are the classical relative momentum and radius vector. Hence in semiclassical approximation the relative acceleration is again given by Eq. (1.4).

The fact that two free particles have a relative acceleration is well known for cosmologists who consider dS symmetry on classical level. This effect is called the dS antigravity. The term antigravity in this context means that the particles repulse rather than attract each other. In the case of the dS antigravity the relative acceleration of two free particles is proportional (not inversely proportional!) to the distance between them. This classical result is a special case of the dS symmetry on quantum level when semiclassical approximation works with a good accuracy.

In dS theory, the spectrum of the free two-body operator (2.22) is not bounded below by $m_1 + m_2$ and a question arises whether this is acceptable or not. In spherical coordinates the internal two-body Hamiltonian corresponding to the non-relativistic mass operator is

$$H_{nr} = \frac{q^2}{2m_{12}} + \frac{i}{R}(q\frac{\partial}{\partial q} + \frac{3}{2}) \tag{2.24}$$

where $q = |\mathbf{q}|$. This operator acts in the space of functions $\psi(q)$ such that

$$\int_0^\infty |\psi(q)|^2 q^2 dq < \infty$$

and the eigenfunction ψ_K of H_{nr} with the eigenvalue K satisfies the equation

$$q\frac{d\psi_K}{dq} = \frac{iRq^2}{m_{12}}\psi_K - (\frac{3}{2} + 2iRK)\psi_K \tag{2.25}$$

The solution of this equation is

$$\psi_K = \sqrt{\frac{R}{\pi}} q^{-3/2} exp(\frac{iRq^2}{2m_{12}} - 2iRKlnq) \tag{2.26}$$

and the normalization condition is $(\psi_K, \psi_{K'}) = \delta(K - K')$. The spectrum of the operator H_{nr} formally belongs to the interval $(-\infty, \infty)$ but this is a consequence of the nonrelativistic approximation and the fact that the square root for the mass operator was calculated in first order in $1/R$. However, the spectrum of H_{nr} for sure has negative values and therefore the spectrum of the mass operator has values less than $m_1 + m_2$.

Suppose that $\psi(q)$ is a wave function of some state. As follows from Eq. (2.26), the probability to have the value of the energy K in this state is defined by the coefficient $c(K)$ such that

$$c(K) = \sqrt{\frac{R}{\pi}} \int_0^\infty exp(-\frac{iRq^2}{2m_{12}} + 2iRKlnq)\psi(q)\sqrt{q}dq \tag{2.27}$$

If $\psi(q)$ does not depend on R and R is very large then $c(K)$ will practically be different from zero only if the integrand in Eq. (2.27) has a stationary point q_0, which is defined by the condition $K = q_0^2/2m_{12}$. Therefore, for negative K, when the stationary point is absent, the value of $c(K)$ will be exponentially small.

This result confirms that, as one might expect from Eq. (2.23), the dS antigravity is not important for local physics when $r \ll R$. At the same time, at cosmological distances the dS antigravity is much stronger than any other interaction (gravitational, electromagnetic *etc.*). Since the spectrum of the energy operator is defined by its behavior at large distances, this means that in dS theory there are no bound states. This does not mean that the theory is unphysical since stationary bound states in standard theory become semistationary with a very large lifetime if R is large. For example, as shown in Eqs. (14) and (19) of Ref. [37], a semiclassical calculation of the probability of the decay of the two-body composite system gives that the probability equals $w = exp(-\pi\epsilon/H)$ where ϵ is the binding energy and H is the Hubble constant. If we replace H by $1/R$ and assume that $R = 10^{26}m$ then for the probability of the decay of the ground state of the hydrogen atom we get that w is of order $exp(-10^{35})$ *i.e.*, an extremely small value. This result is in agreement with our remark after Eq. (2.27).

In Ref. [3] we discussed the following question. In standard quantum mechanics the free Hamiltonian H_0 and the full Hamiltonian H are not always unitarily equivalent since in the presence of bound states they have different spectra. However, in dS theory there are no bound states, the free and full Hamiltonians have the same spectra and therefore they are unitarily equivalent. Hence one can work in the formalism when interaction is introduced not by adding an interaction operator to the free Hamiltonian but by a unitary transformation of this operator.

Although the example of the dS antigravity is extremely simple, we can draw the following very important conclusions.

In our approach the phenomenon of the cosmological acceleration has an extremely simple explanation in the framework of dS quantum mechanics for a system of two free bodies. There is no need to involve dS space and Riemannian geometry since the fact that $\Lambda \neq 0$ should be treated not such that the spacetime background has a curvature (since the notion of the spacetime background is meaningless) but as an indication that the symmetry algebra is the dS algebra rather than the Poincare or AdS algebras. *Therefore for explaining the fact that $\Lambda \neq 0$ there is no need to involve dark energy or other quantum fields.*

Our result is in favor of the argument in Sect. 1.3 that in quantum theory it is possible to reproduce classical results of GR. Indeed, we see that standard classical dS antigravity has been obtained from a quantum operator without introducing any classical background. When the position operator is defined as $\mathbf{r} = i(\partial/\partial\mathbf{q})$ and time is defined by the condition that the Hamiltonian is the evolution operator then one recovers the classical result obtained by considering a motion of particles in classical dS spacetime.

The second conclusion is as follows. We have considered the particles as free, i.e. no interaction into the two-body system has been introduced. However, we have realized that when the two-body system in the dS theory is considered from the point of view of the Galilei invariant theory, the particles interact with each other. Although the reason of the effective interaction in our example is obvious, the existence of the dS antigravity poses the problem whether other interactions, e.g.

gravity, can be treated as a result of transition from a higher symmetry to Poincare or Galilei one.

The third conclusion is that if the dS antigravity is treated as an interaction then it is a true direct interaction since it is not a consequence of the exchange of virtual particles.

Finally, the fourth conclusion is as follows. The result of Eq. (2.26) shows that in the free dS theory the spectrum of the free Hamiltonian is not bounded below by zero and therefore the spectrum of the free mass operator has values less than $m_1 + m_2$ (see also Refs. [2, 4, 8, 1]). Therefore the fact that the spectrum of the free mass operator is not bounded below by the value $m_1 + m_2$, does not necessarily mean that the theory is unphysical. Moreover, if we accept the above arguments that dS symmetry is more relevant than Poincare and AdS ones, the existence of the spectrum below $m_1 + m_2$ is inevitable.

Our final remark is as follows. The consideration in this chapter involves only standard quantum-mechanical notions and in semiclassical approximation the results on the cosmological acceleration are compatible with GR. As argued in Sect. 1.5, the standard coordinate operator has some properties which do not correspond to what is expected from physical intuition; however, at least from mathematical point of view, at cosmological distances semiclassical approximation is valid with a very high accuracy. At the same time, as discussed in the next chapter, when distances are much less than cosmological ones, this operator should be modified. We consider a modification when the wave function contains a rapidly oscillating exponent depending on R. Then the probability to have negative values of K is not exponentially small as it should be in our approach to gravity.

Chapter 3

Algebraic description of irreducible representations

3.1 Construction of IRs in discrete basis

In this section we construct a pure algebraic implementation of IRs such that the basis is characterized only by discrete quantum numbers. This approach is of interest not only in standard dS quantum theory but also because the results can be used in a quantum theory over a Galois field (GFQT). To make relations between standard theory and GFQT more straightforward, we will modify the commutation relations 1.5 by writing them in the form

$$[M^{ab}, M^{cd}] = -2i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{3.1}$$

One might say that these relations are written in units $\hbar/2 = c = 1$. However, as noted in Sect. 1.2, fundamental quantum theory should not involve quantities $\hbar$ and c at all, and Eq. (3.1) indeed does not contain these quantities. The reason for writing the commutation relations in the form (3.1) rather than (1.5) is that in this case the minimum nonzero value of the angular momentum is 1 instead of 1/2. Therefore the spin of fermions is odd and the spin of bosons is even. This will be convenient in GFQT where 1/2 is a very large number (see Chap. 4).

As noted in Sect. 2.2, in this paper we will consider only massive IRs and will neglect spin effects. Therefore our goal is to construct massive spinless IRs in a discrete basis. By analogy with the method of little group in standard theory, one can first choose states which can be treated as rest ones and then obtain the whole representation space by acting on such states by certain linear combinations of representation operators.

Since $\mathbf{B}$ is a possible choice of the dS analog of the momentum operator, one might think that the rest states e_0 can be defined by the condition $\mathbf{B}e_0 = 0$. However, in the general case this is not consistent since, as follows from Eq. (3.1), different components of $\mathbf{B}$ do not commute with each other: as follows from Eq. (3.1) and the definitions of the operators $\mathbf{J}$ and $\mathbf{B}$ in Sect. 2.2, these operators commute with each other according to the rules

$$[J^j,J^k]=[B^j,B^k]=2ie_{jkl}J^l \quad [J^j,B^k]=2ie_{jkl}B^l \tag{3.2}$$

where the indices $j,k.l$ can take the values $1,2,3$, δ_{jk} is the Kronecker symbol, e_{jkl} is the absolutely antisymmetric tensor such that $e_{123}=1$ and a sum over repeated indices is assumed. Therefore a subspace of elements e_0 such that $B^je_0=0$ ($j=1,2,3$) is not closed under the action of the operators B^j.

Let us define the operators $\mathbf{J}'=(\mathbf{J}+\mathbf{B})/2$ and $\mathbf{J}"=(\mathbf{J}-\mathbf{B})/2$. As follows from Eq. (3.1), they satisfy the commutation relations

$$[J'^{j'},J"^k]=0 \quad [J'^j,J'^k]=2ie_{jkl}J'^l \quad [J"^j,J"^k]=2ie_{jkl}J"^l \tag{3.3}$$

Since in Poincare limit $\mathbf{B}$ is much greater than $\mathbf{J}$, as an analog of momentum one can treat $\mathbf{J}'$ instead of $\mathbf{B}$. Then one can define rest states e_0 by the condition that $\mathbf{J}'e_0=0$. In this case the subspace of rest states is defined consistently since it is invariant under the action of the operators $\mathbf{J}'$. Since the operators $\mathbf{J}'$ and $\mathbf{J}"$ commute with each other, one can define the internal angular momentum of the system as a reduction of $\mathbf{J}"$ on the subspace of rest states. In particular, in Ref. [2] we used such a construction for constructing IRs of the dS algebra in the method of $SU(2)\times SU(2)$ shift operators proposed by Hughes for constructing IRs of the SO(5) group [38]. In the spinless case the situation is simpler since for constructing IRs it suffices to choose only one vector e_0 such that

$$\mathbf{J}'e_0=\mathbf{J}"e_0=0 \quad I_2e_0=(w+9)e_0 \tag{3.4}$$

The last requirement reflects the fact that all elements from the representation space are eigenvectors of the Casimir operator I_2 with the same eigenvalue. When the representation operators satisfy Eq. (3.1), the numerical value of the operator I_2 is not as indicated after Eq. (2.16) but

$$I_2=w-s(s+2)+9 \tag{3.5}$$

where $w=m_{dS}^2$. Therefore for spinless particles the numerical value equals $w+9$.

As follows from Eq. (3.1) and the definitions of the operators $(\mathbf{J},\mathbf{N},\mathbf{B},\mathcal{E})$ in Sect. 2.2, in addition to Eqs. 3.2, the following relations are satisfied:

$$[\mathcal{E},\mathbf{N}]=2i\mathbf{B}\ [\mathcal{E},\mathbf{B}]=2i\mathbf{N}\ [\mathbf{J},\mathcal{E}]=0\ [B^j,N^k]=2i\delta_{jk}\mathcal{E}\ [J^j,N^k]=2ie_{jkl}N^l \tag{3.6}$$

We define $e_1=2\mathcal{E}e_0$ and

$$e_{n+1}=2\mathcal{E}e_n-[w+(2n+1)^2]e_{n-1} \tag{3.7}$$

These definitions make it possible to find e_n for any $n=0,1,2....$ As follows from Eqs. (3.2), (3.6) and (3.7), $\mathbf{J}e_n=0$ and $\mathbf{B}^2e_n=4n(n+2)e_n$. We use the notation $J_x=J^1$,

$J_y = J^2$, $J_z = J^3$ and analogously for the operators $\mathbf{N}$ and $\mathbf{B}$. Instead of the (xy) components of the vectors it may be sometimes convenient to use the $\pm$ components such that $J_x = J_+ + J_-$, $J_y = -i(J_+ - J_-)$ and analogously for the operators $\mathbf{N}$ and $\mathbf{B}$. We now define the elements e_{nkl} as

$$e_{nkl} = \frac{(2k+1)!!}{k!l!}(J_-)^l(B_+)^k e_n \tag{3.8}$$

Then a direct calculation using Eqs. (3.2-3.8) gives

$$\begin{aligned}
&\mathcal{E}e_{nkl} = \frac{n+1-k}{2(n+1)}e_{n+1,kl} + \frac{n+1+k}{2(n+1)}[w+(2n+1)^2]e_{n-1,kl}\\
&N_+e_{nkl} = \frac{i(2k+1-l)(2k+2-l)}{8(n+1)(2k+1)(2k+3)}\{e_{n+1,k+1,l} -\\
&[w+(2n+1)^2]e_{n-1,k+1,l}\} -\\
&\frac{i}{2(n+1)}\{(n+1-k)(n+2-k)e_{n+1,k-1,l-2} -\\
&(n+k)(n+1+k)[w+(2n+1)^2]e_{n-1,k-1,l-2}\}\\
&N_-e_{nkl} = \frac{-i(l+1)(l+2)}{8(n+1)(2k+1)(2k+3)}\{e_{n+1,k+1,l+2} -\\
&[w+(2n+1)^2]e_{n-1,k+1,l+2}\} +\\
&\frac{i}{2(n+1)}\{(n+1-k)(n+2-k)e_{n+1,k-1,l} -\\
&(n+k)(n+1+k)[w+(2n+1)^2]e_{n-1,k-1,l}\}\\
&N_ze_{nkl} = \frac{-i(l+1)(2k+1-l)}{4(n+1)(2k+1)(2k+3)}\{e_{n+1,k+1,l+1} -\\
&[w+(2n+1)^2]e_{n-1,k+1,l+1}\} -\\
&\frac{i}{n+1}\{(n+1-k)(n+2-k)e_{n+1,k-1,l-1} -\\
&(n+k)(n+1+k)[w+(2n+1)^2]e_{n-1,k-1,l-1}\}
\end{aligned} \tag{3.9}$$

$$B_+e_{nkl} = \frac{(2k+1-l)(2k+2-l)}{2(2k+1)(2k+3)}e_{n,k+1,l} -$$
$$2(n+1-k)(n+1+k)e_{n,k-1,l-2}$$
$$B_-e_{nkl} = \frac{(l+1)(l+2)}{2(2k+1)(2k+3)}e_{n,k+1,l+2} +$$
$$2(n+1-k)(n+1+k)e_{n,k-1,l}$$
$$B_ze_{nkl} = \frac{(l+1)(2k+1-l)}{2(2k+1)(2k+3)}e_{n,k+1,l+1} -$$
$$4(n+1-k)(n+1+k)e_{n,k-1,l-1}$$
$$J_+e_{nkl} = (2k+1-l)e_{nk,l-1} \quad J_-e_{nkl} = (l+1)e_{nk,l+1}$$
$$J_ze_{nkl} = 2(k-l)e_{nkl} \tag{3.10}$$

where at a fixed value of n, $k = 0, 1, ...n$, $l = 0, 1, ...2k$ and if l and k are not in this range then $e_{nkl} = 0$. Therefore, the elements e_{nkl} form a basis of the spinless IR with a given w.

The next step is to define a scalar product compatible with the Hermiticity of the operators $(\mathcal{E}, \mathbf{B}, \mathbf{N}, \mathbf{J})$. Since $\mathbf{B}^2 + \mathbf{J}^2$ is the Casimir operator for the so(4) subalgebra and

$$(\mathbf{B}^2 + \mathbf{J}^2)e_{nkl} = 4n(n+2)e_{nkl} \tag{3.11}$$

the vectors e_{nkl} with different values of n should be orthogonal. Since $\mathbf{J}^2$ is the Casimir operator of the so(3) subalgebra and $\mathbf{J}^2e_{nkl} = 4k(k+1)e_{nkl}$, the vectors e_{nkl} with different values of k also should be orthogonal. Finally, as follows from the last expression in Eq. (3.10), the vectors e_{nkl} with the same values of n and k and different values of l should be orthogonal since they are eigenvectors of the operator J_z with different eigenvalues. Therefore, the scalar product can be defined assuming that $(e_0, e_0) = 1$ and a direct calculation using Eqs. (3.4-3.8) gives

$$(e_{nkl}, e_{nkl}) = (2k+1)!C_{2k}^l C_n^k C_{n+k+1}^k \prod_{j=1}^{n}[w + (2j+1)^2] \tag{3.12}$$

where $C_n^k = n!/[(n-k)!k!]$ is the binomial coefficient. At this point we do not normalize basis vectors to one since, as will be discussed below, the normalization (3.12) has its own advantages.

Each element of the representation space can be written as

$$x = \sum_{nkl} c(n,k,l)e_{nkl}$$

where the set of the coefficients $c(n,k,l)$ can be called the wave function in the (nkl) representation. As follows from Eqs. (3.9) and (3.10), the action of the representation operators on the wave function can be written as

$$
\begin{aligned}
&\mathcal{E}c(n,k,l) = \frac{n-k}{2n}c(n-1,k,l) + \frac{n+2+k}{2(n+2)}[w+(2n+3)^2] \\
&c(n+1,k,l) \\
&N_+c(n,k,l) = \frac{i(2k+1-l)(2k-l)}{8(2k-1)(2k+1)}\{\frac{1}{n}c(n-1,k-1,l) - \\
&\frac{1}{n+2}[w+(2n+3)^2]c(n+1,k-1,l)\} - \\
&\frac{i(n-1-k)(n-k)}{2n}c(n-1,k+1,l+2) + \\
&\frac{i(n+k+2)(n+k+3)}{2(n+2)}[w+(2n+3)^2]c(n+1,k+1,l+2) \\
&N_-c(n,k,l) = \frac{-i(l-1)l}{8(2k-1)(2k+1)}\{\frac{1}{n}c(n-1,k-1,l-2) - \\
&\frac{1}{n+2}[w+(2n+3)^2]c(n+1,k-1,l-2)\} + \\
&\frac{i(n-1-k)(n-k)}{2n}c(n-1,k+1,l) - \\
&\frac{i(n+k+2)(n+k+3)}{2(n+2)}[w+(2n+3)^2]c(n+1,k+1,l) \\
&N_zc(n,k,l) = \frac{-il(2k-l)}{4(2k-1)(2k+1)}\{\frac{1}{n}c(n-1,k-1,l-1) - \\
&\frac{1}{n+2}[w+(2n+3)^2]c(n+1,k-1,l-1)\} - \\
&\frac{i(n-1-k)(n-k)}{n}c(n-1,k+1,l+1) + \\
&\frac{i(n+k+2)(n+k+3)}{n+2}[w+(2n+3)^2]c(n+1,k+1,l+1)
\end{aligned}
\tag{3.13}
$$

$$B_+c(n,k,l) = \frac{(2k-1-l)(2k-l)}{2(2k-1)(2k+1)}c(n,k-1,l) -$$
$$2(n-k)(n+2+k)c(n,k+1,l+2)$$
$$B_-c(n,k,l) = -\frac{(1-l)l}{2(2k-1)(2k+1)}c(n,k-1,l-2) +$$
$$2(n-k)(n+2+k)c(n,k+1,l)$$
$$B_zc(n,k,l) = -\frac{l(2k-l)}{(2k-1)(2k+1)}c(n,k-1,l-1) -$$
$$4(n-k)(n+2+k)c(n,k+1,l+1)$$
$$J_+c(n,k,l) = (2k-l)c(n,k,l+1) \quad J_-c(n,k,l) = lc(n,k,l-1)$$
$$J_zc(n,k,l) = 2(k-l)c(n,k,l) \tag{3.14}$$

We use $\tilde{e}_{nkl}$ to denote basis vectors normalized to one and $\tilde{c}(n,k,l)$ to denote the wave function in the normalized basis. As follows from Eq. (3.12), the vectors $\tilde{e}_{nkl}$ can be defined as

$$\tilde{e}_{nkl} = \{(2k+1)!C_{2k}^{l}C_{n}^{k}C_{n+k+1}^{k}\prod_{j=1}^{n}[w+(2j+1)^2]\}^{-1/2}e_{nkl} \tag{3.15}$$

As noted in Sects. 2.2 and 2.3, the operator $\mathbf{B}$ is the dS analog of the usual momentum $\mathbf{P}$ such that in Poincare limit $\mathbf{B} = 2R\mathbf{P}$. The operator $\mathbf{J}$ has the same meaning as in Poincare invariant theory. Then it is clear from Eqs. (3.13) and (3.14) that for macroscopic bodies the quantum numbers (nkl) are much greater than 1. With this condition, a direct calculation using Eqs. (3.12-3.15) shows that the action of the representation operators on the wave function in the normalized basis is given by

$$\mathcal{E}\tilde{c}(n,k,l) = \frac{1}{2n}[(n-k)(n+k)(w+4n^2)]^{1/2}$$
$$[\tilde{c}(n+1,k,l) + \tilde{c}(n-1,k,l)]$$
$$N_+\tilde{c}(n,k,l) = \frac{i(w+4n^2)^{1/2}}{8nk}\{(2k-l)[(n+k)\tilde{c}(n-1,k-1,l) -$$
$$(n-k)\tilde{c}(n+1,k-1,l)] + l[(n+k)\tilde{c}(n+1,k+1,l+2) -$$
$$(n-k)\tilde{c}(n-1,k+1,l+2)]\}$$
$$N_-\tilde{c}(n,k,l) = \frac{-i(w+4n^2)^{1/2}}{8nk}\{l[(n+k)\tilde{c}(n-1,k-1,l-2) -$$
$$(n-k)\tilde{c}(n+1,k-1,l-2)] - (2k-l)[(n-k)\tilde{c}(n-1,k+1,l) -$$
$$(n+k)\tilde{c}(n+1,k+1,l)]\}$$
$$N_z\tilde{c}(n,k,l) = \frac{-i[l(2k-l)(w+4n^2)]^{1/2}}{4nk}\{(n+k)\tilde{c}(n-1,k-1,l-1) -$$
$$(n-k)\tilde{c}(n+1,k-1,l-1) + (n-k)\tilde{c}(n-1,k+1,l+1) -$$
$$(n+k)\tilde{c}(n+1,k+1,l+1)\} \tag{3.16}$$

$$B_+\tilde{c}(n,k,l) = \frac{[(n-k)(n+k)]^{1/2}}{2k}\{(2k-l)\tilde{c}(n,k-1,l) -$$
$$l\tilde{c}(n,k+1,l+2)\}$$
$$B_-\tilde{c}(n,k,l) = \frac{[(n-k)(n+k)]^{1/2}}{2k}\{(2k-l)\tilde{c}(n,k+1,l) -$$
$$l\tilde{c}(n,k-1,l-2)\}$$
$$B_z\tilde{c}(n,k,l) = -\frac{1}{k}[l(2k-l)(n-k)(n+k)]^{1/2}\{\tilde{c}(n,k-1,l-1) +$$
$$\tilde{c}(n,k+1,l+1)\}$$
$$J_+\tilde{c}(n,k,l) = [l(2k-l)]^{1/2}\tilde{c}(n,k,l+1)$$
$$J_-\tilde{c}(n,k,l) = [l(2k-l)]^{1/2}\tilde{c}(n,k,l-1)$$
$$J_z\tilde{c}(n,k,l) = 2(k-l)\tilde{c}(n,k,l) \tag{3.17}$$

3.2 Semiclassical approximation

Consider now the semiclassical approximation in the $\tilde{e}_{nkl}$ basis. By analogy with the discussion of the semiclassical approximation in Sects. 1.5 and 2.3, we assume that a state is semiclassical if its wave function has the form

$$\tilde{c}(n,k,l) = a(n,k,l)exp[i(-n\varphi + k\alpha + (l-k)\beta)] \tag{3.18}$$

where $a(n,k,l)$ is an amplitude, which is not small only in some vicinities of $n = n_0$, $k = k_0$ and $l = l_0$. We also assume that when the quantum numbers (nkl) change by one, the main contribution comes from the rapidly oscillating exponent. Then, as follows from the first expression in Eq. (3.16), the action of the dS energy operator can be written as

$$\mathcal{E}\tilde{c}(n,k,l) \approx \frac{1}{n_0}[(n_0-k_0)(n_0+k_0)(w+4n_0^2)]^{1/2}cos(\varphi)\tilde{c}(n,k,l) \tag{3.19}$$

Therefore the semiclassical wave function is approximately the eigenfunction of the dS energy operator with the eigenvalue

$$\frac{1}{n_0}[(n_0-k_0)(n_0+k_0)(w+4n_0^2)]^{1/2}cos\varphi.$$

We will use the following notations. When we consider not the action of an operator on the wave function but its approximate eigenvalue in the semiclassical state, we will use for the eigenvalue the same notation as for the operator and this should not lead to misunderstanding. Analogously, in eigenvalues we will write n, k and l instead of n_0, k_0 and l_0, respectively. By analogy with Eq. (3.19) we can consider eigenvalues of the other operators and the results can be represented as

$$\begin{aligned}
&\mathcal{E} = \frac{1}{n}[(n-k)(n+k)(w+4n^2)]^{1/2}cos\varphi \\
&N_x = (w+4n^2)^{1/2}\{-\frac{sin\varphi}{k}[(k-l)cos\alpha cos\beta + ksin\alpha sin\beta] + \\
&\frac{cos\varphi}{n}[(k-l)sin\alpha cos\beta - kcos\alpha sin\beta]\} \\
&N_y = (w+4n^2)^{1/2}\{-\frac{sin\varphi}{k}[(k-l)cos\alpha sin\beta - ksin\alpha cos\beta] + \\
&\frac{cos\varphi}{n}[(k-l)sin\alpha sin\beta + kcos\alpha cos\beta]\} \\
&N_z = [l(2k-l)(w+4n^2)]^{1/2}(\frac{1}{k}sin\varphi cos\alpha - \frac{1}{n}cos\varphi sin\alpha) \\
&B_x = \frac{2}{k}[(n-k)(n+k)]^{1/2}[(k-l)cos\alpha cos\beta + ksin\alpha sin\beta] \\
&B_y = \frac{2}{k}[(n-k)(n+k)]^{1/2}[(k-l)cos\alpha sin\beta - ksin\alpha cos\beta] \\
&B_z = -\frac{2}{k}[l(2k-l)(n-k)(n+k)]^{1/2}cos\alpha \\
&J_x = 2[l(2k-l)]^{1/2}cos\beta \quad J_y = 2[l(2k-l)]^{1/2}sin\beta \\
&J_z = 2(k-l)
\end{aligned} \tag{3.20}$$

Since $\mathbf{B}$ is the dS analog of $\mathbf{p}$ and in classical theory $\mathbf{J} = \mathbf{r}\times\mathbf{p}$, one might expect that $\mathbf{BJ} = 0$ and, as follows from the above expressions, this is the case. It also follows that $\mathbf{B}^2 = 4(n^2-k^2)$ and $\mathbf{J}^2 = 4k^2$ in agreement with Eq. (3.11).

In Sect. 2.3 we described semiclassical wave functions by six parameters $(\mathbf{r},\mathbf{p})$ while in the basis $\tilde{e}_{nkl}$ the six parameters are $(n,k,l,\varphi,\alpha,\beta)$. Since in the dS theory the ten representation operators are on equal footing, it is also possible to describe a semiclassical state by semiclasscal eigenvalues of these operators. However, we should have four constraints for them. As follows from Eqs. (2.16) and (3.20), the constraints can be written as

$$\mathcal{E}^2 + \mathbf{N}^2 - \mathbf{B}^2 - \mathbf{J}^2 = w \quad \mathbf{N}\times\mathbf{B} = -\mathcal{E}\mathbf{J} \tag{3.21}$$

As noted in Sect. 2.3, in Poincare limit $\mathcal{E} = 2RE$, $\mathbf{B} = 2R\mathbf{p}$ (since we have replaced Eq. (1.5) by Eq. (3.1)) and the values of $\mathbf{N}$ and $\mathbf{J}$ are much less than $\mathcal{E}$ and $\mathbf{B}$. Therefore the first relation in Eq. (3.21) is the Poincare analog of the well known relation $E^2 - \mathbf{p}^2 = m^2$.

The quantities $(nkl\varphi\alpha\beta)$ can be expressed in terms of semiclassical eigenvalues $(\mathcal{E},\mathbf{N},\mathbf{B},\mathbf{J})$ as follows. The quantities (nkl) can be found from the relations

$$\mathbf{B}^2 + \mathbf{J}^2 = 4n^2 \quad \mathbf{J}^2 = 4k^2 \quad J_z = 2(k-l) \tag{3.22}$$

and then the angles $(\varphi\alpha\beta)$ can be found from the relations

$$\begin{aligned}
&cos\varphi = \frac{2\mathcal{E}n}{B(w+4n^2)^{1/2}} \quad sin\varphi = -\frac{\mathbf{BN}}{B(w+4n^2)^{1/2}} \\
&cos\alpha = -JB_z/(BJ_\perp) \quad sin\alpha = (\mathbf{B}\times\mathbf{J})_z/(BJ_\perp) \\
&cos\beta = J_x/J_\perp \quad sin\beta = J_y/J_\perp
\end{aligned} \tag{3.23}$$

where $B = |\mathbf{B}|$, $J = |\mathbf{J}|$ and $J_\perp = (J_x^2 + J_y^2)^{1/2}$. In semiclassical approximation, uncertainties of the quantities (nkl) should be such that $\Delta n \ll n$, $\Delta k \ll k$ and $\Delta l \ll l$. On the other hand, those uncertainties cannot be very small since the distribution in (nkl) should be such that all the ten approximate eigenvalues $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$ should be much greater than their corresponding uncertainties. The assumption is that for macroscopic bodies all these conditions can be satisfied.

In Sect. 2.3 we discussed operators in Poincare limit and corrections of order $1/R$ to them, which lead to the dS antigravity. A problem arises how the dS antigravity can be recovered in the basis defined in this chapter. The first question is how Poincare limit should be defined. In contrast to Sect. 2.3, we can now work not with the unphysical quantities $\mathbf{v}$ or $\mathbf{p} = m\mathbf{v}$ defined on the Lorentz hyperboloid but directly with semiclassical eigenvalues of the representation operators. In contrast to Sect. 2.3, we now *define* $\mathbf{p} = \mathbf{B}/(2R)$, $m = w^{1/2}/(2R)$ and $E = (m^2 + \mathbf{p}^2)^{1/2}$. Then Poincare limit can be defined by the requirement that when R is large, the quantities $\mathcal{E}$ and $\mathbf{B}$ are proportional to R while $\mathbf{N}$ and $\mathbf{J}$ do not depend on R. In this case, as follows from Eq. (3.21), in Poincare limit $\mathcal{E} = 2RE$ and $\mathbf{B} = 2R\mathbf{p}$.

In has been noted in Sect. 2.3 that if $\mathbf{r}$ is defined as $i\partial/\partial\mathbf{p}$ then in semiclassical approximation $\mathbf{N} = -2E\mathbf{r}$. If this result is correct in the formalism of this chapter then it is obvious that the second relation in Eq. (3.21) is the Poincare analog of the relation $\mathbf{J} = \mathbf{r}\times\mathbf{p}$. However a problem arises how $\mathbf{r}$ should be defined in the present formalism and how to prove whether $\mathbf{N} = -2E\mathbf{r}$ or not. If $\mathbf{B}$ and $\mathbf{J}$ are given and $\mathbf{B} \neq 0$ then a requirement that $\mathbf{r}\times\mathbf{p} = \mathbf{J}$ does not define $\mathbf{r}$ uniquely. One can define parallel and perpendicular components of $\mathbf{r}$ as $\mathbf{r} = r_{||}\mathbf{B}/|\mathbf{B}| + \mathbf{r}_\perp$ and analogously $\mathbf{N} = N_{||}\mathbf{B}/|\mathbf{B}| + \mathbf{N}_\perp$. Then the relation $\mathbf{r}\times\mathbf{p} = \mathbf{J}$ defines uniquely only $\mathbf{r}_\perp$ and it follows from the second relation in Eq. (3.21) that $\mathbf{N}_\perp = -2E\mathbf{r}_\perp$. However, it is not clear yet how $r_{||}$ should be defined and whether the last relation is also valid for the parallel components of $\mathbf{N}$ and $\mathbf{r}$. As follows from the second relation in Eq. (3.23), it will be valid if $|sin\varphi| = r_{||}/R$, i.e. φ is the angular coordinate. As noted in Sect. 1.5, semiclassical approximation for a physical quantity can be valid only in states where this quantity is rather large. Therefore if R is very large then φ is very small if the distances are not cosmological (i.e. they are much less than R). Hence the problem arises whether this approximation is valid. This is a very important problem since in standard approach it is assumed that nevertheless φ can be considered semiclassically. We will investigate this problem in the subsequent sections while in this one we assume that this is the case.

Consider now corrections to Poincare limit in the present formalism. Since $\mathbf{B} = 2R\mathbf{p}$ and $\mathbf{J} = \mathbf{r}_\perp \times \mathbf{p}$ then it follows from Eq. (3.22) that in first order in $1/R^2$ $k^2/n^2 = \mathbf{r}_\perp^2/R^2$. Therefore as follows from the first expression in Eq. (3.20), in first order in $1/R^2$ the results on $\mathcal{E}$ and $\mathbf{N}$ can be represented as

$$\mathcal{E} = 2ER(1 - \frac{\mathbf{r}^2}{2R^2}) \quad \mathbf{N} = -2E\mathbf{r} \tag{3.24}$$

The result for the energy is in agreement with Eq. (2.18).

As follows from Eq. (3.5), the two-body operator W, which is an analog of the quantity w can be defined such that if I_2 is the two-body Casimir operator (2.16) then

$$I_2 = W - \mathbf{S}^2 + 9 \tag{3.25}$$

where $\mathbf{S}$ is the two-body spin operator which can be expressed in terms of the two-body Casimir operator of the fourth order. Then as follows from Eqs. (2.16) and (3.5)

$$W = w_1 + w_2 + 2\mathcal{E}_1\mathcal{E}_2 + 2\mathbf{N}_1\mathbf{N}_2 - 2\mathbf{B}_1\mathbf{B}_2 - 2\mathbf{J}_1\mathbf{J}_2 + \mathbf{S}^2 + 9 \tag{3.26}$$

and, taking into account Eq. (3.24) we get

$$W = W_0 - 4E_1E_2\mathbf{r}^2 - 2\mathbf{J}_1\mathbf{J}_2 + \mathbf{S}^2 + 9 \tag{3.27}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $W_0 = 4R^2 I_{2P}$ and I_{2P}, is the Casimir operator of the second order in Poincare invariant theory. If E is the two-body energy operator in Poincare invariant theory and $\mathbf{P}$ is the two-body Poincare momentum then $I_{2P} = E^2 - \mathbf{P}^2$. This operator is sometimes called the mass operator squared although in general I_{2P} is not positive definite (e.g. for tachyons). However, for macroscopic bodies it is positive definite, i.e. can be represented as M_0^2, the classical value of which is $M_0^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2\mathbf{p}_1\mathbf{p}_2$. Let $M^2 = W/4R^2$ be the mass squared in Poincare invariant theory with dS corrections. In the nonrelativistic approximation $M = m_1 + m_2 + H_{nr}$ where H_{nr} is the nonrelativistic Hamiltonian in the c.m. frame. Then it follows from Eq. (3.27) that in first order in $1/R^2$

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - \frac{m_{12}\mathbf{r}^2}{2R^2} \tag{3.28}$$

A question arises why this expression is different from that given by Eq. (2.23). The explanation is as follows. In this and preceding chapters we considered different implementations of IRs of the dS algebra. All such implementations are unitarily equivalent. Therefore if O is the set of operators defined by Eqs. (3.9,3.10) and $\tilde{O}$ is the set of operators defined by Eq. (2.15) then there exists a unitary operator U such that $O = U\tilde{O}U^{-1}$. Let now $(O_j, \tilde{O}_j, U_j)$ $(j = 1, 2)$ be the corresponding operators for particles 1 and 2. In the preceding chapter we discussed a possibility that the representation operators for the two-body system are $\tilde{O}_1 + \tilde{O}_2$ while in this chapter we discussed a possibility that they are represented as $O_1 + O_2$. These possibilities are not equivalent. As follows from the discussion in the subsequent chapters, from the point of view of the approach based on Galois fields, the implementation of IRs in this chapter is more fundamental than in the preceding one. Nevertheless, on classical level they are equivalent since classical equations of motion for the Hamiltonian (3.28) are the same as for the Hamiltonian (2.23). Note the correction to the Hamiltonian is always negative and proportional to m_{12} in the nonrelativistic approximation.

3.3 Two-body relative distance operator

In Sect. 1.5 we discussed semiclassical approximation for the coordinate operator in standard quantum mechanics. In this section we investigate how the relative distance operator can be defined in dS invariant theory. As noted in the preceding section, among the operators of the dS algebra there are no operators which can be identified with the distance operator but there are reasons to think that in semiclassical approximation the values of E and $\mathbf{N}$ are given by Eq. (3.24). From the point of view of our experience in Poincare invariant theory, the dependence of E on $\mathbf{r}$ might seem to be unphysical since the energy depends on the choice of the origin. However, as already noted, only invariant quantities have a physical meaning; in particular the two-body mass can depend only on relative distances which do not depend on the choice of the origin.

In view of Eq. (3.24) one might think that the operator $\tilde{\mathbf{D}} = \mathcal{E}_2\mathbf{N}_1 - \mathcal{E}_1\mathbf{N}_2$ might be a good operator which in semiclassical approximation is proportional to $E_1E_2\mathbf{r}$ at least in main order in $1/R^2$. However, the operator $\mathbf{D}$ defining the relative distance should satisfy the following conditions. First of all, it should not depend on the motion of the two-body system as a whole; in particular it should commute with the operator which is treated as a total momentum in dS theory. As noted in the preceding section, the single-particle operator $\mathbf{J}'$ is a better candidate for the total momentum operator than $\mathbf{B}$. Now we use $\mathbf{J}'$ to denote the total two-particle operator $\mathbf{J}'_1 + \mathbf{J}'_2$. Analogously, we use $\mathbf{J}$" to denote the total two-particle operator $\mathbf{J}_1$" $+\mathbf{J}_2$". As noted in the preceding section, $\mathbf{J}$" can be treated as the internal angular momentum operator. Therefore, since $\mathbf{D}$ should be a vector operator with respect to internal rotations, it should properly commute with $\mathbf{J}$". In summary, the operator $\mathbf{D}$ should satisfy the relations

$$[J'^j, D^k] = 0 \quad [J"^j, D^k] = 2ie_{jkl}D^l \tag{3.29}$$

By using Eqs. (3.2) and (3.6) one can explicitly verify that the operator

$$\mathbf{D} = \mathcal{E}_2\mathbf{N}_1 - \mathcal{E}_1\mathbf{N}_2 - \mathbf{N}_1 \times \mathbf{N}_2 \tag{3.30}$$

indeed satisfies Eq. (3.29). If Poincare approximation is satisfied with a high accuracy then obviously $\mathbf{D} \approx \tilde{\mathbf{D}}$.

In contrast to the situation in quantum mechanics, different components of $\mathbf{D}$ do not commute with each other and therefore are not simultaneously measurable. However, since $[\mathbf{D}^2, \mathbf{J}"] = 0$, by analogy with quantum mechanics one can choose $(\mathbf{D}^2, \mathbf{J}"^2, J_z")$ as a set of diagonal operators. The result of explicit calculations is

$$\mathbf{D}^2 = (\mathcal{E}_1^2 + \mathbf{N}_1^2)(\mathcal{E}_2^2 + \mathbf{N}_2^2) - (\mathcal{E}_1\mathcal{E}_2 + \mathbf{N}_1\mathbf{N}_2)^2 - 4(\mathbf{J}_1\mathbf{B}_2 + \mathbf{J}_2\mathbf{B}_1) - 4\mathbf{J}_1\mathbf{J}_2 \qquad (3.31)$$

It is obvious that in typical situations the last two terms in this expression are much less than the first two terms and for this reason we accept an approximation

$$\mathbf{D}^2 \approx (\mathcal{E}_1^2 + \mathbf{N}_1^2)(\mathcal{E}_2^2 + \mathbf{N}_2^2) - (\mathcal{E}_1\mathcal{E}_2 + \mathbf{N}_1\mathbf{N}_2)^2 \qquad (3.32)$$

At this point no assumption that semiclassical approximation is valid has been made. If Eq. (3.24) is valid then, as follows from Eq. (3.32), in first order in $1/R^2$ $\mathbf{D}^2 = 16E_1^2E_2^2R^2r^2$ where $r = |\mathbf{r}|$. In particular, in the nonrelativistic approximation $\mathbf{D}^2 = 16m_1^2m_2^2R^2r^2$, i.e. $\mathbf{D}^2$ is proportional to r^2 what justifies treating $\mathbf{D}$ as a dS analog of the relative distance operator. On the other hand, as noted in the preceding section, there are reasons to believe that macroscopic wave functions are semiclassical in (k, l) but it is not clear whether their n-dependence is semiclassical. For this reason we consider an approximation when the (k, l) dependence is semiclassical while it should be investigated when the n dependence is semiclassical too.

Note that the operators in Eq. (3.16) act over the variable n while the operators in Eq. (3.17) don't. The formulas defining the action of the operators in Eq. (3.16) contain multipliers $(n + k)$ and $(n - k)$. We expect that $k \ll n$ and therefore one might expect that the main contribution to the operator $\mathbf{N}$ can be obtained if k in $(n + k)$ and $(n - k)$ is neglected. Let $\mathbf{N}_{||}$ be the operator $\mathbf{N}$ obtained in this approximation. Then it follows from Eqs. (3.16) and (3.17) that

$$\mathbf{N}_{||}\tilde{c}(n, k, l) = \frac{i}{4}[\frac{w + 4n^2}{(n - k)(n + k)}]^{1/2}\mathbf{B}[\tilde{c}(n - 1, k, l) - \tilde{c}(n + 1, k, l)] \qquad (3.33)$$

In our approximation we can replace $\mathbf{B}$ by its semiclassical value and take into account (see the preceding section) that $B = 2[(n - k)(n + k)]^{1/2}$ where $B = |\mathbf{B}|$. Then, as follows from Eqs. (3.16) and (3.17), in this approximation

$$\begin{aligned}
\mathbf{N}_{||}\tilde{c}(n, k, l) &= \frac{i}{2}(w + 4n^2)^{1/2}[\tilde{c}(n - 1, k, l) - \tilde{c}(n + 1, k, l)]\frac{\mathbf{B}}{B} \\
\mathbf{N}_{\perp}\tilde{c}(n, k, l) &= -\frac{1}{4n}(w + 4n^2)^{1/2}[\tilde{c}(n - 1, k, l) + \tilde{c}(n + 1, k, l)](\frac{\mathbf{B}}{B} \times \mathbf{J}) \\
\mathcal{E}\tilde{c}(n, k, l) &= \frac{1}{4n}B(w + 4n^2)^{1/2}[\tilde{c}(n - 1, k, l) + \tilde{c}(n + 1, k, l)] \qquad (3.34)
\end{aligned}$$

where $\mathbf{N}_{\perp}$ is the remaining part of the operator $\mathbf{N}$. As follows from the above expression, this part is indeed orthogonal to $\mathbf{B}$. Since we assume that $\mathbf{J}/n$ is of order $1/R$, $\mathbf{N}_{\perp}$ already contains a factor of order $1/R$.

For brevity of notations we will omit the (k,l) dependence of wave functions and will replace $\tilde{c}(n,k,l)$ by $\psi(n)$. Then Eq. (3.34) can be written as

$$\mathbf{N} = -\frac{1}{B}(w+4n^2)^{1/2}[\mathbf{B}\mathcal{A} + \frac{1}{2n}(\mathbf{B}\times\mathbf{J})\mathcal{B}] \quad \mathcal{E} = \frac{1}{2n}B(w+4n^2)^{1/2}\mathcal{B} \qquad (3.35)$$

where the action of the operators $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A}\psi(n) = \frac{i}{2}[\psi(n+1) - \psi(n-1)] \quad \mathcal{B}\psi(n) = \frac{1}{2}[\psi(n+1) + \psi(n-1)] \qquad (3.36)$$

The relations between the operators $\mathcal{A}$, $\mathcal{B}$ and n are

$$[\mathcal{A},n] = i\mathcal{B} \quad [\mathcal{B},n] = -i\mathcal{A} \quad [\mathcal{A},\mathcal{B}] = 0 \quad \mathcal{A}^2 + \mathcal{B}^2 = 1 \qquad (3.37)$$

As noted in Sect. 1.5, in standard quantum theory the semiclassical wave function in momentum space contains a factor $exp(-ipx)$. Since n is now the dS analog of p_zR, we assume that $\psi(n)$ contains a factor $exp(-in\varphi)$, i.e. the angle φ is the dS analof of z/R. It is reasonable to expect that since all the ten representation operators of the dS algebra are angular momenta, in dS theory one should deal only with angular coordinates wich are dimensionless. If $\psi(n) = a(n)exp(-in\varphi)$ and we assume that in semiclassical approximation the main contribution in Eq. (3.36) is given by the exponent then

$$\mathcal{A}\psi(n) \approx sin\varphi\psi(n) \quad \mathcal{B}\psi(n) \approx cos\varphi\psi(n) \qquad (3.38)$$

in agreement with the first two expressions in Eq. (3.23). Therefore if φ is the dS analog of z/R and $z \ll R$, we recover the result that $N_{||} \approx -2Er_{||}$. Eq. (3.38) can be treated in such a way that $\mathcal{A}$ is the operator of the quantity $sin\varphi$ and $\mathcal{B}$ is the operator of the quantity $cos\varphi$. However, the following question arises. As noted in Sect. 1.5, semiclassical approximation for a quantity can be correct only if this quantity is rather large. At the same time, we assume that $\mathcal{A}$ is the operator of the quantity which is very small if R is large.

If φ is small, we have $sin\varphi \approx \varphi$ and in this approximation $\mathcal{A}$ can be treated as the operator of the angular variable φ. This seems natural since in standard theory the operator of the z coordinate is id/dp_z and $\mathcal{A}$ is the finite difference analog of derivative over n (there is no derivative over n since n is the discrete variable and can take only values 0,1,2...). When φ is not small, the argument that $\mathcal{A}$ is the operator of the quantity $sin\varphi$ is as follows. Since

$$arcsin\varphi = \sum_{l=0}^{\infty}\frac{(2l)!\varphi^{2l+1}}{4^l(l!)^2(2l+1)}$$

one might think that

$$\Phi = \sum_{l=0}^{\infty}\frac{(2l)!\mathcal{A}^{2l+1}}{4^l(l!)^2(2l+1)}$$

can be treated as the operator of the quantity φ. Indeed, as follows from this expression and Eq. (3.37), $[\Phi, n] = i$ what is the dS analog of the relation $[z, p_z] = i$.

For a two-body system we define the operator

$$G = 1 - \frac{B_1 B_2}{4n_1 n_2}\mathcal{B}_1\mathcal{B}_2 - \frac{\mathbf{B}_1\mathbf{B}_2}{B_1 B_2}\mathcal{A}_1\mathcal{A}_2 - \frac{\mathbf{B}_1(\mathbf{B}_2\times\mathbf{J}_2)}{2n_2 B_1 B_2}\mathcal{A}_1\mathcal{B}_2 - \frac{(\mathbf{B}_1\times\mathbf{J}_1)\mathbf{B}_2}{2n_1 B_1 B_2}\mathcal{A}_2\mathcal{B}_1 - \frac{(\mathbf{B}_1\times\mathbf{J}_1)(\mathbf{B}_2\times\mathbf{J}_2)}{4n_1 n_2 B_1 B_2}\mathcal{B}_1\mathcal{B}_2 \qquad (3.39)$$

Then, as follows from Eqs. (3.32) and (3.35)

$$\mathcal{E}_1\mathcal{E}_2+\mathbf{N}_1\mathbf{N}_2 = (w_1+4n_1^2)^{1/2}(w_2+4n_2^2)^{1/2}(1-G) \quad \mathbf{D}^2 = (w_1+4n_1^2)(w_2+4n_2^2)(2G-G^2) \qquad (3.40)$$

By analogy with standard theory, we can consider the two-body system in its c.m. frame. Since we choose $\mathbf{B}+\mathbf{J}$ as the dS analog of momentum, the c.m. frame can be defined by the condition $\mathbf{B}_1+\mathbf{J}_2+\mathbf{B}_2+\mathbf{J}_2 = 0$. Therefore, as follows from Eq. (3.22), $n_1 = n_2$. This is an analog of the condition that the magnitudes of particle momenta in the c.m. frame are the same. Another simplification can be achieved if the position of particle 2 is chosen as the origin. Then $\mathbf{J}_2 = 0$, $\mathbf{J}_1 = (\mathbf{r}_\perp \times \mathbf{B}_1)/2R$, $B_2 = 2n_2$ and Eq. (3.39) has a much simpler form:

$$G = 1 - \frac{B_1}{2n_1}(\mathcal{B}_1\mathcal{B}_2 - \mathcal{A}_1\mathcal{A}_2) \qquad (3.41)$$

In the approximation when $\mathcal{B}_i$ can be replaced by $cos\varphi_i$ and $\mathcal{A}_i$ - by $sin\varphi_i$ $(i = 1, 2)$, we can again recover the above result $\mathbf{D}^2 = 16E_1^2E_2^2R^2r^2$ if $|\varphi_1+\varphi_2| = r_{||}/R$.

We conclude that if standard semiclassical approximation is valid then dS corrections to the two-body mass operator are of order $(r/R)^2$. This result is in agreement with standard intuition that dS corrections can be important only at cosmological distances while in the Solar system these corrections are negligible. On the other hand, as it has been already noted, those conclusions are based on belief that the angular distance φ, which is of order r/R, can be considered semiclassically in spite of the fact that it is very small. In the next section we investigate whether this is the case. Since from now on we are interested only in distances which are much less than cosmological ones, we will neglect all corrections of order r/R and greater. In particular, we accept the approximation that $|\mathbf{B}_1| = 2n_1$, $|\mathbf{B}_2| = 2n_2$ and the c.m. frame is defined by the condition $\mathbf{B}_1+\mathbf{B}_2 = 0$.

By analogy with standard theory, it is convenient to consider the two-body mass operator if individual particle momenta n_1 and n_2 are expressed in terms of the total and relative momenta N and n. In the c.m. frame we can assume that $\mathbf{B}_1$ is directed along the positive direction of the z axis and then $\mathbf{B}_2$ is directed along the negative direction of the z axis. Therefore the quantum number N characterizing the total dS momentum can be defined as $N = n_1 - n_2$. In nonrelativistic theory the relative momentum is defined as $\mathbf{q} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1+m_2)$ and in relativistic theory as $\mathbf{q} = (E_2\mathbf{p}_1 - E_1\mathbf{p}_2)/(E_1 + E_2)$. Therefore, taking into account the fact that in the c.m. frame the particle momenta are directed in opposite directions, one might define n as $n = (m_2n_1 + m_1n_2)/(m_1 + m_2)$ or $n = (E_2n_1 + E_1n_2)/(E_1 + E_2)$. These definitions involve Poincare masses and energies. Another possibility is $n = (n_1 + n_2)/2$. In all these cases we have that $n \to (n+1)$ when $n_1 \to (n_1+1)$, $n_2 \to (n_2+1)$ and $n \to (n-1)$ when $n_1 \to (n_1 - 1)$, $n_2 \to (n_2 - 1)$. In what follows, only this feature is important.

Although so far we are working in standard dS quantum theory over complex numbers, we will argue in the next chapters that fundamental quantum theory should be finite. We will consider a version of quantum theory where complex numbers are replaced by a Galois field. In this approach only those functions $\psi_1(n_1)$ and $\psi_2(n_2)$ are physical which have a finite carrier in n_1 and n_2, respectively. Therefore we assume that $\psi_1(n_1)$ can be different from zero only if $n_1 \in [n_{1min}, n_{1max}]$ and analogously for $\psi_2(n_2)$. If $n_{1max} = n_{1min} + \delta_1 - 1$ then a necessary condition that n_1 is semiclassical is $\delta_1 \ll n_1$. At the same time, since δ_1 is the dS analog of Δp_1R and R is very large, we expect that $\delta_1 \gg 1$. We use ν_1 to denote $n_1 - n_{1min}$. Then if $\psi_1(\nu_1) = a_1(\nu_1)exp(-i\varphi_1\nu_1)$, we can expect by analogy with the consideration in Sect. 1.5 that the state $\psi_1(\nu_1)$ will be semiclassical if $|\varphi_1\delta_1| \gg 1$ since in this case the exponent makes many oscillations on $[0, \delta_1]$. Even this condition indicates that φ_1 cannot be extremely small. Analogously we can consider the wave function of particle 2, define δ_2 as the width of its dS momentum distribution and $\nu_2 = n_2 - n_{2min}$. The range of possible values of N and n is shown in Fig. 3.1 where it is assumed

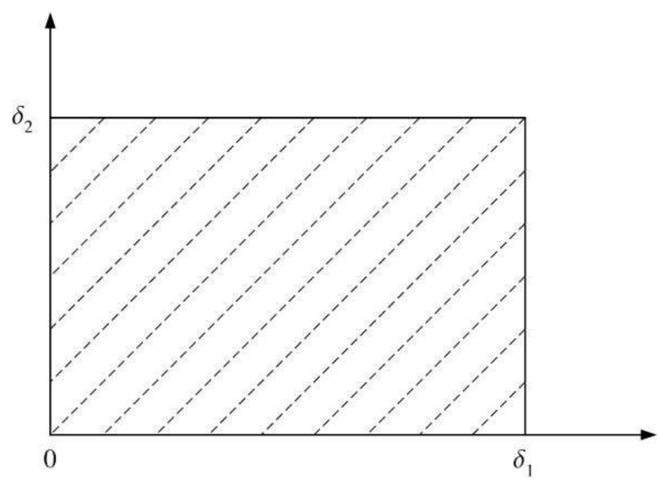

Figure 3.1: Range of possible values of N and n.

that $\delta_1 \geq \delta_2$. The minimum and maximum values of N are $N_{min} = n_{1min} - n_{2max}$ and $N_{max} = n_{1max} - n_{2min}$, respectively. Therefore N can take $\delta_1 + \delta_2$ values. Each incident dashed line represents a set of states with the same value of N and different values of n. We now use n_{min} and n_{max} to define the minimum and maximum values of the relative dS momentum n. For each fixed value of N those values are different, i.e. they are functions of N. Let $\delta(N) = n_{max} - n_{min}$ for a given value of N. It is easy to see that $\delta(N) = 0$ when $N = N_{min}$ and $N = N_{max}$ while for other values of N, $\delta(N)$ is a natural number in the range $(0, \delta_{max}]$ where $\delta_{max} = min(\delta_1, \delta_2)$. The total number of values of (N, n) is obviously $\delta_1\delta_2$, i.e.

$$\sum_{N=Nmin}^{Nmax} \delta(N) = \delta_1\delta_2 \tag{3.42}$$

As follows from Eq. (3.36)

$$(\mathcal{B}_1\mathcal{B}_2 - \mathcal{A}_1\mathcal{A}_2)\psi_1(n_1)\psi_2(n_2) = \frac{1}{2}[\psi_1(n_1+1)\psi_2(n_2+1) + \psi_1(n_1-1)\psi_2(n_2-1)] \tag{3.43}$$

Therefore in terms of the variables N and n

$$(\mathcal{B}_1\mathcal{B}_2 - \mathcal{A}_1\mathcal{A}_2)\psi(N, n) = \frac{1}{2}[\psi(N, n+1) + \psi(N, n-1)] \tag{3.44}$$

Hence the operator $(\mathcal{B}_1\mathcal{B}_2 - \mathcal{A}_1\mathcal{A}_2)$ does not act on the variable N while its action on the variable n is described by the same expressions as the actions of the operators $\mathcal{B}_i$ $(i = 1, 2)$ on the corresponding wave functions. Therefore, considering the two-body system, we will use the notation $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2 - \mathcal{A}_1\mathcal{A}_2$ and formally the action of this operator on the internal wave function is the same as in the second expression in Eq. (3.36). With this notation and with neglecting terms of order r/R and higher, Eqs. (3.27) and (3.41) can be written as

$$G = 1 - \mathcal{B} \quad I_2 = 4R^2M_0^2 - 2(w_1 + 4n_1^2)^{1/2}(w_2 + 4n_2^2)^{1/2}G \tag{3.45}$$

Since both, the operator $\mathbf{D}^2$ and the dS correction to the operator I_2 are defined by the same operator G, physical quantities corresponding to $\mathbf{D}^2$ and I_2 will be semiclassical or not depending on whether the quantity corresponding to G is semiclassical or not. As follows from Eq. (3.37), the spectrum of the operator $\mathcal{B}$ can be only in the range [0,1] and therefore, as follows from Eq. (3.45), the same is true for the spectrum of the operator G. Hence, as follows from Eq. (3.45), any dS correction to the operator I_2 is negative and in the nonrelativistic approximation is proportional to particle masses.

3.4 Validity of semiclassical approximation

Since classical mechanics works with a very high accuracy at macroscopic level, one might think that the validity of semiclassical approximation at this level is beyond any doubts. However, to the best of our knowledge, this question has not been investigated quantitavely. As discussed in Sect. 1.5, such quantities as coordinates and momenta are semiclassicall if their uncertainties are much less than the corresponding mean values. Consider wave functions describing the motion of macroscopic bodies as a whole (say the wave functions of the Sun, the Earth, the Moon etc.). It is obvious that uncertainties of coordinates in these wave functions are much less than the corresponding macroscopic dimensions. What are those uncertainties for the Sun, the Earth, the Moon, etc.? What are the uncertainties of their momenta? In quantum mechanics, the validity of semiclassical approximation is defined by the product $\Delta r\Delta p$ while each uncertainty by itself can be rather large. Do we know what scenario takes place for macroscopic bodies?

In this section we consider several models of the function $\psi(n)$ where it is be possible to explicitly calculate $\bar{G}$ and ΔG and check whether the condition $\Delta G \ll |\bar{G}|$ (showing that the quantity G in the state ψ is semiclassical) is satisfied. As follows from Eq. (3.37), $[G,n]=i\mathcal{A}$ where formally the action of this operator on the internal wave function is the same as in the first expression in Eq. (3.36). Therefore, as follows from Eq. (1.7), $\Delta G\Delta n \geq \bar{\mathcal{A}}/2$.

As noted in Sect. 1.5, one might think that a necessary condition for the validity of semiclassical approximation is that the exponent in the semiclassical wave function makes many oscillations in the region where the wave function is not small. We will consider wave functions $\psi(n)$ containing $exp(-i\varphi n)$ such that $\psi(n)$ can be different from zero only if $n\in[n_{min},n_{max}]$. Then, if $\delta=n_{max}-n_{min}+1$, the exponent makes $|\varphi|\delta/2\pi$ oscillations on $[n_{min},n_{max}]$ and φ should satisfy the condition $|\varphi|\gg 1/\delta$. The problem arises whether this condition is sufficient.

Our first example is such that $\psi(n)=exp(-i\varphi n)/\delta^{1/2}$ if $n\in[n_{min},n_{max}]$. Then a simple calculation gives

$$\bar{G}=1-cos\varphi+\frac{1}{\delta}cos\varphi \quad \Delta G=\frac{(\delta-1)^{1/2}cos\varphi}{\delta} \quad \bar{\mathcal{A}}=(1-\frac{1}{\delta})sin\varphi$$
$$\bar{n}=(n_{min}+n_{max})/2 \quad \Delta n=\delta(\frac{1-1/\delta^2}{12})^{1/2} \tag{3.46}$$

Since φ is of order r/R, we will always assume that $\varphi\ll 1$. Therefore for the validity of the condition $\Delta G\ll\bar{G}$, $|\varphi|$ should be not only much greater than $1/\delta$ but even much greater than $1/\delta^{1/4}$. Note also that $\Delta G\Delta n$ is of order $\delta^{1/2}$, i.e. much greater than $\bar{\mathcal{A}}$. This result shows that the state $\psi(\nu)$ is strongly non-semiclassical. The calculation shows that for ensuring the validity of semiclassical approximation, one should consider functions $\psi(\nu)$ which are small when n is close to n_{min} or n_{max}.

The second example is $\psi(\nu)=const\,C_\delta^\nu exp(-i\varphi\nu)$ where $\nu=n-n_{min}$ and *const* can be defined from the normalization condition. Since $C_\delta^\nu=0$ when $\nu<0$ or $\nu>\delta$, this function is not zero only when $\nu\in[0,\delta]$. The result of calculations is that $const^2=1/C_{2\delta}^\delta$ and

$$\bar{G}=1-cos\varphi+\frac{cos\varphi}{\delta+1} \quad \Delta G=[\frac{sin^2\varphi}{\delta+1}+\frac{2}{\delta^2}+O(\frac{1}{\delta^3})]^{1/2} \quad \bar{\mathcal{A}}=\frac{\delta sin\varphi}{\delta+1}$$
$$\bar{n}=\frac{1}{2}(n_{min}+n_{max}) \quad \Delta n=\frac{\delta}{2(2\delta-1)^{1/2}} \tag{3.47}$$

Now for the validity of the condition $\Delta G \ll \bar{G}$, $|\varphi|$ should be much greater than $1/\delta^{1/2}$ and $\Delta G \Delta n$ is of order $|\bar{\mathcal{A}}|$ which shows that the function is semiclassical. The matter is that $\psi(\nu)$ has a sharp peak at $\nu = \delta/2$ and by using Stirling's formula it is easy to see that the width of the peak is of order $\delta^{1/2}$. It is also clear from the expression for $\bar{G}$ that this quantity equals the semiclassical value $1 - cos\varphi$ with a high accuracy only when $|\varphi| \gg 1/\delta^{1/2}$. This example might be considered as an indication that a semiclassical wave function such that the condition $|\varphi| \gg 1/\delta$ is sufficient, should satisfy the following properties. On one hand the width of the maximum should be of order δ and on the other the function should be small when n is close to n_{min} or n_{max}.

In view of this remark, the third example is $\psi(\nu) = const\, exp(-i\varphi\nu)\nu(\delta - \nu)$ if $n \in [n_{min}, n_{max}]$. Then the normalization condition is $const^2 = [\delta(\delta^4 - 1)/30]^{-1}$ and the result of calculations is

$$\bar{G} = 1 - cos\varphi + \frac{5cos\varphi}{\delta^2} + O(\frac{1}{\delta^3}) \quad \bar{\mathcal{A}} = sin\varphi\,(1 - \frac{5}{\delta^2}) \quad \bar{n} = (n_{min} + n_{max})/2$$
$$\overline{G^2} = (1 - cos\varphi)^2 + \frac{10}{\delta^2}(cos\varphi - cos2\varphi) + \frac{15cos\varphi}{\delta^3} + O(\frac{1}{\delta^4})$$
$$\Delta G = \frac{1}{\delta}[10sin^2\varphi + \frac{15cos\varphi}{\delta} + O(\frac{1}{\delta^2})]^{1/2} \quad \Delta n = \frac{\delta}{2\sqrt{7}} \tag{3.48}$$

Now $\bar{G} \approx 1 - cos\varphi$ if $|\varphi| \gg 1/\delta$ but $\Delta G \ll |\bar{G}|$ only if $|\varphi| \gg 1/\delta^{3/4}$ and $\Delta G \Delta n$ is of order $|\bar{\mathcal{A}}|$ only if $|\varphi| \gg 1/\delta^{1/2}$. The reason why the condition $|\varphi| \gg 1/\delta$ is not sufficient is that $\overline{G^2}$ approximately equals its classical value $(1 - cos\varphi)^2$ only when $|\varphi| \gg 1/\delta^{3/4}$. The term with $1/\delta^3$ in $\overline{G^2}$ arises because when ν is close to 0, $\psi(\nu)$ is proportional only to the first degree of ν and when ν is close to δ, it is proportional to $\delta - \nu$.

Our last example is $\psi(\nu) = const\, exp(-i\varphi\nu)[\nu(\delta - \nu)]^2$ if $n \in [n_{min}, n_{max}]$. It will suffice to estimate sums $\sum_{\nu=1}^{\delta} \nu^k$ by $\delta^{k+1}/(k+1) + O(\delta^k)$. In particular, the normalization condition is $const^2 = 35 \cdot 18/\delta^9$ and the result of calculations is

$$\bar{G} = 1 - cos\varphi + \frac{6cos\varphi}{\delta^2} + O(\frac{1}{\delta^4}) \quad \bar{\mathcal{A}} = sin\varphi\,(1 - \frac{6}{\delta^2}) \quad \bar{n} = (n_{min} + n_{max})/2$$
$$\overline{G^2} = (1 - cos\varphi)^2 + \frac{12}{\delta^2}(cos\varphi - cos2\varphi) + O(\frac{1}{\delta^4})$$
$$\Delta G = \frac{1}{\delta}[12sin^2\varphi + O(\frac{1}{\delta^2})]^{1/2} \quad \Delta n = \frac{\delta}{2\sqrt{11}} \tag{3.49}$$

In this example the condition $|\varphi| \gg 1/\delta$ is sufficient to ensure that $\Delta G \ll |\bar{G}|$ and $\Delta G \Delta n$ is of order $|\bar{\mathcal{A}}|$.

At the same time, the following question arises. If we wish to perform mathematical operations with a physical quantity in classical theory, we should guarantee that not only this quantity is semiclassical but a sufficient number of its powers is semiclassical too. Since the classical value of G is proportional to φ^2 and φ is small, there is no guaranty that for the quantity G this is the case. Consider, for example, whether G^2 is semiclassical. It is clear from Eq. (3.49) that $\overline{G^2}$ is close to its classical value $(1-cos\varphi)^2$ if $|\varphi| \gg 1/\delta$. However, $\Delta(G^2)$ will be semiclassical only if $\overline{G^4}$ is close to its classical value $(1-cos\varphi)^4$. A calculation with the wave function from the last example gives

$$\overline{G^4} = (1-cos\varphi)^4 + \frac{24}{\delta^2}(1-cos\varphi)^3(3+4cos\varphi) + \frac{84}{\delta^4}(1-cos\varphi)^2(64cos^2\varphi + 11cos\varphi - 6) + \frac{35\cdot 9}{2\delta^5} + O(\frac{1}{\delta^6}) \quad (3.50)$$

Therefore $\overline{G^4}$ will be close to its classical value $(1-cos\varphi)^4$ only if $|\varphi| \gg 1/\delta^{5/8}$. Analogously, if $\psi(\nu) = const[\nu(\delta-\nu)]^3$ then G^2 will be semiclassical but G^3 will not. This consideration shows that a suficient number of powers of G will be semiclassical only if $\psi(n)$ is sufficiently small in vicinities of n_{min} and n_{max}. On the other hand, in the example described by Eq. (3.47), the width of maximum is much less than δ and therefore the condition $|\varphi| \gg 1/\delta$ is still insufficient.

The problem arises whether it is possible to find a wave function such that the contributions of the values of ν close to 0 or δ is negligible while the effective width of the maximum is or order δ. For example, it is known that for any segment $[a,b]$ and any $\epsilon < (b-a)/2$ it is possible to find an infinitely differentiable function $f(x)$ on $[a,b]$ such that $f(x) = 0$ if $x \notin [a,b]$ and $f(x) = 1$ if $x \in [a+\epsilon, b-\epsilon]$. However, we cannot use such functions for several reasons. First of all, the values of ν can be only integers: $\nu = 0,1,2,...\delta$. Another reason is that for correspondence with GFQT we can use only rational functions and even $exp(-i\nu\varphi)$ should be expressed in terms of rational functions (see Sect. 3.6).

In view of this discussion, we accept that the functions similar to that described in the second example give the best approximation for semiclassical approximation since in that case the condition $|\varphi| \gg 1/\delta^{1/2}$ guarantees that sufficiently many quantities G^k $(k = 1,2,...)$ will be semiclassical. In this example it is possible to give an explicit formula for $\overline{G^k}$. The calculation involves hypergeometric functions

$$F(-\delta, -\delta+k; k+1; 1) = \sum_{l=0}^{\infty} \frac{(-\delta)_l(-\delta+k)_l}{l!(k+1)_l}$$

where $(k)_l$ is the Pochhammer symbol. Such sums are finite and can be calculated by using the Saalschutz theorem [29]: $F(-\delta, -\delta+k; k+1; 1) = k!(2\delta+k)!/\delta!(\delta+k)!$. As a result,

$$\overline{G^k} = \frac{1}{2^k}C_{2k}^k[1 + 2\sum_{l=1}^{k}(-1)^l cos(l\varphi)\frac{(-k)_l(-\delta)_l}{(k+1)_l(\delta+1)_l}] \quad (3.51)$$

In particular,

$$
\begin{aligned}
&\overline{G^2} = (1-cos\varphi)^2 + \frac{2(1-cos\varphi)(1+2cos\varphi)}{\delta+1} + \frac{3cos2\varphi}{(\delta+1)(\delta+2)}\\
&\overline{G^3} = (1-cos\varphi)^3 + \frac{3(1-cos\varphi)^2(2+3cos\varphi)}{(\delta+1)} +\\
&\frac{9(1-cos\varphi)(4cos^2\varphi+2cos\varphi-1)}{(\delta+1)(\delta+2)} + \frac{15cos3\varphi}{(\delta+1)(\delta+2)(\delta+3)}\\
&\overline{G^4} = (1-cos\varphi)^4 + \frac{4(1-cos\varphi)^3(3+4cos\varphi)}{(\delta+1)} + 6(1-cos\varphi)^2\cdot\\
&\frac{(20cos^2\varphi+18cos\varphi-1)}{(\delta+1)(\delta+2)} + \frac{60(1-cos\varphi)(8cos^3\varphi+4cos^2\varphi-4cos\varphi-1)}{(\delta+1)(\delta+2)(\delta+3)}\\
&+\frac{105cos4\varphi}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)}
\end{aligned}
\tag{3.52}
$$

Since φ is of order r/R, the condition $|\varphi| \gg 1/\delta^{1/2}$ is definitely satisfied at cosmological distances while the problem arises whether it is satisfied in the Solar system. Since δ can be treated as $2R\Delta q$ where Δq is the width of the relative momentum distribution in the internal two-body wave function, $\varphi\delta$ is of order $r\Delta q$. For understanding what the order of magnitude of this quantity is, one should have estimations of Δq for macroscopic wave functions. However, to the best of our knowledge, the existing theory does not make it possible to give reliable estimations of this quantity.

Below we argue that Δq is of order $1/r_g$ where r_g is the gravitational (Schwarzschild) radius of the component of the two-body system which has the greater mass. Then $\varphi\delta$ is of order r/r_g. This is precisely the parameter defining when standard Newtonian gravity is a good approximation to GR. For example, the gravitational radius of the Earth is of order $0.01m$ while the radius of the Earth is $R_E = 6.4\times10^6 m$. Therefore R_E/r_g is of order 10^9. The gravitational radius of the Sun is of order $3000m$, the distance from the Sun to the Earth is or order $150\times10^9 m$ and so r/r_g is of order 10^8. At the same time, the above discussion shows that the condition $\varphi\delta \gg 1$ is not sufficient for ensuring semiclassical approximation while the condition $|\varphi| \gg 1/\delta^{1/2}$ is. Hence we should compare the quantities r/R and $(r_g/R)^{1/2}$. Then it is immediately clear that the requirement $|\varphi| \gg 1/\delta^{1/2}$ will not be satisfied if R is very large. If R is of order $10^{26}m$ then in the example with the Earth r/R is of order 10^{-19} and $(r_g/R)^{1/2}$ is of order 10^{-14} while in the example with the Sun r/R is of order 10^{-15} and $(r_g/R)^{1/2}$ is of order 10^{-10}. Therefore in these examples the requirement $|\varphi| \gg 1/\delta^{1/2}$ is not satisfied. We conclude that for systems of macroscopic bodies, semiclassical approximation can be valid only if standard distance operator is modified.

3.5 Newton's law of gravity

The notion of standard distance operator comes from Poincare invariant particle and nuclear physics. In these theories there is no parameter R; in particular rapidly oscillating exponents do not contain this parameter. One of the reasons why semiclassical approximation with this distance operator can be valid is as follows. It will be argued in the next chapters that in GFQT the width δ of the n-distribution for a macroscopic body is inversely proportional to its mass. Therefore for nuclei and elementary particles the quantity δ is much greater than for macroscopic bodies and the requirement $|\varphi| \gg 1/\delta^{1/2}$ can be satisfied in some situations. On the other hand, such a treatment of the distance operator for macroscopic bodies is incompatible with semiclassical approximation since, as discussed in the preceding section, φ is typically much less than $1/\delta^{1/2}$. For this reason the interpretation of the distance operator should be modified such that the rapidly oscillating exponents are not $exp(-i\varphi n)$ but $exp(-i\chi n)$ where χ is much greater than φ and is a function of r to be determined. Note that when we discussed the operator $\mathbf{D}^2$ compatible with the standard interpretation of the distance operator, we did not neglect $\mathbf{J}$ in this operator and treated $|\varphi|$ as $r_{||}/R$. However, when we neglect all corrections of order $1/R$ and higher, we neglect $\mathbf{J}$ in $\mathbf{D}^2$ and replace φ by χ which does not vanish when $R \to \infty$. As shown in Sect. 3.3, the operator $\mathbf{D}^2$ is rotationally invariant since the internal two-body momentum operator is a reduction of the operator $\mathbf{J}$" on the two-body rest states, $\mathbf{D}$ satisfies Eq. (3.29) and therefore $[\mathbf{J}", \mathbf{D}^2] = 0$. Hence χ can be only a function of r but not $r_{||}$.

Ideally, a physical interpretation of an operator of a physical quantity should be obtained from the quantum theory of mesurements which should describe the operator in terms of a measurement of the corresponding physical quantity. However, although quantum theory is known for 80+ years, the quantum theory of measurements has not been developed yet. Our judgment about operators of different physical quantities can be based only on intuition and comparison of theory and experiment. As noted in Sect. 1.5, in view of our macroscopic experience, it seems unreasonable that if the uncertainty Δr of r does not depend on r then the relative accuracy $\Delta r/r$ in the measurement of r is better when r is greater.

When $exp(-i\varphi n)$ is replaced by $exp(-i\chi n)$, the results obtained in the preceding section remain valid but φ should be replaced by χ. Suppose that $\chi = f(C(\varphi\delta)^\alpha)$ where $\varphi = r/R$, C is a constant and $f(x)$ is a function such that $f(x) = x + o(x)$ where the correction $o(x)$ will be discussed later. Then $\Delta\chi \approx C\varphi^{\alpha-1}\delta^\alpha\Delta\varphi$. If φ is replaced by χ then, as follows from Eq. (3.47), $\Delta(\chi^2)$ is of order $\chi/\delta^{1/2}$ and therefore $\Delta\chi$ is of order $1/\delta^{1/2}$. As a consequence, $\Delta\varphi \approx const \cdot \varphi(\varphi\delta)^{-\alpha}/\delta^{1/2}$. As discussed in the preceding section, the value of $(\varphi\delta)$ is typically much greater than unity. Hence the accuracy of the measurement of φ is better when $\alpha < 0$. In that case the relative accuracy $\Delta\varphi/\varphi$ is better for lesser values of φ and, as noted in Sect. 1.5, this is a desired behavior in view of our macroscopic experience. If $\alpha < 0$ then $\Delta\varphi \approx const \cdot \varphi(\varphi\delta)^{|\alpha|}/\delta^{1/2}$. In view of quantum mechanical experience, one might expect that the accuracy should be better if δ is greater. On the other hand, in our approach δ is inversely proportional to the masses of the bodies under consideration and our macroscopic experience tells us that the accuracy of the measurement of relative distance does not depend on the mass. Indeed, suppose that we measure a distance by sending a light signal. Then the accuracy of the measurement should not depend on whether the signal is reflected by the mass $1kg$ or $1000kg$. Therefore at macroscopic level the accuracy should not depend on δ. Hence the optimal choice is $\alpha = -1/2$. In that case $\Delta\varphi \approx const \cdot \varphi^{3/2}$ and $\chi = f(C/(\varphi\delta)^{1/2})$. Then, if C is of order unity, the condition $\chi \gg 1/\delta^{1/2}$, which, as explained in the preceding section, guarantees that semiclassical approximarion is valid, is automatically satisfied since in the Solar system we always have $(R/r)^{1/2} \gg 1$. We will see below in this section that such a dependence of χ on φ and δ gives a natural explanation of the Newton law of gravity.

As follows from Eqs. (3.47), with φ replaced by χ, the mean value of the operator G is $1 - cos\chi$ with a high accuracy. Consider two-body wave functions having the form $\psi(N,n) = [\delta(N)/(\delta_1\delta_2)]^{1/2}\psi(n)$. As follows from Eq. (3.42), such functions are normalized to one. Then, as follows from Eq. (3.45), the mean value of the operator I_2 can be written as

$$\overline{I_2} = 4R^2M_0^2 + \overline{\Delta I_2} \quad \overline{\Delta I_2} = -2[(w_1+4n_1^2)(w_2+4n_2^2)]^{1/2}F(\delta_1,\delta_2,\varphi)$$
$$F(\delta_1,\delta_2,\varphi) = \frac{1}{\delta_1\delta_2}\sum_{N=Nmin}^{Nmax}\delta(N)\{1 - cos[f(\frac{C}{(\varphi\delta(N))^{1/2}})]\} \tag{3.53}$$

Strictly speaking, the semiclassical form of the wave function $exp(-i\chi n)a(n)$ cannot be used if $\delta(N)$ is very small; in particular, it cannot be used when $\delta(N) = 0$. We assume that in these cases the internal wave function can be modified such that the main contribution to the sum in Eq. (3.53) is given by those N where $\delta(N)$ is not small.

If φ is so large that the argument α of cos in Eq. (3.53) is extremely small, then the correction to Poincare limit is negligible. The next approximation is that this argument is small such we can approximate $cos(\alpha)$ by $1 - \alpha^2/2$. Then, taking into account that $f(\alpha) = \alpha + o(\alpha)$ and that the number of values of N is $\delta_1 + \delta_2$ we get

$$\overline{\Delta I_2} = -C^2[(w_1+4n_1^2)(w_2+4n_2^2)]^{1/2}\frac{\delta_1+\delta_2}{\delta_1\delta_2|\varphi|} \tag{3.54}$$

Now, by analogy with the derivation of Eq. (3.28), it follows that the classical non-relativistic Hamiltonian is

$$H(\mathbf{r},\mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - \frac{m_1m_2RC^2}{2(m_1+m_2)r}(\frac{1}{\delta_1}+\frac{1}{\delta_2}) \tag{3.55}$$

We see that the correction disappears if the width of the dS momentum distribution for each body becomes very large. In standard theory (over complex numbers) there is no limitation on the width of distribution while, as noted in the preceding section, in semiclassical approximation the only limitation is that the width of the dS momentum distribution should be much less than the mean value of this momentum. In the next chapters we argue that in GFQT it is natural that the width of the momentum distribution for a macroscopic body is inversely proportional to its mass. Then we recover the Newton gravitational law. Namely, if

$$\delta_j = \frac{R}{m_j G'} \quad (j = 1, 2), \quad C^2 G' = 2G \tag{3.56}$$

then

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - G\frac{m_1 m_2}{r} \tag{3.57}$$

We conclude that in our approach gravity is simply a dS the correction to the standard nonrelativistic Hamiltonian. This correction is spherically symmetric since, as noted in the beginning of this section, when all corrections of order $1/R$ are neglected, the dependence of $\mathbf{D}^2$ on $\mathbf{J}$ disappears.

3.6 Precession of Mercury's perihelion

It is well known that in GR and other field theories, the N-body system can be described by a Hamiltonian depending only on the degrees of freedom corresponding to these bodies only in order v^2 since even in order v^3 one should take into account other degrees of freedom. In the literature on GR the N-body Hamiltonian is discussed taking into account post-Newtonian corrections to the Hamiltonian (3.57). Among those corrections there is one which does not depend on velocities at all but is quadratic in G/r. Namely, the Hamiltonian with post-Newtonian corrections discussed in a vast literature (see e.g. Ref. [24]) is

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - G\frac{m_1 m_2}{r} + (...) + \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r^2} \tag{3.58}$$

where (...) contains relativistic corrections of order v^2. The last term in this expression is responsible for the precession of the perihelion of Mercury's orbit.

For calculating this effect in our approach, one should choose the form of the function f in Eq. (3.53). A question arises whether one can give arguments in favor of a specific choice. In the subsequent chapters we argue that fundamental quantum theory should be based on Galois fields rather than complex numbers. A question arises whether this imposes any restrictions on the form of wave functions. In standard theory, any complex number can be always written in the form $z = |z|exp(i\alpha)$. However, in our approach there can be no trigonometric functions and square roots and a direct correspondence between GFQT and standard theory takes place only for wave functions represented as rational functions (see the discussion in Sect. 4.1). From this point of view, the function $f(x)$ should be such that $exp[if(x)n] = [exp(if(x))]^n$ is a rational function.

Since $f(x) = x + o(x)$, a possible way to get rid of trigonometric functions is to choose $f(x) = arcsin(x)$. However, in that case $exp(if(x)) = cos(f(x)) + isin(f(x)) = (1-x^2)^{1/2} + ix$ contains a square root what, from the point of view of the above remarks is unacceptable. One can get rid of both, trigonometric functions and square roots, by choosing $f(x) = 2arctg(x/2)$ since in this case

$$exp(if(x)) = \frac{1 - x^2/4 + ix}{1 + x^2/4} \tag{3.59}$$

Then Eq. (3.53) can be written as

$$\overline{\Delta I_2} = 2[(w_1 + 4n_1^2)(w_2 + 4n_2^2)]^{1/2}\{[\frac{1}{\delta_1\delta_2}\sum_{N=Nmin}^{Nmax}\delta(N)\frac{|4\delta(N)\varphi| - C^2}{|4\delta(N)\varphi| + C^2}] - 1\} \tag{3.60}$$

Suppose that one tries to calculate this expression by expending in powers of $C^2/|\delta(N)\varphi|$. Then the term linear in $C^2/|\delta(N)\varphi|$ gives the Newton law (as explained in the preceding section) but the next terms are singular since $\delta(N) = 0$ if $N = N_{min}$ and $N = N_{min}$. As noted above, semiclassical approximation does not apply if $\delta(N)$ is small, so for such values of N Eq. (3.60) should be modified. However, if we consider only a case when $m_2 \gg m_1$ then $\delta_1 \gg \delta_2$ and, as it is clear from Fig. 3.1, the main contribution to Eq. (3.60) is given by those N where $\delta(N) = \delta_2$. Then replacing $\delta(N)$ in Eq. (3.60) by δ_2 we get

$$\overline{\Delta I_2} = -[(w_1 + 4n_1^2)(w_2 + 4n_2^2)]^{1/2}\frac{C^2}{|\varphi|}(\frac{1}{\delta_1} + \frac{1}{\delta_2})\frac{1}{1 + C^2/(4\delta_2|\varphi|)} \tag{3.61}$$

Then, as follows from Eq. (3.56), in the nonrelativistic approximation

$$M^2 = M_0^2 - \frac{2}{r}Gm_1m_2(m_1 + m_2)\frac{1}{1 + Gm_2/2r} \tag{3.62}$$

where M^2 is the mean value of the operator $I_2/4R^2$ and M_0^2 is the mean value of the mass operator squared in Poincare invariant theory. Now we can expand this expression in powers of G/r and take into account only the linear and quadratic terms. Then calculating the square root from the both parts of this expression with the same accuracy, we get that if $m_2 \gg m_1$ then the nonrelativistic Hamiltonian is

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - G\frac{m_1m_2}{r} + \frac{G^2m_1m_2^2}{2r^2} \tag{3.63}$$

This result is in agreement with Eq. (3.58) if $m_2 \gg m_1$.

It is well known that the contribution of the last term of Eq. (3.58) to the precession of Mercury's perihelion is 43" per century. This is less than 1% of the total precession 5600". It is believed that the main contributions to the total precession (the precession of the equinoxes and the gravitational tugs of the other planets) are known with a very high accuracy. Nevertheless, in the literature there are different opinions on whether, the contribution 43" of GR fully explains the data or not. In our approach this result has been recovered from the point of view of correspondence between standard theory and GFQT. Therefore our consideration can be treated as an argument in favor of the result obtained in GR.

In summary, we have shown that if $\chi = f(C/(\varphi\delta)^{1/2})$ then in the framework of our approach, the first term of expansion of $\overline{\Delta I_2}$ in G/r reproduces standard Newtonian gravity while the second term reproduces the result of GR for the precession of Mercury's perihelion if the width of the dS momentum distribution for a macroscopic body is inversely proportional to its mass. In the subsequent chapters we argue that in GFQT this property has a natural explanation.

3.7 Remarks on the problem of evolution in de Sitter invariant quantum theory

In Sects. 3.5 and 3.6 it has been shown how the terms in the classical Hamiltonian responsible for Newton's law of gravity and the precession of Mercury's perihelion can be recovered in the framework of our approach. However, classical equations of motions have not been discussed. In classical mechanics, if the classical Hamiltonian is known then the evolution can be described by the Hamilton equations or the Hamilton-Jacobi equation. Since we believe that quantum theory is more general than classical one, this conclusion should be substantiated from the point of view of semiclassical approximation in quantum theory. However, as noted in Sect. 1.3, the problem of time is an unsolved problem of quantum theory since there is no operator corresponding to t.

In standard quantum mechanics it is assumed that t is a parameter describing the evolution of a quantum system via the Schroedinger equation. Then in a special case when a semiclassical wave function contains a rapidly oscillating exponent $exp(iS)$ where S is the classical action, one recovers the Hamilton-Jacobi equation from the Schroedinger equation. However, the consideration is Sect. 3.4 shows that the presence of rapidly oscillating exponent does not always guarantee that semiclassical approximation is valid. In our approach the Newton law and the result for the precession of Mercury's perihelion have been obtained from wave functions containing a rapidly oscillating exponent $exp(-i\chi n)$ instead of $exp(-i\varphi n)$ in standard theory. Since χ depends on the width of the dS momentum distribution, the index of the exponent can no longer be treated as a classical action.

Another problem in describing evolution in dS quantum theory is as follows. The classical Hamiltonian, which is believed to be responsible for classical evolution, is extracted from the operator $M = (I_2/4R^2)^{1/2}$. This operator is treated as the mass operator, i.e. the Hamiltonian in the c.m. frame. Therefore M defines the evolution in the c.m. frame such that if ψ is the internal wave function then its dependence on time t is defined by

$$i\frac{\partial\psi(t)}{\partial t} = M\psi \tag{3.64}$$

However, in dS theory the operator M^{40}, which can be called the dS Hamiltonian, is not distinguished among the other operators. The operator I_2 has a clear meaning as a Casimir operator, i.e. as the operator commuting with all the dS operators M^{ab}. In the general case this operator is not even positive definite. This situation has a clear analogy with standard relativistic theory where there is no law that the Casimir operator $E^2 - \mathbf{P}^2$ should be positive definite.

In view of these remarks, it is natural to think that in dS theory the evolution should be described by a parameter τ such that

$$i\frac{\partial\psi(\tau)}{\partial\tau} = I_2\psi \tag{3.65}$$

Then it is clear that τ and t have even different dimensions. In the nonrelativistic approximation, Eqs. (3.64) and (3.65) are equivalent if the relation between t and τ is $t = 4R^2(m_1 + m_2)\tau$. They are also equivalent in the case when only one particle is nonrelativistic and the mass of this particle is much greater than the energy of the other particle (e.g. when particle 1 is a photon and particle 2 is a macroscopic body). However, in the general case those equations are not equivalent. This poses a problem of how the evolution parameter should be defined.

3.8 Remarks on gravitational experiments with light

As already noted, in GR and other field theories, a closed description of a system with a fixed number of particles is possible only with the accuracy $(v/c)^2$. In particular, there is no closed description of two-body systems where one of the bodies is a photon. However, in our approach the particles are free and there is no problem to write down all the ten operators of the dS algebra for such systems.

The results of Sects. 3.3-3.6 give grounds to believe that the operator $\mathbf{D}$ is a natural dS generalization of the distance operator in standard theory. However, as noted in Sects. 1.3 and 1.5, the notion of the distance operator for the photon is problematic. From the formal point of view, in the consideration of the operator $\mathbf{D}$ it has not been assumed that the particle in question is nonrelativistic. Therefore one might try to see what happens if the formal results involving $\mathbf{D}$ are applied to a two-body system where particle 1 is relativistic (e.g. it is a photon) while particle 2 is nonrelativistic and has the mass much greater than the energy of particle 1. It is not clear whether semiclassical approximation can be applied to the photon but in any case, if $\delta_1 \gg \delta_2$ then the details of the photon wave function are not important and by using Eq. (3.56) we obtain by analogy with Eq. (3.57) that

$$M = M_0 - G\frac{E_1 m_2}{r} \tag{3.66}$$

where M is the mean value of the two-body mass in standard units, M_0 is the mean value of the two-body mass in Poincare invariant theory and $E_1 = (w_1 + 4n_1^2)^{1/2}/2R$ is the energy of particle 1 in standard units. Therefore we have a full analogy with the Newton gravitational law but the mass of particle 1 is replaced by its energy.

Consider first the case when the photon travels in the radial direction from the Earth surface to the height H. In that case one can formally define the potential energy of the photon near the Earth surface by $U(H) = E_1 gH/c^2$ in standard units. Therefore when the photon travels in the radial direction from the Earth surface to the height H, the relative change of its kinetic energy is $\Delta E_1/E_1 = gH/c^2$. We have a full analogy with classical mechanics but now the change of the energy is small. Nevertheless, from the formal point of view, the result is in agreement with GR and the usual statement is that this small effect has been measured in the famous Pound-Rebka experiment.

The conventional interpretation of the above effect has been criticized by L.B. Okun in Ref. [39]. In his opinion, "*a presumed analogy between a photon and a stone*" is wrong. The reason is that "*the energy of the photon and hence its frequency $\omega = E/\hbar$ do not depend on the distance from the gravitational body, because in the static case the gravitational potential does not depend on the time coordinate t. The reader who is not satisfied with this argument may look at Maxwell's equations as given e.g. in section 5.2 of ref. [40]. These equations with time independent metric have solutions with frequencies equal to those of the emitter*". In Ref. [39] the result of the Pound-Rebka experiment is explained such that not the photon looses its kinetic energy but the differences between the atom energy levels on the height H are greater than on the Earth surface and "*As a result of this increase the energy of a photon emitted in a transition of an atom downstairs is not enough to excite a reverse transition upstairs. For the observer upstairs this looks like a redshift of the photon. Therefore for a competent observer the apparent redshift of the photon is a result of the blueshift of the clock.*".

As noted in Ref. [39], "*A naive (but obviously wrong!) way to derive the formula for the redshift is to ascribe to the photon with energy E a mass $m_\gamma = E/c^2$ and to apply to the photon a non-relativistic formula $\Delta E = -m_\gamma \Delta\phi$ treating it like a stone. Then the relative shift of photon energy is $\Delta E/E = -\Delta\phi/c^2$, which coincides with the correct result. But this coincidence cannot justify the absolutely thoughtless application of a nonrelativistic formula to an ultrarelativistic object.*"

However, in our approach no nonrelativistic formulas for the photon have been used and the result $\Delta E_1/E_1 = gH/c^2$ has been obtained in a fully relativistic approach. As already noted, the only problematic point in deriving this result is that the notion of the coordinate operator for the photon is not clear. In the framework of our approach a stone and a photon are simply particles with different masses; that is why the stone is nonrelativistic and the photon is ultrarelativistic. Therefore there is no reason to think that in contrast to the stone, the photon will not loose its kinetic energy. At the same time, we believe that Ref. [39] gives strong arguments that energy levels on the Earth surface and on the height H are different.

We believe that the following point in the arguments of Ref. [39] is not quite consistent. A stone, a photon and other particles can be characterized by their energies, momenta and other quantities for which there exist well defined operators. Those quantities might be measured in collisions of those particles with other particles. At the same time, the notions of "frequency of a photon" or "frequency of a stone" have no physical meaning. The terms "wave function" and "particle-wave duality" have arisen at the beginning of quantum era in efforts to explain quantum behavior in terms of classical waves but now it is clear that no such explanation exists. The notion of wave is purely classical; it has a physical meaning only as a way of describing systems of many particles by their average characteristics. In particular, such notions as frequency and wave length can be applied only to classical waves, i.e. to systems consisting of many particles. If a particle wave function (or rather a state vector is a better name) contains $exp[i(px-Et)/\hbar]$ then by analogy with the theory of classical waves one might say that the particle is a wave with the frequency $\omega = E/\hbar$ and the wave length $\lambda = 2\pi\hbar/p$. However, such defined quantities ω and λ are not real frequencies and the wave lengths measured e.g. in spectroscopic experiments. The term "wave function" may be misleading since in quantum theory it defines not amplitudes of waves but only amplitudes of probabilities. In what follows, speaking about ω and λ we will mean only frequencies and wave lengths measured in experiments with classical waves. Those quantities necessarily involve classical space and time. Then the relation $E = \hbar\omega$ between the energies of particles in classical waves and frequencies of those waves is only an assumption that those different quantities are related in such a way. This relation has been first proposed by Planck for the description of the blackbody radiation and the experimental data indicate that it is valid with a high accuracy. However, there is no guaranty that this relation is always valid with the absolute accuracy, as the author of Ref. [39] assumes. In spectroscopic experiments not energies and momenta of emitted photons are measured but wave lengths of the radiation obtained as a result of transitions between different energy levels. In particular, there is no experiment confirming that the relation $E = \hbar\omega$ is always exact, e.g. on the Earth surface and on the height H. In summary, the Pound-Rebka experiment cannot be treated as a model-independent confirmation of GR.

Another experiment widely discussed in the literature is the deflection of light by the Sun. The result is usually represented such that if ρ is the impact parameter and r_g is the gravitational radius of the Sun then the deflection angle is $\theta = (1+\gamma)r_g/\rho$ where γ depends on the theory and in GR $\gamma = 1$. It is believed that $\gamma = 1$ has been experimentally confirmed with the accuracy better than 1%. A question arises how this result can be obtained in the framework of a quantum approach. If we take the result given by Eq. (3.66) and assume that the evolution is defined by the Hamilton equations with M or I_2 in place of the classical Hamiltonian (see the discussion in the preceding section) then the result corresponds to $\gamma = 0$. This is probably an indication that, as already noted, the notion of the coordinate operator for a photon is problematic. In the texbook [41], the deflection is treated as a consequence of one-graviton exchange. The author defines the vertices responsible for the interaction of a virtual graviton with a scalar nonrelativistic particle and with a photon and in that case the cross-section of the process described by the one-graviton exchange corresponds to the result with $\gamma = 1$. The problem is that there is no other way of testing the photon-graviton vertex and we believe that it is highly unrealistic that when the photon travels in the x direction from $-\infty$ to $+\infty$, it exchanges only by one virtual graviton with the Sun. Therefore a problem of how to recover the result with $\gamma = 1$ in quantum theory remains open.

Chapter 4

Why is GFQT more pertinent physical theory than standard one?

4.1 What mathematics is most pertinent for quantum physics?

Since the absolute majority of physicists are not familiar with Galois fields, our first goal in this chapter is to convince the reader that the notion of Galois fields is not only very simple and elegant, but also is a natural basis for quantum physics. If a reader wishes to learn Galois fields on a more fundamental level, he or she might start with standard textbooks (see e.g. Ref. [42]).

In view of the present situation in modern quantum physics, a natural question arises why, in spite of big efforts of thousands of highly qualified physicists for many years, the problem of quantum gravity has not been solved yet. We believe that a possible answer is that they did not use the most pertinent mathematics.

For example, the problem of infinities remains probably the most challenging one in standard formulation of quantum theory. As noted by Weinberg [22], *'Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day'*. The title of the recent Weinberg's paper [43] is "Living with infinities". A desire to have a theory without divergences is probably the main motivation for developing modern theories extending QFT, e.g. loop quantum gravity, noncommutative quantum theory, string theory etc. On the other hand, in theories over Galois fields, infinities cannot exist in principle since any Galois field is finite.

The key ingredient of standard mathematics is the notions of infinitely small and infinitely large. The notion of infinitely small is based on our everyday experience that any macroscopic object can be divided by two, three and even a million parts. But is it possible to divide by two or three the electron or neutrino? It seems obvious that the very existence of elementary particles indicates that standard division has only a limited meaning. Indeed, consider, for example, the gram-molecule of water having the mass 18 grams. It contains the Avogadro number of molecules $6 \cdot 10^{23}$. We can divide this gram-molecule by ten, million, billion, but when we begin to divide by numbers greater than the Avogadro one, the division operation loses its meaning.

If we accept that the notion of infinitely small can be only approximate in some situations then we have to acknowledge that fundamental physics cannot be based on continuity, differentiability, geometry, topology etc. We believe it is rather obvious that these notions are based on our macroscopic experience. For example, the water in the ocean can be described by equations of hydrodynamics but we know that this is only an approximation since matter is discrete. The reason why modern quantum physics is based on these notions is probably historical: although the founders of quantum theory and many physicists who contributed to it were highly educated scientists, discrete mathematics was not (and still is not) a part of standard physics education.

The notion of infinitely large is based on our belief that *in principle* we can operate with any large numbers. In standard mathematics this belief is formalized in terms of axioms about infinite sets (e.g. Zorn's lemma or Zermelo's axiom of choice) which are accepted without proof. Our belief that these axioms are correct is based on the fact that sciences using standard mathematics (physics, chemistry etc.) describe nature with a very high accuracy. It is believed that this is much more important than the fact that, as follows from Goedel's incompleteness theorems, standard mathematics cannot be a selfconsistent theory since no system of axioms can ensure that all facts about natural numbers can be proved.

Standard mathematics contains statements which seem to be counterintuitive. For example, the interval $(0, 1)$ has the same cardinality as $(-\infty, \infty)$. Another example is that the function tgx gives a one-to-one relation between the intervals $(-\pi/2, \pi/2)$ and $(-\infty, \infty)$. Therefore one can say that a part has the same number of elements as a whole. One might think that this contradicts common sense but in standard mathematics the above facts are not treated as contradicting.

Another example is that we cannot verify that $a + b = b + a$ for any numbers a and b. At the same time, in the spirit of quantum theory there should be no statements accepted without proof (and based only on belief that they are correct); only those statements should be treated as physical, which can be experimentally verified, at least in principle. Suppose we wish to verify that 100+200=200+100. In the spirit of quantum theory it is insufficient to just say that 100+200=300 and 200+100=300. We should describe an experiment where these relations can be verified. In particular, we should specify whether we have enough resources to represent the numbers 100, 200 and 300. We believe the following observation is very important: although standard mathematics is a part of our everyday life, people typically do not realize that *standard mathematics is implicitly based on the assumption that* *one can have any desirable amount of resources.*

Suppose, however that our Universe is finite. Then the amount of resources cannot be infinite. In particular, it is impossible in principle to build a computer operating with any number of bits. In this scenario it is natural to assume that there exists a fundamental number p such that all calculations can be performed only modulo p. Then it is natural to consider a quantum theory over a Galois field with the characteristic p. Since any Galois field is finite, the fact that arithmetic in this field is correct can be verified (at least in principle) by using a finite amount of resources.

Let us look at mathematics from the point of view of the famous Kronecker expression: "*God made the natural numbers, all else is the work of man*". Indeed, the natural numbers 0, 1, 2... have a clear physical meaning. However only two operations are always possible in the set of natural numbers: addition and multiplication. In order to make addition reversible, we introduce negative integers -1, -2 etc. Then, instead of the set of natural numbers we can work with the ring of integers where three operations are always possible: addition, subtraction and multiplication. However, the negative numbers do not have a direct physical meaning (we cannot say, for example, "I have minus two apples"). Their only role is to make addition reversible.

The next step is the transition to the field of rational numbers in which all four operations except division by zero are possible. However, as noted above, division has only a limited meaning.

In mathematics the notion of linear space is widely used, and such important notions as the basis and dimension are meaningful only if the space is considered over a field or body. Therefore if we start from natural numbers and wish to have a field, then we have to introduce negative and rational numbers. However, if, instead of all natural numbers, we consider only p numbers 0, 1, 2, ... $p-1$ where p is prime, then we can easily construct a field without adding any new elements. This construction, called Galois field, contains nothing that could prevent its understanding even by pupils of elementary schools.

Let us denote the set of numbers 0, 1, 2,...$p-1$ as F_p. Define addition and multiplication as usual but take the final result modulo p. For simplicity, let us consider the case $p=5$. Then F_5 is the set 0, 1, 2, 3, 4. Then $1+2=3$ and $1+3=4$ as usual, but $2+3=0$, $3+4=2$ etc. Analogously, $1\cdot 2=2$, $2\cdot 2=4$, but $2\cdot 3=1$, $3\cdot 4=2$ etc. By definition, the element $y\in F_p$ is called opposite to $x\in F_p$ and is denoted as $-x$ if $x+y=0$ in F_p. For example, in F_5 we have -2=3, -4=1 etc. Analogously $y\in F_p$ is called inverse to $x\in F_p$ and is denoted as $1/x$ if $xy=1$ in F_p. For example, in F_5 we have 1/2=3, 1/4=4 etc. It is easy to see that addition is reversible for any natural $p>0$ but for making multiplication reversible we should choose p to be a prime. Otherwise the product of two nonzero elements may be zero modulo p. If p is chosen to be a prime then indeed F_p becomes a field without introducing any new objects (like negative numbers or fractions). For example, in this field each element can obviously be treated as positive and negative *simultaneously*!

The above example with division might also be an indication that, in the spirit of Ref. [44], the ultimate quantum theory will be based even not on a Galois field but on a finite ring (this observation was pointed out to me by Metod Saniga).

One might say: well, this is beautiful but impractical since in physics and everyday life 2+3 is always 5 but not 0. Let us suppose, however that fundamental physics is described not by "usual mathematics" but by "mathematics modulo p" where p is a very large number. Then, operating with numbers which are much less than p we will not notice this p, at least if we only add and multiply. We will feel a difference between "usual mathematics" and "mathematics modulo p" only while operating with numbers comparable to p.

We can easily extend the correspondence between F_p and the ring of integers Z in such a way that subtraction will also be included. To make it clearer we note the following. Since the field F_p is cyclic (adding 1 successively, we will obtain 0 eventually), it is convenient to visually depict its elements by the points of a circle of the radius $p/2\pi$ on the plane (x, y). In Fig. 4.1 only a part of the circle near the origin is depicted. Then the distance between neighboring elements of the field is

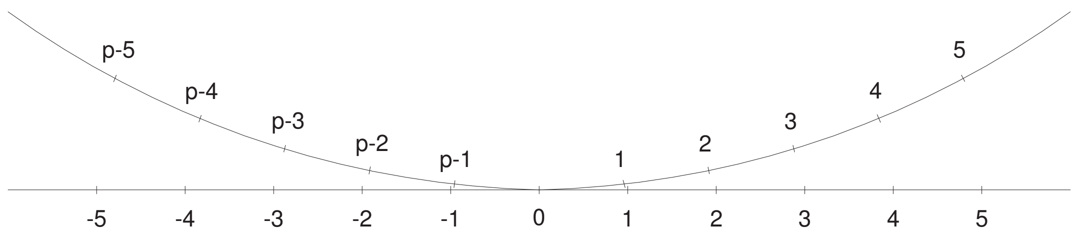

Figure 4.1: Relation between F_p and the ring of integers

equal to unity, and the elements 0, 1, 2,... are situated on the circle counterclockwise. At the same time we depict the elements of Z as usual such that each element $z \in Z$ is depicted by a point with the coordinates $(z, 0)$. We can denote the elements of F_p not only as 0, 1,... $p-1$ but also as 0, ± 1, ± 2,,...$\pm(p-1)/2$, and such a set is called the set of minimal residues. Let f be a map from F_p to Z, such that the element $f(a) \in Z$ corresponding to the minimal residue a has the same notation as a but is considered as the element of Z. Denote $C(p) = p^{1/(lnp)^{1/2}}$ and let U_0 be the set of elements $a \in F_p$ such that $|f(a)| < C(p)$. Then if $a_1, a_2, ...a_n \in U_0$ and n_1, n_2 are such natural numbers that

$$n_1 < (p-1)/2C(p), \; n_2 < ln((p-1)/2)/(lnp)^{1/2} \tag{4.1}$$

then

$$f(a_1 \pm a_2 \pm ...a_n) = f(a_1) \pm f(a_2) \pm ...f(a_n)$$

if $n \leq n_1$ and

$$f(a_1a_2...a_n) = f(a_1)f(a_2)...f(a_n)$$

if $n \leq n_2$. Thus though f is not a homomorphism of rings F_p and Z, but if p is sufficiently large, then for a sufficiently large number of elements of U_0 the addition, subtraction and multiplication are performed according to the same rules as for elements $z \in Z$ such that $|z| < C(p)$. Therefore f can be treated as a local isomorphism of rings F_p and Z.

The above discussion has a well known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature R. Analogously one might think that the set of numbers describing physics has a curvature defined by a very large number p but we do not notice it when we deal only with numbers much less than p. This number should be treated as a fundamental constant describing laws of physics in our Universe.

One might argue that introducing a new fundamental constant is not justified. However, the history of physics tells us that new theories arise when a parameter, which in the old theory was treated as infinitely small or infinitely large, becomes finite. For example, from the point of view of nonrelativistic physics, the velocity of light c is infinitely large but in relativistic physics it is finite. Analogously, from the point of view of classical theory, the Planck constant $\hbar$ is infinitely small but in quantum theory it is finite. Therefore it is natural to think that in the future quantum physics the quantity p will be not infinitely large but finite.

Let us note that even for elements from U_0 the result of division in the field F_p differs generally speaking, from the corresponding result in the field of rational number Q. For example the element $1/2$ in F_p is a very large number $(p+1)/2$. For this reason one might think that physics based on Galois fields has nothing to with the reality. We will see in the subsequent section that this is not so since the spaces describing quantum systems are projective. It is also clear that in general the meaning of square root in F_p is not the same as in Q. For example, even if $\sqrt{2}$ in F_p exists, it is a very large number of the order at least $p^{1/2}$. Another obvious fact is that GFQT cannot involve exponents and trigonometric functions since they are represented by infinite sums. Therefore a direct correspondence between wave functions in GFQT and standard theory can exist only for rational functions. This remark has been used in Sect. 3.6 for choosing the form of the wave function describing the precession of Mercury's perihelion.

By analogy with the field of complex numbers, we can consider a set F_{p^2} of p^2 elements $a + bi$ where $a, b \in F_p$ and i is a formal element such that $i^2 = -1$. The question arises whether F_{p^2} is a field, i.e. we can define all the four operations except division by zero. The definition of addition, subtraction and multiplication in F_{p^2} is obvious and, by analogy with the field of complex numbers, one could define division as $1/(a + bi) = a/(a^2 + b^2) - ib/(a^2 + b^2)$. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in F_p for any $a, b \in F_p$ i.e. $a^2 + b^2$ is not divisible by p. Therefore the definition is meaningful only if p *cannot* be represented as a sum of two squares and is meaningless otherwise. We will not consider the case $p = 2$ and therefore p is necessarily odd. Then we have two possibilities: the value of $p\,(mod\,4)$ is either 1 or 3. The well known result of number theory (see e.g. the textbooks [42]) is that a prime number p can be represented as a sum of two squares only in the former case and cannot in the latter one. Therefore the above construction of the field F_{p^2} is correct only if $p\,(mod\,4) = 3$. By analogy with the above correspondence between F_p and Z, we can define a set U in F_{p^2} such that $a + bi \in U$ if $a \in U_0$ and $b \in U_0$. Then if $f(a + bi) = f(a) + f(b)i$, f is a local homomorphism between F_{p^2} and $Z + Zi$.

In general, it is possible to consider linear spaces over any fields. Therefore a question arises what Galois field should be used in GFQT. It is well known (see e.g. Ref. [42]) that any Galois field can contain only p^n elements where p is prime and n is natural. Moreover, the numbers p and n define the Galois field up to isomorphism. It is natural to require that there should exist a correspondence between any new theory and the old one, i.e. at some conditions the both theories should give close predictions. In particular, there should exist a large number of quantum states for which the probabilistic interpretation is valid. Then, in view of the above discussion, the number p should necessarily be very large and we have to understand whether there exist deep reasons for choosing a particular value of p or this is simply an accident that our Universe has been created with this value. In any case, if we accept that p is a universal constant then the problem arises what the value of n is. Since we treat GFQT as a more general theory than standard one, it is desirable not to postulate that GFQT is based on F_{p^2} (with $p = 3\ (mod\,4)$) because standard theory is based on complex numbers but vice versa, explain the fact that standard theory is based on complex numbers since GFQT is based on F_{p^2}. Therefore we should find a motivation for the choice of F_{p^2} with $p = 3\ (mod\,4)$. Arguments in favor of such a choice are discussed in Refs. [5, 6, 7] and in this paper we will consider only this choice.

4.2 Correspondence between GFQT and standard theory

For any new theory there should exist a correspondence principle that at some conditions this theory and standard well tested one should give close predictions. Well known examples are that classical nonrelativistic theory can be treated as a special case of relativistic theory in the formal limit $c \to \infty$ and a special case of quantum mechanics in the formal limit $\hbar \to 0$. Analogously, Poincare invariant theory is a special case of dS or AdS invariant theories in the formal limit $R \to \infty$. We treat standard quantum theory as a special case of GFQT in the formal limit $p \to \infty$. Therefore a question arises which formulation of standard theory is most suitable for its generalization to GFQT.

A well-known historical fact is that quantum mechanics has been originally proposed by Heisenberg and Schroedinger in two forms which seemed fully incompatible with each other. While in the Heisenberg operator (matrix) formulation quantum states are described by infinite columns and operators — by infinite matrices, in the Schroedinger wave formulations the states are described by functions and operators — by differential operators. It has been shown later by Born, von Neumann and others that the both formulations are mathematically equivalent. In addition, the path integral approach has been developed.

In the spirit of the wave or path integral approach one might try to replace classical spacetime by a finite lattice which may even not be a field. In that case the problem arises what the natural quantum of spacetime is and some of physical quantities should necessarily have the field structure. However, as argued in Sect. 1.3, fundamental physical theory should not be based on spacetime.

We treat GFQT as a version of the matrix formulation when complex numbers are replaced by elements of a Galois field. We will see below that in that case the columns and matrices are automatically truncated in a certain way, and therefore the theory becomes finite-dimensional (and even finite since any Galois field is finite).

In conventional quantum theory the state of a system is described by a vector $\tilde{x}$ from a separable Hilbert space H. We will use a "tilde" to denote elements of Hilbert spaces and complex numbers while elements of linear spaces over a Galois field and elements of the field will be denoted without a "tilde".

Let $(\tilde{e}_1, \tilde{e}_2, ...)$ be a basis in H. This means that $\tilde{x}$ can be represented as

$$\tilde{x} = \tilde{c}_1\tilde{e}_1 + \tilde{c}_2\tilde{e}_2 + ... \tag{4.2}$$

where $(\tilde{c}_1, \tilde{c}_2, ...)$ are complex numbers. It is assumed that there exists a complete set of commuting selfadjoint operators $(\tilde{A}_1, \tilde{A}_2, ...)$ in H such that each $\tilde{e}_i$ is the eigenvector of all these operators: $\tilde{A}_j\tilde{e}_i = \tilde{\lambda}_{ji}\tilde{e}_i$. Then the elements $(\tilde{e}_1, \tilde{e}_2, ...)$ are mutually orthogonal: $(\tilde{e}_i, \tilde{e}_j) = 0$ if $i \neq j$ where (...,...) is the scalar product in H. In that case the coefficients can be calculated as

$$\tilde{c}_i = \frac{(\tilde{e}_i, \tilde{x})}{(\tilde{e}_i, \tilde{e}_i)} \tag{4.3}$$

Their meaning is that $|\tilde{c}_i|^2(\tilde{e}_i, \tilde{e}_i)/(\tilde{x}, \tilde{x})$ represents the probability to find $\tilde{x}$ in the state $\tilde{e}_i$. In particular, when $\tilde{x}$ and the basis elements are normalized to one, the probability equals $|\tilde{c}_i|^2$.

Let us note that the Hilbert space contains a big redundancy of elements, and we do not need to know all of them. Indeed, with any desired accuracy we can approximate each $\tilde{x} \in H$ by a finite linear combination

$$\tilde{x} = \tilde{c}_1\tilde{e}_1 + \tilde{c}_2\tilde{e}_2 + ...\tilde{c}_n\tilde{e}_n \tag{4.4}$$

where $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$ are rational complex numbers. This is a consequence of the well known fact that the set of elements given by Eq. (4.4) is dense in H. In turn, this set is redundant too. Indeed, we can use the fact that Hilbert spaces in quantum theory are projective: ψ and $c\psi$ represent the same physical state. Then we can multiply both parts of Eq. (4.4) by a common denominator of the numbers $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$. As a result, we can always assume that in Eq. (4.4) $\tilde{c}_j = \tilde{a}_j + i\tilde{b}_j$ where $\tilde{a}_j$ and $\tilde{b}_j$ are integers.

The meaning of the fact that Hilbert spaces in quantum theory are projective is very clear. The matter is that not the probability itself but the relative probabilities of different measurement outcomes have a physical meaning. We believe, the notion of probability is a good illustration of the Kronecker expression about natural numbers (see Sect. 4.1). Indeed, this notion arises as follows. Suppose that conducting experiment N times we have seen the first event n_1 times, the second event n_2 times etc. such that $n_1 + n_2 + ... = N$. We define the quantities $w_i(N) = n_i/N$ (these quantities depend on N) and $w_i = lim\, w_i(N)$ when $N \to \infty$. Then w_i is called the probability of the *ith* event. We see that all the information about the experiment is given by a set of natural numbers, and in real life all those numbers are finite. However, in order to define probabilities, people introduce additionally the notion of rational numbers and the notion of limit. Another example is the notion of mean value. Suppose we measure a physical quantity such that in the first event its value is q_1, in the second event - q_2 etc. Then the mean value of this quantity is defined as $(q_1n_1 + q_2n_2 + ...)/N$ if N is very large. Therefore, even if all the q_i are integers, the mean value might be not an integer. We again see that rational numbers arise only as a consequence of our convention on how the results of experiments should be interpreted.

The Hilbert space is an example of a linear space over the field of complex numbers. Roughly speaking this means that one can multiply the elements of the space by the elements of the field and use the properties $\tilde{a}(\tilde{b}\tilde{x}) = (\tilde{a}\tilde{b})\tilde{x}$ and $\tilde{a}(\tilde{b}\tilde{x}+\tilde{c}\tilde{y}) = \tilde{a}\tilde{b}\tilde{x} + \tilde{a}\tilde{c}\tilde{y}$ where $\tilde{a}, \tilde{b}, \tilde{c}$ are complex numbers and $\tilde{x}, \tilde{y}$ are elements of the space. The fact that complex numbers form a field is important for such notions as linear dependence and the dimension of spaces over complex numbers.

By analogy with conventional quantum theory, we require that in GFQT linear spaces V over F_{p^2}, used for describing physical states, are supplied by a scalar product (...,...) such that for any $x, y \in V$ and $a \in F_{p^2}$, (x, y) is an element of F_{p^2} and the following properties are satisfied:

$$(x, y) = \overline{(y, x)}, \quad (ax, y) = \bar{a}(x, y), \quad (x, ay) = a(x, y) \tag{4.5}$$

We will always consider only finite dimensional spaces V over F_{p^2}. Let $(e_1, e_2, ...e_N)$ be a basis in such a space. Consider subsets in V of the form $x = c_1e_1 + c_2e_2 + ...c_ne_n$ where for any i, j

$$c_i \in U, \quad (e_i, e_j) \in U \tag{4.6}$$

On the other hand, as noted above, in conventional quantum theory we can describe quantum states by subsets of the form Eq. (4.4). If n is much less than p,

$$f(c_i) = \tilde{c}_i, \quad f((e_i, e_j)) = (\tilde{e}_i, \tilde{e}_j) \tag{4.7}$$

then we have the correspondence between the description of physical states in projective spaces over F_{p^2} on one hand and projective Hilbert spaces on the other. This means that if p is very large then for a large number of elements from V, linear combinations with the coefficients belonging to U and scalar products look in the same way as for the elements from a corresponding subset in the Hilbert space.

In the general case a scalar product in V does not define any positive definite metric and thus there is no probabilistic interpretation for all the elements from V. In particular, $(e, e) = 0$ does not necessarily imply that $e = 0$. However, the probabilistic interpretation exists for such a subset in V that the conditions (4.7) are satisfied. Roughly speaking this means that for elements $c_1e_1 + ...c_ne_n$ such that $(e_i, e_i), c_i\bar{c}_i \ll p$, $f((e_i, e_i)) > 0$ and $c_i\bar{c}_i > 0$ for all $i = 1, ...n$, the probabilistic interpretation is valid. It is also possible to explicitly construct a basis $(e_1, ...e_N)$ such that $(e_j, e_k) = 0$ for $j \neq k$ and $(e_j, e_j) \neq 0$ for all j (see the subsequent chapter). Then $x = c_1e_1 + ...c_Ne_N$ $(c_j \in F_{p^2})$ and the coefficients are uniquely defined by $c_j = (e_j, x)/(e_j, e_j)$.

As usual, if A_1 and A_2 are linear operators in V such that

$$(A_1x, y) = (x, A_2y) \quad \forall x, y \in V \tag{4.8}$$

they are said to be conjugated: $A_2 = A_1^*$. It is easy to see that $A_1^{**} = A_1$ and thus $A_2^* = A_1$. If $A = A^*$ then the operator A is said to be Hermitian.

If $(e, e) \neq 0$, $Ae = ae$, $a \in F_{p^2}$, and $A^* = A$, then it is obvious that $a \in F_p$. In the subsequent section (see also Refs. [5, 6]) we will see that there also exist situations when a Hermitian operator has eigenvectors e such that $(e, e) = 0$ and the corresponding eigenvalue is pure imaginary.

Let now $(A_1, ...A_k)$ be a set of Hermitian commuting operators in V, and $(e_1, ...e_N)$ be a basis in V with the properties described above, such that $A_je_i = \lambda_{ji}e_i$. Further, let $(\tilde{A}_1, ...\tilde{A}_k)$ be a set of Hermitian commuting operators in some Hilbert space H, and $(\tilde{e}_1, \tilde{e}_2, ...)$ be some basis in H such that $\tilde{A}_je_i = \tilde{\lambda}_{ji}\tilde{e}_i$. Consider a subset $c_1e_1 + c_2e_2 + ...c_ne_n$ in V such that, in addition to the conditions (4.7), the elements e_i are the eigenvectors of the operators A_j with λ_{ji} belonging to U and such that $f(\lambda_{ji}) = \tilde{\lambda}_{ji}$. Then the action of the operators on such elements have the same form as the action of corresponding operators on the subsets of elements in Hilbert spaces discussed above.

Summarizing this discussion, we conclude that if p is large then there exists a correspondence between the description of physical states on the language of Hilbert spaces and selfadjoint operators in them on one hand, and on the language of linear spaces over F_{p^2} and Hermitian operators in them on the other.

The field of complex numbers is algebraically closed (see standard textbooks on modern algebra, e.g. Ref. [42]). This implies that any equation of the *nth* order in this field always has n solutions. This is not, generally speaking, the case for the field F_{p^2}. As a consequence, not every linear operator in the finite-dimensional space over F_{p^2} has an eigenvector (because the characteristic equation may have no solution in this field). One can define a field of characteristic p which is algebraically closed and contains F_{p^2}. However such a field will necessarily be infinite and we will not use it. We will see in this chapter that uncloseness of the field F_{p^2} does not prevent one from constructing physically meaningful representations describing elementary particles in GFQT.

In physics one usually considers Lie algebras over R and their representations by Hermitian operators in Hilbert spaces. It is clear that analogs of such representations in our case are representations of Lie algebras over F_p by Hermitian operators in spaces over F_{p^2}. Representations in spaces over a field of nonzero characteristics are called modular representations. There exists a wide literature devoted to such representations; detailed references can be found for example in Ref. [45] (see also Ref. [5]). In particular, it has been shown by Zassenhaus [46] that all modular IRs are finite-dimensional and many papers have dealt with the maximum dimension of such representations. At the same time, it is worth noting that usually mathematicians consider only representations over an algebraically closed field.

From the previous, it is natural to expect that the correspondence between ordinary and modular representations of two Lie algebras over R and F_p, respectively, can be obtained if the structure constants of the Lie algebra over F_p - c^j_{kl}, and the structure constants of the Lie algebra over R - $\tilde{c}^j_{kl}$, are such that $f(c^j_{kl}) = \tilde{c}^j_{kl}$ (the Chevalley basis [47]), and all the c^j_{kl} belong to U_0. In Refs. [5, 2, 48] modular analogs of IRs of su(2), sp(2), so(2,3), so(1,4) algebras and the osp(1,4) superalgebra have been considered. Also modular representations describing strings have been briefly mentioned. In all these cases the quantities $\tilde{c}^j_{kl}$ take only the values $0, \pm 1, \pm 2$ and the above correspondence does take place.

It is obvious that since all physical quantities in GFQT are discrete, this theory cannot involve any dimensionful quantities and any operators having the continuous spectrum. We have seen in the preceding chapter than the so(1,4) invariant theory is dimensionless and it is possible to choose a basis such that all the operators have only discrete spectrum. For this reason one might expect that this theory is a natural candidate for its generalization to GFQT. In what follows, we consider a generalization of dS invariant theory to GFQT. This means that symmetry is defined by the commutation relations (3.1) which are now considered not in standard Hilbert spaces but in spaces over F_{p^2}. We will see in this chapter that there exists a correspondence in the above sense between modular IRs of the finite field analog of the so(1,4) algebra and IRs of the standard so(1,4) algebra. At the same time, there is no natural generalization of the Poincare invariant theory to GFQT.

Since the main problems of QFT originate from the fact that local fields interact at the same point, the idea of all modern theories aiming to improve QFT is to replace the interaction at a point by an interaction in some small space-time region.

From this point of view, one could say that those theories involve a fundamental length, explicitly or implicitly. Since GFQT is a fully discrete theory, one might wonder whether it could be treated as a version of quantum theory with a fundamental length. Although in GFQT all physical quantities are dimensionless and take values in a Galois field, on a qualitative level GFQT can be thought to be a theory with the fundamental length in the following sense. The maximum value of the angular momentum in GFQT cannot exceed the characteristic of the Galois field p. Therefore the Poincare momentum cannot exceed p/R. This can be interpreted in such a way that the fundamental length in GFQT is of order R/p.

One might wonder how continuous transformations (e.g. time evolution or rotations) can be described in the framework of GFQT. A general remark is that if theory $\mathcal{B}$ is a generalization of theory $\mathcal{A}$ then the relation between them is not always straightforward. For example, quantum mechanics is a generalization of classical mechanics, but in quantum mechanics the experiment outcome cannot be predicted unambiguously, a particle cannot be always localized etc. As noted in Sect. 1.3, even in the framework of standard quantum theory, time evolution is well-defined only on macroscopic level. Suppose that this is the case and the Hamiltonian H_1 in standard theory is a good approximation for the Hamiltonian H in GFQT. Then one might think that $exp(-iH_1t)$ is a good approximation for $exp(-iHt)$. However, such a straightforward conclusion is problematic for the following reasons. First, there can be no continuous parameters in GFQT. Second, even if t is somehow discretized, it is not clear how the transformation $exp(-iHt)$ should be implemented in practice. On macroscopic level the quantity Ht is very large and therefore the Taylor series for $exp(-iHt)$ contains a large number of terms which should be known with a high accuracy. On the other hand, one can notice that for computing $exp(-iHt)$ it is sufficient to know Ht only modulo 2π but in this case the question about the accuracy for π arises. We see that a direct correspondence between the standard quantum theory and GFQT exists only on the level of Lie algebras but not on the level of Lie groups.

4.3 Modular IRs of dS algebra and spectrum of dS Hamiltonian

Consider modular analogs of IRs constructed in Sect. 3.1. We noted that the basis elements of this IR are e_{nkl} where at a fixed value of n, $k = 0, 1, ...n$ and $l = 0, 1, ...2k$. In standard case, IR is infinite-dimensional since n can be zero or any natural number. A modular analog of this IR can be only finite-dimensional. The basis of the modular IR is again e_{nkl} where at a fixed value of n the numbers k and l are in the same range as above. The operators of such IR can be described by the same expressions as in Eqs. (3.9-3.14) but now those expressions should be understood as relations in a space over F_{p^2}. However, the quantity n can now be only in the range $0, 1, ...N$

where N can be found from the condition that the algebra of operators described by Eqs. (3.9) and (3.10) should be closed. It follows from these expressions, that this is the case if $w + (2N + 3)^2 = 0$ in F_p and $N + k + 2 < p$. Therefore we have to show that such N does exist.

In the modular case w cannot be written as $w = \mu^2$ with $\mu \in F_p$ since the equality $a^2 + b^2 = 0$ in F_p is not possible if $p = 3\ (mod\,4)$. In terminology of number theory, this means that w is a quadratic nonresidue. Since -1 also is a quadratic nonresidue if $p = 3\ (mod\,4)$, w can be written as $w = -\tilde{\mu}^2$ where $\tilde{\mu} \in F_p$ and for $\tilde{\mu}$ obviously two solutions are possible. Then N should satisfy one of the conditions $N + 3 = \pm\tilde{\mu}$ and one should choose that with the lesser value of N. Let us assume that both, $\tilde{\mu}$ and $-\tilde{\mu}$ are represented by $0, 1, ...(p-1)$. Then if $\tilde{\mu}$ is odd, $-\tilde{\mu} = p - \tilde{\mu}$ is even and *vice versa*. We choose the odd number as $\tilde{\mu}$. Then the two solutions are $N_1 = (\tilde{\mu} - 3)/2$ and $N_2 = p - (\tilde{\mu} + 3)/2$. Since $N_1 < N_2$, we choose $N = (\tilde{\mu} - 3)/2$. In particular, this quantity satisfies the condition $N \leq (p-5)/2$. Since $k \leq N$, the condition $N + k + 2 < p$ is satisfied and the existence of N is proved. In any realistic scenario, w is such that $w \ll p$ even for macroscopic bodies. Therefore the quantity N should be at least of order $p^{1/2}$. The dimension of IR is

$$Dim = \sum_{n=0}^{N}\sum_{k=0}^{n}(2k+1) = (N+1)(\frac{1}{3}N^2 + \frac{7}{6}N + 1) \qquad (4.9)$$

and therefore Dim is at least of order $p^{3/2}$.

The relative probabilities are defined by $||c(n,k,l)e_{nkl}||^2$. In standard theory the basis states and wave functions can be normalized to one such that the normalization condition is $\sum_{nkl} |\tilde{c}(n,k,l)|^2 = 1$. Since the values $\tilde{c}(n,k,l)$ can be arbitrarily small, wave functions can have an arbitrary carrier belonging to $[0,\infty)$. However, in GFQT the quantities $|c(n,k,l)|^2$ and $||e_{nkl}||^2$ belong F_p. Roughly speaking, this means that if they are not zero then they are greater or equal than one. Since for probabilistic interpretation we should have that $\sum_{nkl} ||c(n,k,l)e_{nkl}||^2 \ll p$, the probabilistic interpretation may take place only if $c(n,k,l) = 0$ for $n > n_{max},\ n_{max} \ll N$. That is why in Chap. 3 we discussed only wave functions having the carrier in the range $[n_{min}, n_{max}]$.

As follows from the spectral theorem for selfadjoint operators in Hilbert spaces, any selfadjoint operator A is fully decomposable, i.e. it is always possible to find a basis, such that all the basis elements are eigenvectors (or generalized eigenvectors) of A. As noted in Sect. 4.2, in GFQT this is not necessarily the case since the field F_{p^2} is not algebraically closed. However, it can be shown [42] that for any equation of the Nth order, it is possible to extend the field such that the equation will have $N+1$ solutions. A question arises what is the minimum extension of F_{p^2}, which guarantees that all the operators $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$ are fully decomposable.

The operators $(\mathbf{B}, \mathbf{J})$ describe a representation of the so(4) = su(2)×su(2) subalgebra. It is easy to show (see also the subsequent section) that the operators of the representations of the su(2) algebra are fully decomposable in the field F_{p^2}. Therefore it is sufficient to investigate the operators $(\mathcal{E}, \mathbf{N})$. They represent components of the so(4) vector operator $M^{0\nu}$ $(\nu = 1,2,3,4)$ and therefore it is sufficient to investigate the dS energy operator $\mathcal{E}$, which with our choice of the basis has a rather simpler form (see Eqs. (3.9) and (3.13)). This operator acts nontrivially only over the variable n and its nonzero matrix elements are given by

$$\mathcal{E}_{n-1,n} = \frac{n+1+k}{2(n+1)}[w + (2n+1)^2] \quad \mathcal{E}_{n+1,n} = \frac{n+1-k}{2(n+1)} \tag{4.10}$$

Therefore, for a fixed value of k it is possible to consider the action of $\mathcal{E}$ in the subspace with the basis elements e_{nkl} $(n = k, k+1, ...N)$.

Let $A(\lambda)$ be the matrix of the operator $\mathcal{E} - \lambda$ such that $A(\lambda)_{qr} = \mathcal{E}_{q+k,r+k} - \lambda\delta_{qr}$. We use $\Delta_q^r(\lambda)$ to denote the determinant of the matrix obtained from $A(\lambda)$ by taking into account only the rows and columns with the numbers $q, q+1, ...r$. With our definition of the matrix $A(\lambda)$, its first row and column have the number equal to 0 while the last ones have the number $K = N - k$. Therefore the characteristic equation can be written as

$$\Delta_0^K(\lambda) = 0 \tag{4.11}$$

In general, since the field F_{p^2} is not algebraically closed, there is no guaranty that we will succeed in finding even one eigenvalue. However, we will see below that in a special case of the operator with the matrix elements (4.10), it is possible to find all $K + 1$ eigenvalues.

The matrix $A(\lambda)$ is three-diagonal. It is easy to see that

$$\Delta_0^{q+1}(\lambda) = -\lambda\Delta_0^q(\lambda) - A_{q,q+1}A_{q+1,q}\Delta_0^{q-1}(\lambda) \tag{4.12}$$

Let λ_l be a solution of Eq. (4.11). We denote $e_q \equiv e_{q+k,kl}$. Then the element

$$\chi(\lambda_l) = \sum_{q=0}^{K}\{(-1)^q\Delta_0^{q-1}(\lambda_l)e_q/[\prod_{s=0}^{q-1} A_{s,s+1}]\} \tag{4.13}$$

is the eigenvector of the operator $\mathcal{E}$ with the eigenvalue λ_l. This can be verified directly by using Eqs. (3.13) and (4.10-4.13).

To solve Eq. (4.12) we have to find the expressions for $\Delta_0^q(\lambda)$ when $q = 0, 1, ...K$. It is obvious that $\Delta_0^0(\lambda) = -\lambda$, and as follows from Eqs. (4.10) and (4.12),

$$\Delta_0^1(\lambda) = \lambda^2 - \frac{w + (2k+3)^2}{2(k+2)} \tag{4.14}$$

If $w = -\tilde{\mu}^2$ then it can be shown that $\Delta_0^q(\lambda)$ is given by the following expressions. If

q is odd then

$$\Delta_0^q(\lambda) = \sum_{l=0}^{(q+1)/2} C_{(q+1)/2}^l \prod_{s=1}^{l}[\lambda^2 + (\tilde{\mu} - 2k - 4s + 1)^2](-1)^{(q+1)/2-l}$$
$$\prod_{s=l+1}^{(q+1)/2} \frac{(2k+2s+1)(\tilde{\mu} - 2k - 4s + 1)(\tilde{\mu} - 2k - 4s - 1)}{2(k + (q+1)/2 + s)} \tag{4.15}$$

and if q is even then

$$\Delta_0^q(\lambda) = (-\lambda) \sum_{l=0}^{q/2} C_{q/2}^l \prod_{s=1}^{l}[\lambda^2 + (\tilde{\mu} - 2k - 4s + 1)^2](-1)^{q/2-l}$$
$$\prod_{s=l+1}^{(q+1)/2} \frac{(2k+2s+1)(\tilde{\mu} - 2k - 4s - 1)(\tilde{\mu} - 2k - 4s - 3)}{2(k + q/2 + s + 1)} \tag{4.16}$$

Indeed, for $q = 0$ Eq. (4.16) is compatible with $\Delta_0^0(\lambda) = -\lambda$, and for $q = 1$ Eq. (4.15) is compatible with Eq. (4.14). Then one can directly verify that Eqs. (4.15) and (4.16) are compatible with Eq. (4.12).

With our definition of $\tilde{\mu}$, the only possibility for K is such that

$$\tilde{\mu} = 2K + 2k + 3 \tag{4.17}$$

Then, as follows from Eqs. (4.15) and (4.16), when K is odd or even, only the term with $l = [(K+1)/2]$ (where $[(K+1)/2]$ is the integer part of $(K+1)/2$) contributes to $\Delta_0^K(\lambda)$ and, as a consequence

$$\Delta_0^K(\lambda) = (-\lambda)^{r(K)} \prod_{k=1}^{[(K+1)/2]} [\lambda^2 + (\tilde{\mu} - 2j - 4k + 1)^2] \tag{4.18}$$

where $r(K) = 0$ if K is odd and $r(K) = 1$ if K is even. If $p = 3$ $(mod\ 4)$, this equation has solutions only if F_p is extended, and the minimum extension is F_{p^2}. Then the solutions are given by

$$\lambda = \pm i(\tilde{\mu} - 2k - 4s + 1) \quad (s = 1, 2...[(K+1)/2]) \tag{4.19}$$

and when K is even there also exists an additional solution $\lambda = 0$. When K is odd, solutions can be represented as

$$\lambda = \pm 2i,\ \pm 6i, ... \pm 2iK \tag{4.20}$$

while when K is even, the solutions can be represented as

$$\lambda = 0,\ \pm 4i,\ \pm 8i, ... \pm 2iK \tag{4.21}$$

Therefore the spectrum is equidistant and the distance between the neighboring elements is equal to $4i$. As follows from Eqs. (4.17), all the roots are simple and then, as follows from Eq. (4.13), the operator $\mathcal{E}$ is fully decomposable. It can be shown by a direct calculation [6] that the eigenvectors e corresponding to pure imaginary eigenvalues are such that $(e, e) = 0$ in F_p. Such a possibility has been mentioned in the preceding section.

Our conclusion is that if $p = 3\,(mod\,4)$ then all the operators $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$ are fully decomposable if F_p is extended to F_{p^2} but no further extension is necessary. This might be an argument explaining why standard theory is based on complex numbers. On the other hand, our conclusion is obtained by considering states where n is not necessarily small in comparison with $p^{1/2}$ and standard physical intuition does not work in this case. One might think that the solutions (4.20) and (4.21) for the eigenvalues of the dS Hamiltonian indicate that GFQT is unphysical since the Hamiltonian cannot have imaginary eigenvalues. However, such a conclusion is premature since in standard quantum theory the Hamiltonian of a free particle does not have normalized eigenstates (since the spectrum is pure continuous) and therefore for any realistic state the width of the energy distribution cannot be zero.

If A is an operator of a physical quantity in standard theory then the distribution of this quantity in some state can be calculated in two ways. First, one can find eigenvectors of A, decompose the state over those eigenvectors and then the coefficients of the decomposition describe the distribution. Another possibility is to calculate all moments of A, i.e. the mean value, the mean square deviation etc. Note that the moments do not depend on the choice of basis since they are fully defined by the action of the operator on the given state. A standard result of the probability theory (see e.g. Ref. [49]) is that the set of moments uniquely defines the moment distribution function, which in turn uniquely defines the distribution. However in practice there is no need to know all the moments since the number of experimental data is finite and knowing only several first moments is typically quite sufficient.

In GFQT the first method does not necessarily defines the distribution. In particular, the above results for the dS Hamiltonian show that its eigenvectors $\sum_{nkl} c(n,k,l)e_{nkl}$ are such that $c(n,k,l) \neq 0$ for all $n = k, ...N$, where N is at least of order $p^{1/2}$. Since the $c(n,k,l)$ are elements of F_{p^2}, their formal modulus cannot be less than 1 and therefore the formal norm of such eigenvectors cannot be much less than p (the equality $(e,e) = 0$ takes place since the scalar product is calculated in F_p). Therefore eigenvectors of the dS Hamiltonian do not have a probabilistic interpretation. On the other hand, as already noted, we can consider states $\sum_{nkl} c(n,k,l)e_{nkl}$ such that $c(n,k,l) \neq 0$ only if $n_{min} \leq n \leq n_{max}$ where $n_{max} \ll N$. Then the probabilistic interpretation for such states might be a good approximation if at least several first moments give reasonable physical results (see the discussion of probabilities in Sect. 4.1). In Chap. 3 we discussed semiclassical approximation taking into account only the first two moments: the mean value and mean square deviation.

Chapter 5

Semiclassical states in modular representations

5.1 Semiclassical states in representations of su(2) algebra

The uncertainty relations between the coordinate and momentum and between the angular coordinate and angular momentum are widely discussed in the literature. However, to the best of our knowledge, the uncertainty relation between different components of the angular momentum is not widely discussed. This problem is especially important in de Sitter invariant theories where all the representation operators are angular momenta. In this section we consider the simplest case of the uncertainty relations between the operators (J_x, J_y, J_z) in representations of the su(2) algebra. The commutation relations between these operators are given by Eq. (3.6). The discussion in this section is applied both, in the standard and modular cases.

As follows from Eq. (3.10), the operators (J_+, J_-, J_z) do not change the values of n and k. Therefore is $s = 2k$ is fixed, the basis of IR of the su(2) algebra can be written as e_l where $l = 0, 1, ...s$,

$$J_+e_l = (s+1-l)e_{l-1} \quad J_-e_l = (l+1)e_{l+1} \quad J_ze_l = (s-2l)e_l \quad (e_l, e_l) = C_s^l \tag{5.1}$$

anf $C_s^l = s!/(l!(s-l)!)$ is the binomial coefficient. In particular, e_l is the eigenvector of J_z with the eigenvalue $s-2l$. The Casimir operator of the second order for the su(2) algebra is $\mathbf{J}^2$ and in the representation (5.1) all the vectors from the representation space are eigenvectors of $\mathbf{J}^2$ with the eigenvalue $s(s+2)$.

Let $e_l^{(x)}$ be an analog of e_l in the basis when J_x is diagonalized, i.e. $J_xe_l^{(x)} = (s-2l)e_l^{(x)}$ and $e_l^{(y)}$ be an analog of e_l in the basis when J_y is diagonalized, i.e. $J_ye_l^{(y)} = (s-2l)e_l^{(y)}$. A possible expression for $e_l^{(x)}$ is

$$e_l^{(x)} = \frac{(-i)^l}{2^{s/2}}C_s^l\sum_{q=0}^{s}F(-l,-q;-s;2)e_q \tag{5.2}$$

where F is the standard hypergeometric function. This can be varified by using Eq. (5.1) and the relation [29]

$$(-s+q)F(-l,-q-1;-s;2)+(s-2l)F(-l,-q;-s;2)- \\ qF(-l,-q+1;-s;2)=0 \tag{5.3}$$

Analogously one can verify that a possible expression for $e_l^{(y)}$ is

$$e_l^{(y)}=\frac{C_s^l}{2^{s/2}}\sum_{q=0}^{s}F(-l,-q;-s;2)i^q e_q \tag{5.4}$$

By using the relation [29]

$$\sum_{q=0}^{s}C_s^q F(-l,-q;-s;2)F(-l',-q;-s;2)=2^s\delta_{ll'}/C_s^l \tag{5.5}$$

and Eqs. (5.2) and (5.4), it is easy to show that the normalization of the vectors $e_l^{(x)}$ and $e_l^{(y)}$ is the same as the vectors e_l, i.e.

$$(e_l^{(x)},e_{l'}^{(x)})=(e_l^{(y)},e_{l'}^{(y)})=C_s^l\delta_{ll'} \tag{5.6}$$

If $c^{(x)}(l)$ is the wave function in the basis $e_l^{(x)}$ and $c^{(y)}(l)$ is the wave function in the basis $e_l^{(y)}$ then it follows from Eqs. (5.2) and (5.4) that

$$c^{(x)}(l)=\frac{i^l}{2^{s/2}}\sum_{q=0}^{s}C_s^q F(-l,-q;-s;2)c(q)$$
$$c^{(y)}(l)=\frac{1}{2^{s/2}}\sum_{q=0}^{s}(-i)^q C_s^q F(-l,-q;-s;2)c(q) \tag{5.7}$$

Our goal is to construct states, which are semiclassical in all the three components of the angular momentum. According to a convention adopted in Sect. 3.1, for the approximate semiclassical eigenvalues of the operators (J_x,J_y,J_z) we will use the same notations (J_x,J_y,J_z), respectively. In the modular case we require additionally that those numbers are integers such that their magnitude is much less than p (more rigorously, we should require that those numbers belong to the set U_0 discussed in Sect. 4.1).

Since the values of (J_x,J_y,J_z) in semiclassical states are very large, we can work in the approximation $J_x^2+J_y^2+J_z^2\approx s^2$. By using the above results one can show that

$$\sum_q C_s^q z^q F(-q,-l;-s;2)=(1+z)^{s-l}(1-z)^l \tag{5.8}$$

Then, as a consequence of Eqs. (5.7) and (5.8), a possible choice of the wave function is

$$\begin{aligned}
c^{(x)}(l) &= [(s+J_x+J_z+iJ_y)(s+J_y)]^s(s+J_x)^{s-l}(J_y+iJ_z)^l \\
c^{(y)}(l) &= [(s+J_y+J_z-iJ_x)(s+J_x)]^s(s+J_y)^{s-l}(J_z+iJ_x)^l \\
c(l) &= 2^{s/2}[(s+J_x)(s+J_y)]^s(s+J_z)^{s-l}(J_x+iJ_y)^l
\end{aligned} \tag{5.9}$$

Note that in standard case the dependence of $c(l)$ on l is in agreement with Eqs. (3.20) and (3.23).

Consider the distribution of probabilities over l in $c(l)$. As follows from Eqs. (5.1) and (5.9), the normalization sum for $c(l)$ is

$$\rho = \sum_{l=0}^{s} \rho(l) = [2(s+J_x)(s+J_y)]^{2s}[s(s+J_z)]^s \tag{5.10}$$

where

$$\rho(l) = 2^s C_s^l [(s+J_x)(s+J_y)]^{2s}(s+J_z)^{2s-l}(s-J_z)^l \tag{5.11}$$

Since there in no nontrivial division in this expression, it follows from Eq. (5.10) that in the modular case the probabilistic interpretation is valid if $\rho \ll p$. Since the number s for macroscopic bodies is very large, this condition will be satisfied if $slns \ll lnp$. We see that not only p should be very large but even lnp should be very large.

As follows from Eqs. (5.10) and (5.11),

$$\sum_{l=0}^{s} \rho(l)(s-2l) = J_z\rho \tag{5.12}$$

and therefore with our notations the number J_z is the exact mean value of the operator J_z. The fact that in the modular case the probabilistic interpretation is valid, implies that even in this case we can use standard mathematics for qualitative understanding of the distribution (5.11). In particular, we can use the Stirling formula for the binomial coefficient in this expression and formally consider l as a continuous variable. Then it follows from Eq. (5.11) that the maximum of the function $\rho(l)$ is at $l = l_0$ such that $l_0 = (s - J_z)/2$, and in the vicinity of the maximum

$$\rho(l) \approx \rho[2\pi l_0(s-l_0)/s]^{1/2} exp[-\frac{s(l-l_0)^2}{2l_0(s-l_0)}] \tag{5.13}$$

Therefore in the vicinity of the maximum the distribution is Gaussian with the width $[l_0(s-l_0)/s]^{1/2}$. If l_0 and $s - l_0$ are of order s (i.e. l_0 is not close to zero or $s/2$), this quantity is of order $s^{1/2}$.

In standard quantum mechanics, the semiclassical wave function contains a factor $exp(i\mathbf{pr})$, which does not depend on the choice of the quantization axis.

The reason for choosing the wave functions in the form (5.9) is to have an analogous property in our case. As seen from these expressions, if the quantization axis changes then the dependence of the wave function on l with the new quantization axis can be obtained from the original dependence by using a cyclic permutation of indices (x, y, z). Therefore, if the quantization axis is x or y, the distribution over l is again given by Eq. (5.13) but l_0 is such that $l_0 = (s - J_x)/2$ or $l_0 = (s - J_y)/2$, respectively.

In the above example, the carrier of the wave function $c(l)$ contains all integers in the range $[0, s]$ but $|c(l)|^2$ has a sharp maximum with the width of order $s^{1/2}$. In GFQT it is often important that the carrier should have a width which is much less than the corresponding mean value. Since properties of the state defined by the wave function $c(l)$ depend mainly on the behavior of $c(l)$ in the region of maximum, one can construct states which have properties similar to those discussed above but the carrier of $c(l)$ will belong to the range $[l_{min}, l_{max}]$ where $l_{max} - l_{min}$ is of order $s^{1/2}$.

Our conclusion is as follows. It is possible to construct states, which are simultaneously semiclassical in all the three components of the angular momentum if all the quantities (J_x, J_y, J_z) are of order s. Then the uncertainty of each component is of order $s^{1/2}$. The requirement that neither of the components (J_x, J_y, J_z) should be small is analogous to the well known requirement in standard quantum mechanics that in semiclassical states neither of the momentum components should be small.

5.2 Semiclassical states in GFQT

In Sect. 3.2 we discussed semiclassical states in standard theory and noted that they can be defined by ten numbers $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$, which are subject to constraints (3.21). For semiclassical states all those numbers are very large and the numbers $(\mathcal{E}, \mathbf{B})$ are very large even for elementary particles. Semiclassical wave functions can be described by parameters $(nkl\varphi\alpha\beta)$, which can be expressed in terms of $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$ by using Eqs. (3.22) and (3.23).

In GFQT one should use the basis defined by Eq. (3.8) and the coefficients $c(n, k, l)$ should be elements of F_{p^2}. Therefore, a possible approach to constructing a semiclassical wave function in GFQT is to express those coefficients in terms of $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$. First of all, since the numbers $(\mathcal{E}, \mathbf{N}, \mathbf{B}, \mathbf{J})$ are very large, we can assume that they are integers. Then, in general, the relations (3.21) cannot be exact but can be valid with a high accuracy. As noted in Chap. 4, a probabilistic interpretation can be possible only if $c(n, k, l) \neq 0$ for $n \in [n_{min}, n_{max}]$, $k \in [k_{min}, k_{max}]$ and $l \in [l_{min}, l_{max}]$. Therefore our task is obtain integer values of $c(n, k, l)$ at such conditions.

As noted in Sect. 3.2, a semiclassical wave function should be such that the amplitude is a function, which is significant only in a relatively small region, which can be called the region of maximum. It cannot be extremely narrow since in the region of maximum the change of the wave function should be mainly governed by the exponents in Eq. (3.18). It follows from these considerations and Eq. (3.23) that the semiclassical wave function in the region of maximum should have a factor

$$(2\mathcal{E}n_0 - i\mathbf{BN})^{n_{max}-n}[-JB_z - i(\mathbf{B}\times\mathbf{J})_z]^{k_{max}-k}(J_x + iJ_y)^{l-k}$$

where n_0 is some value of n inside the interval $[n_{min}, n_{max}]$ and J is an integer close to $(J_x^2 + J_y^2 + J_z^2)^{1/2}$. At the same time, the norm of $c(n,k,l)e_{nkl}$ should be a slowly changing function of (nkl) in the region of maximum. Our nearest aim is to show that a possible semiclassical wave function can be written as

$$\begin{aligned}
&c(n,k,l) = 2^{n-n_{min}+l_{max}-l}(2\mathcal{E}n_0 - i\mathbf{BN})^{n_{max}-n}[-JB_z - i(\mathbf{B}\times\mathbf{J})_z]^{k_{max}-k} \\
&\frac{(n-k)!}{(n_{min}-k)!}\frac{(2k_{max})!k!}{(2k)!k_{max}!}(J_x - iJ_y)^{k-k_{min}}(J_x + iJ_y)^{l-l_{min}} \\
&(J+J_z)^{k-k_{min}}\frac{(2k-l)!}{(2k-l_{max})!}a(n,k,l) \qquad (5.14)
\end{aligned}$$

where the amplitude $a(n,k,l)$ is a slowly changing function in the region of its maximum. Since $(2k)! = 2^k k!(2k-1)!!$, this expression does not contain nontrivial divisions in F_p and therefore the correspondence principle with standard theory is satisfied if $|c(n,k,l)|^2 \ll p$.

By using Eqs. (3.12) and (3.20) one can explicitly verify that in the region of maximum $||c(n,k,l)e_{nkl}||^2 = \tilde{\rho}(n,k,l)|a(n,k,l)|^2$ where

$$\begin{aligned}
&\tilde{\rho}(n,k,l) = 4^{n-n_{min}+l_{max}-l}B^{2(n_{max}-n+k_{max}-k)}(J_x^2 + J_y^2)^{k_{max}-k_{min}+l-l_{min}} \\
&(J+J_z)^{2(k-k_{min})}(2k+1)[\frac{(2k_{max})!}{(k_{max}!)^2}][\frac{(2k_{max})!}{(2k_{max}-l_{max})!l_{max}!}][\frac{l_{max}!}{l!}][\frac{(2k-l)!}{(2k-l_{max})!}] \\
&[\frac{(2k_{max}-l_{max})!}{(2k-l_{max})!}][\frac{(n-k)!}{(n_{min}-k)!}][\frac{n!}{(n_{min}-k)!}][\frac{(n+k+1)!}{(n+1)!}] \\
&(w+4n_0^2)^{n_{max}-n}[\prod_{j=1}^{n}(w+(2j+1)^2] \qquad (5.15)
\end{aligned}$$

This expression is written in the form showing that multipliers in each square brackets do not contain nontrivial divisions in F_p. Then by using Eq. (3.20), it is easy to show that in the region of maximum

$$\tilde{\rho}(n,k,l) \approx \tilde{\rho}(n+1,k,l) \approx \tilde{\rho}(n,k+1,l) \approx \tilde{\rho}(n,k,l+1)$$

Therefore the norm of $c(n,k,l)e_{nkl}$ is indeed a slowly changing function of (nkl) in the region of maximum.

Since Eq. (5.15) does not contain a nontrivial division, there is a chance that a probabilistic interpretation in GFQT will be valid. As noted in Sect. 4.2, only ratios of probabilities have a physical meaning. Therefore the problem arises whether it is possible to find a constant C such that $\tilde{\rho}(n,k,l) = C\rho(n,k,l)$, for all $n \in [n_{min}, n_{max}]$, $k \in [k_{min}, k_{max}]$ and $l \in [l_{min}, l_{max}]$, the conditions $\rho(n,k,l) \ll p$,

$a(n,k,l)^2 \ll p$ are satisfied and the sum $\sum_{nkl}\rho(n,k,l)|a(n,k,l)|^2$ also is much less than p. It is clear that for this purpose it is desirable to obtain for $\rho(n,k,l)$ the least possible value.

It is immediately seen from Eq. (5.15), that a factor

$$C_1 = (J_x^2+J_y^2)^{k_{max}-k_{min}} \prod_{j=1}^{n_{min}} (w+(2j+1)^2)[\frac{(2k_{max})!}{(k_{max}!)^2}][\frac{(2k_{max})!}{(2k_{max}-l_{max})!l_{max}!}]$$

can be included into C. The next observation is as follows. If $|\mathbf{p}|$ is the magnitude of standard momentum then, as noted in Sect. 3.1 (see Eq. (3.11)), n is of order $|\mathbf{p}|R$ and k is of order $|\mathbf{p}|r$. Therefore one might expect that in situations we are interested in, the conditions $k \ll n$ and $\Delta k \ll \Delta n$ are satisfied, where $\Delta k = k_{max} - k_{min}$ and $\Delta n = n_{max} - n_{min}$. However, although R is very large, the relation $\Delta n \gg k$ is valid only if R is extremely large.

We first consider the case $\Delta n \ll k$. Since

$$\frac{(n+1+k)!}{(n+1)!} = [(n+1+k)\cdots(n+2+k_{min})][(n+1+k_{min})\cdots(n_{min}+2+ k_{min})][(n_{min}+1+k_{min})\cdots(n_{max}+2)][(n_{max}+1)\cdots(n+2)] \quad (5.16)$$

the factor $C_2 = (n_{min}+1+k_{min})\cdots(n_{max}+2)$ can be included into C. Analogously, since

$$\frac{n_{min}!}{(n_{min}-k)!} = [n_{min}\cdots(n_{min}+1-k_{min})][(n_{min}-k_{min})\cdots(n_{min}+1-k)] \quad (5.17)$$

the factor $C_3 = [n_{min}\cdots(n_{min}+1-k_{min})]$ can be included into C. Then a direct calculation gives

$$\rho(n,k,l) = 4^{n-n_{min}} B^{2(n_{max}-n+k_{max}-k)} (J_x^2+J_y^2)^{l-l_{min}} (J+J_z)^{2(k-k_{min})} (2k+1)(l+1)_{(l_{max}-l)}(2k+1-l_{max})_{(l_{max}-l)}(2k+1-l_{max})_{(2k_{max}-2k)} (n_{min}+1-k)_{(n-n_{min})}(n_{min}+1)_{(n-n_{min})}(n_{min}+1-k)_{(k-k_{min})} (n+2+k_{min})_{(k-k_{min})}(n_{min}+2+k_{min})_{(n-n_{min})}(n+2)_{(n_{max}-n)} (w+4n_0^2)^{n_{max}-n}[\prod_{j=n_{min}}^{n}(w+(2j+1)^2] \quad (5.18)$$

where $(a)_n$ is the Pochhammer symbol.

It follows from this expression that in the region of maximum

$$\rho(n,k,l) \approx 4^{\Delta n}(J_x^2+J_y^2)^{\Delta l}(J+J_z)^{2\Delta k}(n_{min}+1-k)_{\Delta n}(n_{min}+1)_{\Delta n} (n_{min}+1-k)_{\Delta k}(n+2-k_{min})_{\Delta k}(n_{min}+2+k_{min})_{\Delta n}(w+4n_0^2)^{\Delta n} \quad (5.19)$$

where $\Delta n = n_{max} - n_{min}$, $\Delta k = k_{max} - k_{min}$ and $\Delta l = l_{max} - l_{min}$. Now we take into account that $\Delta n \gg \Delta k$, $\Delta n \gg \Delta l$ and in the nonrelativistic approximation $w \gg n^2$. Then the condition $\rho(n,k,l) \ll p$ can be approximately written in the form

$$\Delta n ln w \ll ln p \tag{5.20}$$

If $\Delta n \gg k$ or Δn and k are of the same order, this estimation is valid too.

Therefore not only the number p should be very large, but even lnp should be very large. As a consequence, if $ln(|a(n,k,l)|) \ll p$ the condition

$$\sum_{nkl} \rho(n,k,l)|a(n,k,l)|^2 \ll p$$

is satisfied since $ln(\Delta n \Delta k \Delta l) \ll lnp$.

5.3 Many-body systems in GFQT and gravitational constant

In quantum theory, state vectors of a system of N bodies belong to the Hilbert space which is the tensor product of single-body Hilbert spaces. This means that state vectors of the N-body systems are all possible linear combinations of functions

$$\psi(n_1, k_1, l_1, ...n_N, k_N, l_N) = \psi_1(n_1, k_1, l_1) \cdots \psi_N(n_N, k_N, l_N) \tag{5.21}$$

By definition, the bodies do not interact if all representation operators of the symmetry algebra for the N-body systems are sums of the corresponding single-body operators. For example, the energy operator $\mathcal{E}$ for the N-body system is a sum $\mathcal{E}_1 + \mathcal{E}_2 + ... + \mathcal{E}_N$ where the operator $\mathcal{E}_i$ $(i = 1, 2, ...N)$ acts nontrivially over its "own" variables (n_i, k_i, l_i) while over other variables it acts as the identity operator.

If we have a system of noninteracting bodies in standard quantum theory, each $\psi_i(n_i, k_i, l_i)$ in Eq. (5.21) is fully independent of states of other bodies. However, in GFQT the situation is different. Here, as shown in the preceding section, a necessary condition for a wave function to have a probabilistic interpretation is given by Eq. (5.20). Since we assume that p is very large, this is not a serious restriction. However, if a system consists of N components, a necessary condition that the wave function of the system has a probabilistic interpretation is

$$\sum_{i=1}^{N} \delta_i ln w_i \ll ln p \tag{5.22}$$

where $\delta_i = \Delta n_i$ and $w_i = 4R^2 m_i^2$ where m_i is the mass of the subsystem i. This condition shows that in GFQT the greater the number of components is, the stronger is the restriction on the width of the dS momentum distribution for each component.

This is a crucial difference between standard theory and GFQT. A naive explanation is that if p is finite, the same set of numbers which was used for describing one body is now shared between N bodies. In other words, if in standard theory each body in the free N-body system does not feel the presence of other bodies, in GFQT this is not the case. This might be treated as an effective interaction in the free N-body system.

In Chaps. 2 and 3 we discussed a system of two free bodies such their relative motion can be described in the framework of semiclassical approximation. We have shown that the mean value of the mass operator for this system differs from the expression given by standard Poincare theory. The difference describes an effective interaction which we treat as the dS antigravity at very large distances and gravity when the distances are much less than cosmological ones. In the latter case the result depends on the total dS momentum distribution for each body (see Eq. (3.55)). Since the interaction is proportional to the masses of the bodies, this effect is important only in situations when at least one body is macroscopic. Indeed, if neither of the bodies is macroscopic, their masses are small and their relative motion is not described in the framework of semiclassical approximation. In particular, in this approach, gravity between two elementary particles has no physical meaning.

The existing quantum theory does not make it possible to reliably calculate the width of the total dS momentum distribution for a macroscopic body and at best only a qualitative estimation of this quantity can be given. The above discussion shows that the greater is the mass of the macroscopic body, the stronger is the restriction on the dS momentum distribution for each subsystem of this body. Suppose that a body with the mass M can be treated as a composite system consisting of similar subsystems with the mass m. Then the number of subsystems is $N = M/m$ and, as follows from Eq. (5.22), the width δ of their dS momentum distributions should satisfy the condition $N\delta lnw \ll lnp$ where $w = 4R^2m^2$. Since the greater the value of δ is, the more accurate is the semiclassical approximation, a reasonable scenario is that each subsystem tends to have the maximum possible δ but the above restriction allows to have only such value of δ that it is of the order of magnitude not exceeding $lnp/(Nlnw)$.

The next question is how to estimate the width of the total dS momentum distribution for a macroscopic body. For solving this problem one has to change variables from individual dS momenta of subsystems to total and relative dS momenta. Now the total dS momentum and relative dS momenta will have their own momentum distributions which are subject to a restriction similar to that given by Eq. (5.22). If we assume that all the variables share this restriction equally then the width of the total momentum distribution also will be a quantity not exceeding $lnp/(Nlnw)$. Suppose that $m = N_1m_0$ where m_0 is the nucleon mass. The value of N_1 should be such that our subsystem still can be described by semiclassical approximation. Then the estimation of δ is

$$\delta = N_1m_0lnp/[2Mln(2RN_1m_0)] \qquad (5.23)$$

Suppose that N_1 can be taken to be the same for all macroscopic bodies. For example, it is reasonable to expect that when N_1 is of order of 10^3, the subsystems still can be described by semiclassical approximation but probably this is the case even for smaller values of N_1.

In summary, although calculation of the width of the total dS momentum distribution for a macroscopic body is a very difficult problem, GFQT gives a reasonable qualitative explanation why this quantity is inversely proportional to the mass of the body. With the estimation (5.23), the result given by Eq. (3.55) can be written in the form (3.57) where

$$G = \frac{2const\, Rln(2RN_1m_0)}{N_1m_0lnp} \tag{5.24}$$

In Chaps. 1 and 4 we argued that in theories based on dS invariance and/or Galois fields, neither the gravitational nor cosmological constant can be fundamental. In particular, in units $\hbar/2 = c = 1$, the dimension of G is $length^2$ and its numerical value is l_P^2 where l_P is the Planck length ($l_P \approx 10^{-35}m$). Eq. (5.24) is an additional indication that this is the case since G depends on R (or the cosmological constant) and there is no reason to think that it does not change with time. One might think that since $G\Lambda$ is dimensionless in units $\hbar/2 = c = 1$, it is possible that only this combination is fundamental. Let $\mu = 2Rm_0$ be the dS nucleon mass and $\Lambda = 3/R^2$ be the cosmological constant. Then Eq. (5.24) can be written as

$$G = \frac{12const\, ln(N_1\mu)}{\Lambda N_1\mu lnp} \tag{5.25}$$

As noted in Sect. 1.2, standard cosmological constant problem arises when one tries to explain the value of Λ from quantum theory of gravity assuming that this theory is QFT, G is fundamental and the dS symmetry is a manifestation of dark energy (or other fields) on flat Minkowski background. Such a theory contains strong divergences and the result depends on the value of the cutoff momentum. With a reasonable assumption about this value, the quantity Λ is of order $1/G$ and this is reasonable since G is the only parameter in this theory. Then Λ is by more than 120 orders of magnitude greater than its experimental value. However, in our approach we have an additional parameter p which is treated as a fundamental constant. Eq. (5.25) shows that $G\Lambda$ is not of order unity but is very small since not only p but even lnp is very large. For a rough estimation, we assume that the values of $const$ and N_1 in this expression are of order unity. Then assuming that R is of order $10^{26}m$, we have that μ is of order 10^{42} and lnp is of order 10^{80}. Therefore p is a huge number of order $exp(10^{80})$. In the preceding chapter we argued that standard theory can be treated as a special case of GFQT in the formal limit $p \to \infty$. The above discussion shows that restrictions on the width of the total dS momentum arise because p is not infinitely large. It is seen from Eq. (5.25) that gravity disappears in the above formal limit. Therefore in our approach gravity is a consequence of the fact that dS symmetry is considered over a Galois field rather than the field of complex numbers.

Chapter 6

Discussion and conclusion

As noted in Sect. 1.1, the main idea of this work is that gravity might be not an interaction but simply a manifestation of de Sitter invariance over a Galois field. This is obviously not in the spirit of mainstream approaches that gravity is a manifestation of the graviton exchange or holographic principle. Our approach does not involve General Relativity, quantum field theory (QFT), string theory, loop quantum gravity or other sophisticated theories. We consider only systems of two *free* bodies in de Sitter invariant quantum mechanics.

We argue that quantum theory should be based on the choice of symmetry algebra and should not involve spacetime at all. Then the fact that we observe the cosmological repulsion is a strong argument that the de Sitter (dS) symmetry is a more pertinent symmetry than Poincare or anti de Sitter (AdS) ones. As shown in Refs. [4, 1] and in the present paper, the phenomenon of the cosmological repulsion can be easily understood by considering semiclassical approximation in standard dS invariant quantum mechanics of two free bodies. In the framework of this consideration it becomes immediately clear that the cosmological constant problem does not exist and there is no need to involve dark energy or other fields. This phenomenon can be easily explained by using only standard quantum-mechanical notions without involving dS space, metric, connections or other notions of Riemannian geometry. One might wonder why such a simple explanation has not been widely discussed in the literature. According to our observations, this is a manifestation of the fact that even physicists working on dS QFT are not familiar with basic facts about irreducible representations (IRs) of the dS algebra. It is difficult to imagine how standard Poincare invariant quantum theory can be constructed without involving well known results on IRs of the Poincare algebra. Therefore it is reasonable to think that when Poincare invariance is replaced by dS one, IRs of the Poincare algebra should be replaced by IRs of the dS algebra. However, physicists working on QFT in curved spacetime believe that fields are more fundamental than particles and therefore there is no need to involve IRs.

The assumption that quantum theory should be based on dS symmetry implies several far reaching consequences. First of all, in contrast to Poincare and AdS symmetries, the dS one does not have a supersymmetric generalization. Moreover, as argued in our papers [4, 1], in dS invariant theories only fermions can be fundamental.

One might say that a possibility that only fermions can be elementary is not attractive since such a possibility would imply that supersymmetry is not fundamental. There is no doubt that supersymmetry is a beautiful idea. On the other hand, one might say that there is no reason for nature to have both, elementary fermions and elementary bosons since the latter can be constructed from the former. A well know historical analogy is that the simplest covariant equation is not the Klein-Gordon equation for spinless fields but the Dirac and Weyl equations for the spin 1/2 fields since the former is the equation of the second order while the latter are the equations of the first order.

The key difference between IRs of the dS algebra on one hand and IRs of the Poincare and AdS algebras on the other is that in the former case one IR describes a particle and its antiparticle simultaneously while in the latter case a particle and its antiparticle are described by different IRs. As a consequence, in dS invariant theory there are no neutral elementary particles and transitions particle$\leftrightarrow$antiparticle are not prohibited. As a result, the electric charge and the baryon and lepton quantum numbers can be only approximately conserved. These questions are discussed in details in Ref. [1].

In the present paper, another feature of IRs of the dS algebra is important. In contrast to IRs of the Poincare and AdS algebras, in IRs of the dS algebra the particle mass *is not* the lowest value of the dS Hamiltonian which has the spectrum in the range $(-\infty, \infty)$. As a consequence, the free mass operator of the two-particle system is not bounded below by $(m_1 + m_2)$ where m_1 and m_2 are the particle masses. The discussion in Sect. 2.3 shows that this property by no means implies that the theory is unphysical.

In 2000, Clay Mathematics Institute announced seven Millennium Prize Problems. One of them is called "Yang-Mills and Mass Gap" and the official description of this problem can be found in Ref. [50]. In this description it is stated that the Yang-Mills theory should have three major properties where the first one is as follows: "It must have a "mass gap;" namely there must be some constant $\Delta > 0$ such that every excitation of the vacuum has energy at least Δ." The problem statement assumes that quantum Yang-Mills theory should be constructed in the framework of Poincare invariance. However, as follows from the above discussion, this invariance can be only approximate and dS invariance is more general. Meanwhile, in dS theory the mass gap does not exist. Therefore we believe that the problem has no solution.

Since in Poincare and AdS invariant theories the spectrum of the free mass operator is bounded below by $(m_1 + m_2)$, in these theories it is impossible to obtain the correction $-Gm_1m_2/r$ to the mean value of this operator. However, in dS theory there is no law prohibiting such a correction. It is not a problem to indicate internal two-body wave functions for which the mean value of the mass operator contains

$-Gm_1m_2/r$ with possible post-Newtonian corrections. The problem is to show that such wave functions are semiclassical with a high accuracy. As shown in Chaps. 2 and 3, in semiclassical approximation any correction to the standard mean value of the mass operator is negative and proportional to the energies of the particles. In particular, in the nonrelativistic approximation it is proportional to m_1m_2.

Our consideration poses a very important question of how the distance operator should be defined. In standard quantum mechanics the coordinate and momentum are canonically conjugated and the relation between the coordinate and momentum representations are given by the Fourier transform. This definition of the coordinate operator works in atomic and nuclear physics but the problem arises whether it is physical at macroscopic distances. In Chap. 3 we argue that it is not and that the coordinate operator should be defined differently. We propose a modification of the coordinate operator which has correct properties, reproduces Newton's gravity, and the precession of Mercury's perihelion if the width of the de Sitter momentum distribution for a macroscopic body is inversely proportional to its mass. In Sects. 3.7 and 3.8 we also discuss a problem of evolution and gravitational experiments with light.

In Chaps. 4 and 5 we argue that quantum theory should be based on Galois fields rather than complex numbers. We tried to make the presentation as simple as possible without assuming that the reader is familiar with Galois fields. Our version of a quantum theory over a Galois field (GFQT) gives a natural qualitative explanation why the width of the total dS momentum distribution of the macroscopic body is inversely proportional to its mass. In this approach neither G nor Λ can be fundamental physical constants. We argue that only $G\Lambda$ might have physical meaning. The calculation of this quantity is a very difficult problem since it requires a detailed knowledge of wave functions of many-body systems. However, GFQT gives clear indications that $G\Lambda$ contains a factor $1/lnp$ where p is the characteristic of the Galois field. We treat standard theory as a special case of GFQT in the formal limit $p \to \infty$. Therefore gravity disappears in this limit. Hence in our approach gravity is a consequence of the fact that dS symmetry is considered over a Galois field rather than the field of complex numbers. In Chap. 5 we give a very rough estimation of G which shows that lnp is of order 10^{80}. Therefore p is a huge number of order $exp(10^{80})$.

In our approach gravity is a phenomenon which has a physical meaning only in situations when at least one body is macroscopic and can be described in the framework of semiclassical approximation. The result (3.53) shows that gravity depends on the width of the total dS momentum distributions for the bodies under consideration. However, when one mass is much greater than the other, the momentum distribution for the body with the lesser mass is not important. In particular, this is the case when one body is macroscopic and the other is the photon. At the same time, the phenomenon of gravity in systems consisting only of elementary particles has no physical meaning since gravity is not an interaction but simply a kinematical manifestation of dS invariance over a Galois field in semiclassical approximation. In this connection a problem arises what is the minimum mass when a body can be treated as macroscopic. This problem requires understanding of the structure of the many-body wave function.

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