The Many Novel Physical Consequences of Born's Reciprocal Relativity in Phase-Spaces

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

April, 2011

Abstract

We explore the many *novel* physical consequences of Born's reciprocal Relativity theory in flat phase-space and to generalize the theory to the curved phase-space scenario. We provide with *six* specific novel physical results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalize by constructing a Born reciprocal general relativity theory in curved phase-spaces which requires the introduction of a complex Hermitian metric, torsion and nonmetricity.

1 Introduction

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *curved* phase space case scenario has been analyzed by Brandt [9] within the context of the Finsler geometry of the $8D$ cotangent bundle of spacetime where there is a limiting value to the proper acceleration and such that generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. A pedagogical monograph on Finsler geometry can be found in [21] where, in particular, Clifford/spinor structures were defined with respect to nonlinear connections associated with certain nonholonomic modifications of Riemann–Cartan gravity.

Born's reciprocal "duality" principle is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i.e. namely, a small/large radius duality, a winding modes/ Kaluza-Klein modes duality symmetry in string compactifications and the ultraviolet/infrared entanglement in noncommutative field theories. The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where $X^i, T, E, P^i; i = 1, 2, 3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

Invariant actions for a point-particle in reciprocal Relativity involving Casimir group invariant quantities were studied in [27]. Casimir invariant field equations; unitary irreducible representations based on Mackey's theory of induced representations; the relativistic harmonic oscillator and coherent states can be found in [2]. The granular cellular structure of spacetime, the Schrodinger-Robertson inequality, multi-mode squeezed states, a "non-commutative" relativistic phase space geometry, in which position and momentum are interchangeable and frame-dependent, was studied by [3]. Born's reciprocity principle in atomic physics and galactic motion based on $(1/r)+(b/p)$ potentials was studied recently by [4] with little effect on atomic physics but with relevant effects on galactic rotation without invoking dark matter.

This approach differs from the pseudo-complex Lorentz group description by [22] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of deformed (double) Special Relativity [10] that were motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [16, 17, 18]. Related to the minimal Planck scale, an upper limit on the maximal acceleration principle in Nature was proposed long ago by Cainello [5]. Maximal-acceleration physics has ben studied by [18], [6], [7], [23] among others; its relation to the Double Special Relativity programs was investigated by [8].

The purpose of this work is to explore the many novel physical consequences of Born's reciprocal Relativity theory in flat phase-space and to generalize the theory to the curved phase-space scenario. We provide with **6** specific novel physical results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are :

(i) momentum-dependent time delay in the emission and detection of photons; (ii) energy-dependent notion of locality; (iii) superluminal behavior; (iv) relative rotation of photon trajectories due to the aberration of light; (v) invariance of areas-cells in phase-space and Planck areas; (vi) modified dispersion relations.

A completely different approach to the notion of "relativity of locality" and energy-dependent time-delay of photons based on a curved momentum space has been undertaken by [15], [25]. The role of curved momentum space and torsion in deformations of Special Relativity were analyzed extensively by [13]. These authors found that the natural framework for embedding the ideas of deformed, or doubly, special relativity (DSR) into a curved spacetime is a generalisation of Einstein-Cartan theory, considered by Stelle and West. Instead of interpreting the noncommuting "spacetime coordinates" of the Snyder algebra as endowing spacetime with a fundamentally noncommutative structure, they were led to consider a connection with torsion in this framework. The relationship among conformal theories, curved phase spaces, relativistic wavelets and the geometry of complex domains was examined thoroughly by [12]. An adaptation of Born's Reciprocity Principle to conformal relativity, the replacement of space-time by the 8-dimensional conformal domain at short distances, the existence of a maximal acceleration was put forward by [19]. Also relevant, was the revision of the Unruh effect (vacuum radiation in uniformly relativistic accelerated frames) in a group-theoretical setting by constructing a conformal $SO(4,2)$-invariant quantum field theory and studying its spontaneous breakdown.

Born reciprocal general relativity theory in curved phase-spaces (without the need to introduce star products) can be defined as a local gauge theory of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* generators $[Z_a, Z_b] \neq 0$; i.e. noncommutative coordinates and momenta. The Hermitian metric is complex-valued with symmetric and nonsymmetric components : $g_{(\mu\nu)} + i g_{[\mu\nu]}$. If one sets the nonmetricity to zero, there are *two* different complex-valued Hermitian Ricci tensors $\mathcal{R}_{\mu\nu}, \mathcal{S}_{\mu\nu}$. The deformed Born's reciprocal gravitational action linear in the Ricci scalars $\mathcal{R}, \mathcal{S}$ with Torsion-squared terms and BF terms is provided in section **3** after reviewing our prior results in [26].

2 Born's Reciprocal Relativity in Phase-Spaces

2.1 Invariance under the $U(1,3)$ Group

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = - dt \wedge dp^0 + \delta_{ij} dx^i \wedge dp^j;\ i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 - (dp_x)^2 - (dp_y)^2 - (dp_z)^2 \right) \tag{2.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details. These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \ = \ (dT)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ (dP)^2}{b^2} =$$

$$(d\tau)^2 \ \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right) \qquad (2.2)$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows [3]

$$\lambda_t \ = \ \sqrt{\frac{\hbar}{bc}}; \ \ \lambda_l \ = \ \sqrt{\frac{\hbar \ c}{b}}; \ \ \lambda_p \ = \ \sqrt{\frac{\hbar \ b}{c}}; \ \ \lambda_e \ = \ \sqrt{\hbar \ b \ c} \qquad (2.3)$$

The gravitational constant can be written as $G = \alpha_G \ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (2.3) coincide with the *Planck* time, length, momentum and energy, respectively. An interesting numerical relation involving the Planck scale and Hubble radius is $\mathcal{F}_{\max} = m_P \ \frac{c^2}{L_P} \sim M_{Universe} \ \frac{c^2}{R_H}$, hence in [23] we suggested that a certain large (Hubble) /small (Planck) scale *duality* was operating in this Born's reciprocal relativity theory reminiscent of the T-duality in string theory compactifications. Such duality was also compatible with Mach's principle. The authors [14] have proposed a relativity theory based on the de Sitter group that requires an invariant scale, besides the speed of light. The authors [14] have argued that such scale might bear a connection to the cosmological constant; i.e. to the quantity $(R_{Hubble})^{-2}$ observed today.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2.2) invariant are [2]

$$T' \ = \ T \ cosh\xi \ + \ (\frac{\xi_v \ X}{c^2} \ + \ \frac{\xi_a \ P}{b^2}) \ \frac{sinh\xi}{\xi} \qquad (2.4a)$$

$$E' \ = \ E \ cosh\xi \ + \ (-\xi_a \ X \ + \ \xi_v P) \ \frac{sinh\xi}{\xi} \qquad (2.4b)$$

$$X' \ = \ X \ cosh\xi \ + \ (\xi_v \ T \ - \ \frac{\xi_a \ E}{b^2}) \ \frac{sinh\xi}{\xi} \qquad (2.4c)$$

$$P' \ = \ P \ cosh\xi \ + \ (\frac{\xi_v \ E}{c^2} \ + \ \xi_a \ T) \ \frac{sinh\xi}{\xi} \qquad (2.4d)$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) = \frac{f}{F_{max}} \tag{2.5}$$

The net *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(2.4) is defined in terms of the velocity and force (acceleration) rapidity boosts parameters ξ_v, ξ_a as

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}} \tag{2.6a}$$

Straightforward algebra allows to verify that these transformations leave the interval $(d\omega)^2$ of eq-(2.2) in classical phase space invariant. When on sets $\xi_a = 0$ in eqs-(2.4) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $c^2(dT)^2 - (dX)^2 = (d\tau)^2$ and $((dE)^2 - c^2(dP)^2)/b^2$.

When one sets $\xi_v = 0$ we obtain the transformations rules from one reference-frame into another $non-inertial$ frame of reference whose force (acceleration) boost rapidity parameter is

$$\xi \equiv \frac{\xi_a}{b}; \quad tanh\xi = tanh(\frac{\xi_a}{b}) = \frac{f}{b}. \tag{2.6b}$$

The transformations for force (acceleration) boosts are

$$T' = T\ cosh\xi + \frac{P}{b}\ sinh\xi. \tag{2.7a}$$

$$E' = E\ cosh\xi - b\ X\ sinh\xi. \tag{2.7b}$$

$$X' = X\ cosh\xi - \frac{E}{b}\ sinh\xi \tag{2.7c}$$

$$P' = P\ cosh\ \xi + b\ T\ sinh\xi \tag{2.7d}$$

It is straightforwad to verify that the transformations (2.7) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2} \right) \tag{2.8}$$

is truly left invariant under force (acceleration) boosts (2.7). The composition of two succesive force (acceleration)-boosts is another force (acceleration)-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of forces (accelerations) follows the usual relativistic composition rule

$$tanh\xi'' \ = \ tanh(\xi + \xi') \ = \ \frac{tanh\xi \ + \ \tanh\xi'}{1 \ + \ tanh\xi \ tanh\xi'} \Rightarrow \frac{f''}{b} \ = \ \frac{\frac{f}{b} \ + \ \frac{f'}{b}}{1 \ + \ \frac{f \ f'}{b^2}}. \quad (2.10)$$

and in this fashion the upper limiting force (acceleration) is never *surpassed* like it happens with the speed of light in the ordinary Special Relativistic addition of velocities. The composition properties of both velocity and force (acceleration) boosts (2.4) requires much more algebra [23] to analyze. A careful study reveals that the group composition *rules* of two successive transformations of the form (2.4) are *preserved* if, and only if, the $\xi; \xi'; \xi''......$ parameters obey the suitable *collinear* relations [23]

$$\xi \ = \ \sqrt{\frac{(\xi_v)^2}{c^2} \ + \ \frac{\xi_a^2}{b^2}}; \quad \xi' \ = \ \sqrt{\frac{(\xi'_v)^2}{c^2} \ + \ \frac{(\xi'_a)^2}{b^2}}; \quad \xi'' \ = \ \sqrt{\frac{(\xi''_v)^2}{c^2} \ + \ \frac{(\xi''_a)^2}{b^2}} \quad (2.11a)$$

$$\frac{\xi_a}{\xi} \ = \ \frac{\xi'_a}{\xi'} \ = \ \frac{\xi''_a}{\xi''}; \quad \frac{\xi_v}{\xi} \ = \ \frac{\xi'_v}{\xi'} \ = \ \frac{\xi''_v}{\xi''}; \quad \xi''_v \ = \ \xi_v + \xi'_v; \quad \xi''_a \ = \ \xi_a + \xi'_a; \quad \xi'' \ = \ \xi + \xi' \quad (2.11b)$$

The relations (2.11) are required to be satisfied in order to have the proper $U(1,1)$ group composition law involving both velocity and force (acceleration) boosts transformations (2.4) and resulting in a phase-space change of coordinates (in the cotangent bundle of a $2D$ spacetime).

2.2 The Many Novel Physical Consequences

• Momentum-dependent time delay in the emission and detection of photons

From eqs-(2.7) one learns

$$\Delta T' \ = \ \Delta T \ cosh\xi \ + \ \frac{\Delta P}{b} \ sinh\xi \quad (2.12)$$

If two photons of different momentum P_1, P_2 are emitted *simultaneously* in a given reference frame, $\Delta T = 0$, there is a time *delay* in the emission times as measured with respect to an accelerated frame of reference. The time delay is

$$\Delta T' \ = \ \frac{\Delta P}{b} \ sinh\xi \ = \ T'_2 \ - \ T'_1 \ = \ \frac{P_2 - P_1}{b} \ sinh\xi \quad (2.13)$$

A momentum dependent *delay* in the emission times of photons will also cause a time delay in their detection as measured with respect to an accelerated frame of reference.

• Energy-dependent notion of locality in the emission of photons

From eqs-(2.7) one also has

$$\Delta X' \;=\; \Delta X \; cosh\xi \;-\; \frac{\Delta E}{b} \; sinh\xi \tag{2.14}$$

If two photons of different energy E_1, E_2 are emitted from the *same* location in a given reference frame, $\Delta X = 0$, they are *not* emitted from the same location in the accelerated frame of reference

$$\Delta X' \;=-\; \frac{\Delta E}{b} \; sinh\xi \;=\; X'_2 \;-\; X'_1 \;=\; -\; \frac{E_2 - E_1}{b} \; sinh\xi \tag{2.15}$$

thus the notion of *locality* is now frame-dependent; i.e. it is *relative* to the observers.

Due to the facts that $\Delta T' \neq 0, \Delta X' \neq 0$, despite that $\Delta T = \Delta X = 0$, one arrives at the conclusion that the notion of spatio-temporal *locality* is no longer an *invariant* concept as it was in special relativity; it is *relative* with respect to the non-inertial (accelerated) frames of reference. A completely different approach to the notion of "relativity of locality" and energy-dependent time-delay of photons based on a curved momentum space has been undertaken by [15], [25].

- **Superluminal behavior in Born's Reciprocal Relativity**

Let us study the notion of generalized proper vectors in flat phase-space. In units of $\hbar = c = 1$, the generalized phase-space coordinates are

$$Z^\alpha \;=\; (\; x^0,\; x^1,\; \frac{p^0}{b},\; \frac{p^1}{b}\;); \quad F_{max} \;=\; b; \quad x^0 = T,\; x^1 = X,\; p^0 = E,\; p^1 = P \tag{2.16}$$

in these units of $\hbar = c = 1$, b has dimensions of $(mass)^2$ (string tension). When $d\omega \neq 0$, the generalized velocity vector has for components

$$\prod \;=\; \frac{dZ^\alpha}{d\omega} \;=\; (\; \frac{dx^0}{d\omega},\; \frac{dx^1}{d\omega},\; \frac{1}{b}\frac{dp^0}{d\omega},\; \frac{1}{b}\frac{dp^1}{d\omega}\;), \quad d\omega \neq 0 \tag{2.17}$$

where the $U(1,1)$-invariant proper displacement is defined in terms of the proper time τ and proper-force squared F^2 as

$$d\omega \;=\; d\tau \; \sqrt{1 - (\frac{F^2}{F^2_{max}})}, \quad -\; F^2 \;=\; (\frac{dE}{d\tau})^2 \;-\; (\frac{dP}{d\tau})^2 \;\leq\; 0 \tag{2.18}$$

A *constant* generalized velocity (in a given frame of reference) by definition has *constant* components a_o, a_1, a_2, a_3

$$\frac{dx^0}{d\omega} = a_o,\; \frac{dx^1}{d\omega} = a_1,\; \frac{1}{b}\frac{dp^0}{d\omega} = a_2,\; \frac{1}{b}\frac{dp^1}{d\omega} = a_3 \tag{2.19}$$

and from which one can infer that

$$\frac{dp^0}{d\omega} = (\frac{dp^0}{d\tau})(\frac{d\tau}{d\omega}) = \frac{1}{\sqrt{1-(\frac{F^2}{F^2_{max}})}}(\frac{dp^0}{d\tau}) = a_2$$

$$\frac{dp^1}{d\omega} = (\frac{dp^1}{d\tau})(\frac{d\tau}{d\omega}) = \frac{1}{\sqrt{1-(\frac{F^2}{F^2_{max}})}}(\frac{dp^1}{d\tau}) = a_3 \Rightarrow$$

$$\frac{1}{1-(\frac{F^2}{F^2_{max}})}[(\frac{dp^0}{d\tau})^2 - (\frac{dp^1}{d\tau})^2] = -\frac{F^2}{1-(\frac{F^2}{F^2_{max}})} =$$

$$(a_2)^2 - (a_3)^2 = constant \Rightarrow F^2 = constant \quad (2.20)$$

hence, when $F^2 = constant$ one arrives at the (Lorentz covariant) conditions

$$\frac{dp^0}{d\tau} = constant; \ \frac{dp^1}{d\tau} = constant; \ \frac{dx^0}{d\tau} = constant; \ \frac{dx^1}{d\tau} = constant \quad (2.21)$$

expressed in terms of the ordinary proper time τ in special relativity. In general, $F^2(\tau) \neq constant$ and the generalized velocities are *not* constant either.

If $d\omega \neq 0$, one has the trivial identity

$$(\prod)^2 = (\frac{dE}{d\omega})^2 - (\frac{dP}{d\omega})^2 + (\frac{dx^0}{d\omega})^2 - (\frac{dx^1}{d\omega})^2 = (\frac{d\omega}{d\omega})^2 = 1 \quad (2.22)$$

when the interval is null, $d\omega = 0$, one must take the derivatives with respect to an affine parameter κ ($d\kappa \neq 0$) such that one has the proper null condition associated with a null generalized vector in flat phase-space

$$(\prod)^2 = (\frac{dE}{d\kappa})^2 - (\frac{dP}{d\kappa})^2 + (\frac{dx^0}{d\kappa})^2 - (\frac{dx^1}{d\kappa})^2 = (\frac{d\omega}{d\kappa})^2 = 0 \quad (2.23)$$

Such generalized $U(1,1)$-invariant null condition in phase space does *not* necessarily imply, *separately*, the two *restricted* and special conditions

$$(\frac{dE}{d\kappa})^2 - (\frac{dP}{d\kappa})^2 = 0, \quad (\frac{dx^0}{d\kappa})^2 - (\frac{dx^1}{d\kappa})^2 = 0 \quad (2.24)$$

In ordinary special relativity, the above separate two-conditions (2.24) are by themselves Lorentz invariant. A null line infinitesimal interval in phase space obeys in general the condition

$$d\omega = d\tau\sqrt{1-(\frac{F^2}{F^2_{max}})} = 0 \Rightarrow F^2 = F^2_{max}; \ or \ d\tau = 0, \ or \ both \quad (2.25)$$

If $d\tau \neq 0$ in eq-(2.25), one ends up with

$$-F^2 = (\frac{dE}{d\tau})^2 - (\frac{dP}{d\tau})^2 = -F^2_{max} \quad (2.26)$$

In this case, a null line path in phase space corresponds to a maximal proper-force trajectory; whereas a null line path in ordinary special relativity corresponds to a photon (geodesic) trajectory moving at the maximal speed of light c. If $d\tau = 0$ and $(dE)^2 - (dP)^2 < 0 \Rightarrow (d\omega)^2 < 0$ which is unphysical, it is the analog of a "tachyonic" interval.

Given the null condition $(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = 0$, dividing by $(dT)^2$, yields

$$1 - (\frac{dX}{dT})^2 + (\frac{1}{b})^2 (\frac{dE}{dT})^2 - (\frac{1}{b})^2 (\frac{dP}{dT})^2 = 0 \Rightarrow$$

$$1 - (v)^2 + (\frac{1}{b})^2 (f_o)^2 - (\frac{1}{b})^2 (f_1)^2 = 0 \Rightarrow v = \pm \sqrt{1 + (\frac{1}{b})^2 (f_0)^2 - (\frac{1}{b})^2 (f_1)^2} \tag{2.27}$$

where $v = \frac{dX}{dT}$ is the coordinate velocity; the analog of power and force are respectively $f_o = \frac{dE}{dT} \neq \frac{dE}{d\tau} = F_0$; $f_1 = \frac{dP}{dT} \neq \frac{dP}{d\tau} = F_1$. Reinserting the speed of light c (that was set to unity) one arrives at [2]

$$v = \pm c \sqrt{1 + (\frac{1}{bc})^2 (f_0)^2 - (\frac{1}{b})^2 (f_1)^2} = \pm c \sqrt{1 + (\frac{c}{b})^2 (\frac{dM}{dT})^2} \tag{2.28}$$

where the infinitesimal mass-displacement is defined as

$$c^2 (dM)^2 = (\frac{1}{c^2}) (dE)^2 - (dP)^2 \tag{2.29}$$

Taking the positive sign under the square root, when $(\frac{c}{b})^2 (\frac{dM}{dT})^2 < 0$, one arrives at the interesting conclusion that at the *null* hypersurface in $phase - space$ one can have points such that $v < c$. However, if $(\frac{c}{b})^2 (\frac{dM}{dT})^2 > 0$ one can have *superluminal* $v > c$ behavior in this case, despite having a null hypersurface in phase-space. When $(\frac{c}{b})^2 (\frac{dM}{dT})^2 = 0$, one recovers $v = c$ as it occurs in Special Relativity. Superluminal behavior in the underlying Minkowski space may occur also in the Extended Relativity Theory in Clifford spaces [24].

- **Relative Rotation of photon trajectories due to the Aberration of light**

The addition of velocities in Special Relativity ($c = 1$) can be derived by performing, for example, a velocity-boost transformation with velocity $v = v_1$ along the x-axis of the proper velocity's components associated to the moving object : $(U_0 = \frac{dx_0}{d\tau}; U_1 = \frac{dx_1}{d\tau}; U_2 = \frac{dx_2}{d\tau})$ in $2 + 1$ dimensions (for simplicity)

$$\begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} V_0 & V_1 & 0 \\ V_1 & V_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix}, \quad V_0 = \frac{1}{\sqrt{1 - v^2}}, \quad V_1 = \frac{v}{\sqrt{1 - v^2}}, \tag{2.30}$$

giving

$$W_0 \;=\; \frac{dt'}{d\tau} \;=\; U_0\; V_0 \;+\; U_1\; V_1$$

$$W_1 \;=\; \frac{dx'_1}{d\tau} \;=\; U_0\; V_1 \;+\; U_1\; V_0$$

$$W_2 \;=\; \frac{dx'_2}{d\tau} \;=\; U_2; \quad d\tau' = d\tau = invariant \tag{2.31}$$

The new coordinate velocities in the prime frame of reference are

$$u'_1 \;=\; \frac{dx'_1}{dt'} \;=\; \frac{W_1}{W_0} \;=\; \frac{U_0\; V_1 \;+\; U_1\; V_0}{U_0\; V_0 \;+\; U_1\; V_1} \;=$$

$$\frac{\frac{V_1}{V_0} \;+\; \frac{U_1}{U_0}}{1 \;+\; \frac{U_1}{U_0}\; \frac{V_1}{V_0}} \;=\; \frac{v_1 \;+\; u_1}{1 \;+\; u_1\; v_1} \tag{2.32a}$$

$$u'_2 \;=\; \frac{dx'_2}{dt'} \;=\; \frac{W_2}{W_0} \;=\; \frac{U_2}{U_0\; V_0 \;+\; U_1\; V_1} \;=\; \frac{\frac{U_2}{U_0 V_0}}{1 \;+\; \frac{U_1}{U_0}\; \frac{V_1}{V_0}} \;=$$

$$\frac{1}{V_0}\; \frac{u_2}{1 \;+\; u_1\; v_1} \;=\; \sqrt{1 - v^2}\; \frac{u_2}{1 \;+\; u_1\; v_1}; \quad v = v_1 \tag{2.32b}$$

If one reinstates the usual value of c into eqs-(2.32) one recovers the addition formula of velocities in Special Relativity. One of the consequences of the addition of velocities is the aberration of light. The angle corresponding to the direction of a photon measured along the line of sight of an observer moving with constant velocity $\vec{v}$ with respect to the light source (star), is *not* the same as the angle measured by an observer at rest. Taking the velocity $\vec{v}$ of the moving observer to lie along the x-direction, and the star at rest hovering above and in front of the observer, the addition formula of the velocities components associated with the x-direction, and corresponding to the photon and the moving observer, leads to

$$-c\; cos(\theta_{obs}) \;=\; \frac{v \;-\; c\; cos(\theta_{source})}{1 \;-\; \frac{v}{c}\; cos(\theta_{source})} \;\Rightarrow$$

$$cos(\theta_{obs}) \;=\; \frac{cos(\theta_{source}) - \frac{v}{c}}{1 \;-\; \frac{v}{c}\; cos(\theta_{source})} \Rightarrow\; \theta_{obs} \;\neq\; \theta_{source} \tag{2.33}$$

When there is an observer in an accelerated frame of reference, he will also experience an aberration of light. Furthermore, there is also a *new* effect as well. The directions of two photons, A, B, with $different$ energy-momentum content, that are emitted along the same line of sight of an observer, at rest with respect to a light source (star), will appear to be *rotated* with respect to an accelerated observer ; i.e. the two angles corresponding to the directions of two photons, measured along the line of sight of an observer in the accelerated frame of reference (with respect to the light source) are *not* the same.

A force (acceleration) boost transformation of the generalized velocity's components $(dZ^\alpha/d\omega)$ in phase space can be written as

$$\begin{pmatrix} U'_0 \\ U'_1 \\ \frac{F'_0}{b} \\ \frac{F'_1}{b} \end{pmatrix} = \begin{pmatrix} cosh\xi & 0 & 0 & sinh\xi \\ 0 & cosh\xi & -sinh\xi & 0 \\ 0 & -sinh\xi & cosh\xi & 0 \\ cosh\xi & 0 & 0 & cosh\xi \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ \frac{F_0}{b} \\ \frac{F_1}{b} \end{pmatrix}; \quad tanh(\xi) = \frac{f}{b}. \tag{2.34}$$

giving

$$\begin{aligned} U'_0 &= U_0\ cosh\xi\ +\ \frac{F_1}{b}\ sinh\xi \\ U'_1 &= U_1\ cosh\xi\ -\ \frac{F_0}{b}\ sinh\xi \\ \frac{F'_0}{b} &= -\ U_1\ sinh\xi\ +\ \frac{F_0}{b}\ cosh\xi \\ \frac{F'_1}{b} &= U_0\ sinh\xi\ +\ \frac{F_1}{b}\ cosh\xi \end{aligned} \tag{2.35}$$

In units of $\hbar = c = 1$, one has

$$\frac{U'_1}{U'_0} = cos(\theta_{obs}) = \frac{cos(\theta_{source})\ -\ (\frac{dE}{dx^0})\ (\frac{tanh\xi}{b})}{1\ +\ (\frac{dP}{dx^0})\ (\frac{tanh\xi}{b})} \Rightarrow \theta_{obs} \neq \theta_{source} \tag{2.36}$$

Hence, one also experiences an aberration of light in the accelerated frame of reference. Furthermore, if one has two photons A, B with $different$ energy-momentum content, due to the fact that in the most general case one has

$$(\frac{dE}{dx^0})_A \neq (\frac{dE}{dx^0})_B; \quad (\frac{dP}{dx^0})_A \neq (\frac{dP}{dx^0})_B \tag{2.37}$$

and one arrives at the conclusion that $(\theta_{obs})_A \neq (\theta_{obs})_B$ despite the fact that the two photons were originally emitted parallel to each other in the rest frame of reference of the light source (star) $(\theta_{source})_A = (\theta_{source})_B$. Therefore, the two angles corresponding to the directions of two photons, measured along the line of sight of an observer in the accelerated frame of reference (with respect to the light source) are *not* the same. Their directions will appear to be *rotated* with respect to each other. A completely different approach to the relative rotation of photon trajectories in curved momentum space due to the presence of torsion (in momentum space) was proposed by [25].

• Invariance of Fundamental Area-cells in Phase Space and Planck areas

In Special Relativity, under velocity boost transformations after recurring to eqs-(2.4) when the force (acceleration) boost rapidity parameter is set to zero $\xi_a = 0$, one has in this case $\xi = \frac{\xi_v}{c}$ such that

$$\Delta T' \ = \ \Delta T \ cosh\xi \ + \ \frac{\Delta X}{c} \ sinh\xi; \quad \Delta X' \ = \ \Delta X \ cosh\xi \ + \ c \ \Delta T \ sinh\xi \ \Rightarrow$$

$$\Delta X' \ \Delta T' \ \neq \ \Delta X \ \Delta T \tag{2.38a}$$

the space-time areas $\Delta X \Delta T$ are not invariant. At first sight this appears to contradict the well known consequences of Special Relativity : if there is a time dilation and a length contraction then one may naively conclude that the space-time areas are invariant. The underlying reason behind this apparent contradiction occurs because by time dilation it is meant that $\Delta T' = \gamma \Delta T = cosh\xi \ \Delta T = \frac{\Delta T}{\sqrt{1-(v/c)^2}}$ when the reading of the clocks ΔT occurs at the *same* location in the frame of reference which is at *rest* $\Delta X = 0$. Whereas by length contraction, it is meant $\Delta X' = (\Delta X/\gamma)$ when the length of the (contracted) rod is measured at the *same* time in the *moving* frame of reference $\Delta T' = 0$.

The time dilation and a length contraction findings can be seen simply by replacing $T' \leftrightarrow T, \ X' \leftrightarrow X$ and $\xi \leftrightarrow -\xi$ in (2.38a) yielding

$$\Delta T \ = \ \Delta T' \ cosh\xi - \frac{\Delta X'}{c} \ sinh\xi; \quad \Delta X \ = \ \Delta X' \ cosh\xi - c \ \Delta T' \ sinh\xi \tag{2.38b}$$

so that $\Delta T' = \gamma \Delta T$ when $\Delta X = 0$, and $\Delta X' = (\Delta X/\gamma)$ when $\Delta T' = 0$. Having resolved this apparent contradiction, one finds that $\Delta X' \Delta T' \neq \Delta X \Delta T$ in Special Relativity despite the time dilation and length contraction phenomena.

The group transformations rules of the coordinates in Phase space permitted us to show why force (acceleration) boosts preserve Planck-Scale *Areas* [23] when $b = (1/L_P^2)$ ($\hbar = c = 1$ and $\alpha_G = 1$). In the most general case areas can be preserved if certain conditions are met.

From eqs-(2.7) one obtains the transformation rules of the finite intervals $\Delta X, \Delta T, \Delta E, \Delta P$, from one inertial reference frame to another $non-inertial$ frame of reference under force (acceleration) boosts with rapidity parameter ξ

$$\Delta T' \ = \ \Delta T \ cosh\xi \ + \ \frac{\Delta P}{b} \ sinh\xi; \quad \Delta E' \ = \ \Delta E \ cosh\xi \ - b \ \Delta X \ sinh\xi \tag{2.39a}$$

$$\Delta X' \ = \ \Delta X \ cosh\xi \ - \ \frac{\Delta E}{b} \ sinh\xi; \quad \Delta P' \ = \ \Delta P \ cosh\xi \ + b \ \Delta T \ sinh\xi \tag{2.39b}$$

If, and only if, the conditions

$$\Delta X \ \Delta P \ = \ \Delta T \ \Delta E; \quad \Delta T \ \Delta X \ = \ \frac{\Delta E \ \Delta P}{b^2} \tag{2.40}$$

are obeyed, due to the identity $cosh^2\xi - sinh^2\xi \ = \ 1$, one can see that the following $area - cells$ will be *invariant* under force (acceleration) boosts

$$\Delta X' \ \Delta P' \ = \ \Delta X \ \Delta P \ (cosh^2\xi \ - \ sinh^2\xi) \ = \ \Delta X \ \Delta P \tag{2.41a}$$

$$\Delta T' \ \Delta E' \ = \ \Delta T \ \Delta \ E \ (cosh^2\xi \ - \ sinh^2\xi) \ = \ \Delta T \ \Delta E \tag{2.41c}$$

$$\Delta X' \; \Delta T' \;=\; \Delta X \; \Delta T \; (cosh^2\xi \;-\; sinh^2\xi) \;=\; \Delta X \; \Delta T \qquad (2.41c)$$

$$\Delta P' \; \Delta E' \;=\; \Delta P \; \Delta E \; (cosh^2\xi \;-\; sinh^2\xi) \;=\; \Delta P \; \Delta E \qquad (2.41d)$$

The conditions (2.40) are *trivially* satisfied for the very *special* case that

$$\lambda_l \;=\; \Delta X; \quad \lambda_t \;=\; \Delta T; \quad \lambda_p \;=\; \Delta P; \quad \lambda_e \;=\; \Delta E \qquad (2.42)$$

or any judicious multiples of those fundamental scales, where $\lambda_l, \lambda_t, \lambda_p, \lambda_e$ are the four scales provided by eq-(2.3). From the discussion after eq-(2.3), one learned that if $\alpha_G = 1$ one recovers the four Planck scales of length, time, momentum and energy in eq-(2.42), respectively, so that their corresponding Planck areas (2.41) are *invariant* under force (acceleration) boosts. However, one must emphasize that in general the conditions (2.40) are *not* always obeyed so that eqs-(2.41) are not always obeyed either (only in special cases).

The symplectic 2-form $\Omega \;=-\; dT \wedge dE + dX \wedge dP$ is *always* invariant under the transformations (2.4, 2.7), *irrespective* if the conditions (2.40) are obeyed or not. By recurring to the antisymmetry $-dT \wedge dE = dE \wedge dT; dX \wedge dP = -dP \wedge dX, ...$ one can verify that under force (acceleration) boosts one has

$$\Delta X' \wedge \Delta P' \;-\; \Delta T' \wedge \Delta E' = \; \Delta X \wedge \Delta P \; cosh^2\xi \;+\; \Delta T \wedge \Delta E \; sinh^2\xi \;+$$

$$(sinh\xi \; cosh\xi) \; (b \, \Delta X \wedge \Delta T - \frac{\Delta E \wedge \Delta P}{b}) - \Delta T \wedge \Delta E \; cosh^2\xi + \Delta P \wedge \Delta X \; sinh^2\xi +$$

$$(sinh\xi \; cosh\xi) \; (b \; \Delta T \wedge \Delta X \;-\; \frac{\Delta P \wedge \Delta E}{b}) \;=$$

$$\Delta X \wedge \Delta P \; (cosh^2\xi - sinh^2\xi) - \Delta T \wedge \Delta E \; (cosh^2\xi - sinh^2\xi) \;=\; \Delta X \wedge \Delta P - \Delta T \wedge \Delta E \qquad (2.43)$$

therefore $\Omega = \Omega'$ remains invariant under force (acceleration) boosts. The $8D$ phase space symplectic 2-form $\Omega \;=\; -\; dt \wedge dp^0 \;+\; \delta_{ij} dx^i \wedge dp^j; \; i,j = 1,2,3$ is $U(1,3)$ invariant. In a phase-space of $2n$ dimensions, the symplectic 2-form Ω is $U(1, n-1)$ invariant [2] due to the fact that the (pseudo) unitary group $U(1, n-1)$ can be interpreted as the intersection of $\mathcal{D} \otimes Sp(2n)$ and $O(2n)$, where $\mathcal{D}$ is the pure dilations/scalings group, $Sp(2n)$ is the symplectic group and $O(2n)$ the orthogonal group in $2n$-dimensions. The fact that Planck-scale *Areas* (when $\alpha_G = 1$) are invariant under force (acceleration) boosts could reveal very important information about Black holes Entropy and Holography. Minimal areas (in Planck units) was an important consequence in Loop Quantum Gravity.

- **Modification of the Dispersion Relations**

Multiplying the generalized velocity vector (2.17) in phase-space by an invariant mass parameter M (not to be confused with the proper rest mass m of a particle in ordinary Special Relativity) allows to define a generalized momentum vector

$$
M \frac{dZ^\alpha}{d\omega} = M \left(\frac{dx^0}{d\omega}, \frac{dx^1}{d\omega}, \frac{1}{b}\frac{dp^0}{d\omega}, \frac{1}{b}\frac{dp^1}{d\omega} \right) =
$$

$$
\frac{M}{\sqrt{1-(F^2/b^2)}} \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{1}{b}\frac{dp^0}{d\tau}, \frac{1}{b}\frac{dp^1}{d\tau} \right) =
$$

$$
\frac{M}{\sqrt{1-(F^2/b^2)}} \left(\frac{p^0}{m}, \frac{p^1}{m}, \frac{F^0}{b}, \frac{F^1}{b} \right); \quad d\omega \neq 0 \tag{2.44}
$$

Given the last term of eq-(2.44), and after recurring to eqs-(2.18, 2.22), one has that the norm-squared of the generalized momentum vector coincides with the invariant M^2

$$
(M \frac{dZ^\alpha}{d\omega})^2 = M^2 \Rightarrow M^2 = \frac{1}{1-(F^2/b^2)} \left((\frac{M}{m})^2 (p^0)^2 - (\frac{M}{m})^2 (p^1)^2 - \frac{M^2}{b^2} F^2 \right) \Rightarrow
$$

$$
(1 - \frac{F^2}{b^2}) M^2 + M^2 \frac{F^2}{b^2} = M^2 = (\frac{M}{m})^2 [(p^0)^2 - (p^1)^2] \Rightarrow (p^0)^2 - (p^1)^2 = m^2 \tag{2.45}
$$

From eq-(2.45) one arrives at the standard dispersion relation $(p^0)^2 - (p^1)^2 = m^2$ in Special Relativity (in units $c = 1$) if m is identified as the proper (rest) mass. However, one must emphasize that m is *not* an *invariant* in this theory [2]; M *is* the true invariant quantity.

To explain this further, one introduces the Quaplectic group $\mathcal{Q}(1,3)$ [2] which is given by the semidirect product of $U(1,3) \otimes_s \mathcal{H}(4)$, where $\mathcal{H}(4)$ is the Weyl-Heisenberg group. In ordinary Special Relativity the Poincare group $SO(1,3) \otimes_s T_4$ is the semidirect product of the Lorentz group with the translation group in Minkowski spacetime. The invariant proper mass is related to the quadratic Casimir invariant $p^\mu p_\mu = m^2$ of the Poincare group. This is no *longer* the case for the Quaplectic group [2]. The mass-squared m^2 is no longer a quadratic Casimir invariant of $\mathcal{Q}(1,3)$. In this theory the $non-commuting$ space is the coset $\mathcal{Q}(1,3)/SU(1,3)$ which is the non-commuting analog of Minkowski Space $(SO(1,3) \otimes_s T_4)/SO(1,3)$. The quadratic Casimir invariant of the Quaplectic group is [2]

$$
\frac{1}{2} \frac{1}{\lambda_t^2} \left(T^2 + \frac{E^2}{b^2 c^2} - \frac{X^2}{c^2} - \frac{P^2}{b^2} + \frac{2 \hbar I}{b c} (\frac{Y}{b c} - 2) \right) = \mathcal{C}_2 \tag{2.46}
$$

where λ_t is the invariant temporal parameter in (2.3); I is the center of the Weyl-Heisenberg group and Y is the $U(1)$ generator contained in $U(1,3) = U(1) \otimes SU(1,3)$. $X^2 = (x^1)^2 + (x^2)^2 + (x^3)^2; P^2 = (p^1)^2 + (p^2)^2 + (p^3)^2$. Hence, from the quadratic Casimir invariant in eq-(2.46) one can define the invariant mass-squared as a suitable multiple of the Casimir : $M^2 = (b\hbar/c^3)\ \mathcal{C}_2$. The quantity $(b\hbar/c^3)$ coincides with the Planck mass-squared $(m_P)^2$ when $\alpha_G = 1$.

Therefore, from eq-(2.46) one learns that the combination $\frac{1}{b^2}[\frac{E^2}{c^2} - P^2]$ is only a piece of the $\mathcal{Q}(1,3)$ group Casimir invariant so that $\frac{E^2}{c^4} - \frac{P^2}{c^2} = m^2$

is not an invariant and, consequently, there is *no* Quaplectic group invariant notion of m. The relation (2.46) represents the Quaplectic group analog of the energy-momentum dispersion relation for the Poincare group. From this point of view, the energy-momentum dispersion relations are indeed $modified$, fact which is compatible with the $non - commuting$ nature of the coset space $\mathcal{Q}(1,3)/SU(1,3)$. $Modified$ energy-momentum dispersion relations occur in deformed (double) special Relativity due to the quantum group (Hopf algebraic) kappa-$deformations$ of the Poincare algebra [11], [10] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [16, 17, 18].

Modified dispersion relations occur also in Clifford spaces [24]. In C-space (Clifford space), the invariant M^2 is identified with the norm-squared of the poly-momentum associated with the poly-particle [24]. The norm-squared of the Clifford-valued momentum (a poly-vector with antisymmetric components of different rank) is

$$M^2 = \pi^2 + p^\mu p_\mu + p^{\mu\nu} p_{\mu\nu} + p^{\mu\nu\rho} p_{\mu\nu\rho} + p^{\mu\nu\rho\sigma} p_{\mu\nu\rho\sigma} \qquad (2.47)$$

in order to match dimensions in (2.47) one requires to introduce suitable powers of a mass parameter, like the Planck mass. From (2.47) one concludes that $p^\mu p_\mu = m^2$ is *not* an *invariant* under the most general $Cl(1,3)$-algebra valued transformations (poly-rotations) in C-space. π is the scalar component of the Clifford-valued momentum. Therefore in C-space m is a *variable* and, in this aspect, a poly-particle shares similar properties to the notion of *unparticles* of *variable* mass in a regime where conformal invariance operates. The $4D$ conformal algebra $su(2,2) \sim so(4,2)$ admits an explicit realization in terms of Clifford algebra generators. Also the $4D$ superconformal algebra can be realized as well in terms Clifford algebra generators via 5×5 matrices and the charge conjugation matrix as shown by [28].

3 Born's Reciprocal Relativity in Curved Phase Space and Noncommutative Gravity

Born's reciprocal relativity in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. For the sake of completeness, in this last section we review our construction [26] where we extended Born's theory to the case of curved spacetimes and construct a $deformed$ Born reciprocal general relativity theory in curved spacetimes (without the need to introduce star products) as a local gauge theory of the $deformed$ Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* coordinates and momenta.

The deformed Weyl-Heisenberg algebra involves the generators

$$Z_a \;=\; \frac{1}{\sqrt{2}}\;(\;\frac{X_a}{\lambda_l}\;-\;i\frac{P_a}{\lambda_p}\;);\quad \bar{Z}_a \;=\; \frac{1}{\sqrt{2}}\;(\;\frac{X_a}{\lambda_l}\;+\;i\frac{P_a}{\lambda_p}\;);\quad a=1,2,3,4. \tag{3.1}$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The gauge theory is constructed in the fiber bundle over the base manifold which is a $4D$ curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$. The (deformed) Quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore we must *not* confuse the *deformed* complex gravity constructed here with the noncommutative gravity work in the literature [29] where the spacetime coordinates x^μ are not commuting.

The Hermitian generators $Z_{ab}, Z_a, \bar{Z}_a, I$ of the $U(1,3)$ algebra and the *deformed* Weyl-Heisenberg algebra obey the relations

$$(Z_{ab})^\dagger \;=\; Z_{ab};\quad (Z_a)^\dagger \;=\; \bar{Z}_a;\quad I^\dagger = I;\quad a,b=1,2,3,4. \tag{3.2}$$

The standard Quaplectic group [2] is given by the semi-direct product of the $U(1,3)$ group and the unmodified Weyl-Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, \bar{Z}_a, I$ with $a,b = 1,2,3,4$. A careful analysis reveals that the complex generators $Z_a, \bar{Z}_a$ (with Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl-Heisenberg algebra can be defined in terms of the Hermitian $U(1,4)$ algebra generators Z_{AB} , where $A,B = 1,2,3,4,5$; $a,b = 1,2,3,4$; $\eta_{AB} = diag\ (+,-,-,-,-)$, as follows

$$Z_a \;=\; (-i\,)^{1/2}\;(\;Z_{a5}\;-\;iZ_{5a}\;);\;\bar{Z}_a\;=\;(\;i\;)^{1/2}\;(\;Z_{a5}\;+\;iZ_{5a}\;);\;Z_{55}=\frac{\mathcal{I}}{2} \tag{3.3}$$

the Hermitian generators are $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}_a^b,\;\mathcal{E}_c^d]\;=\;-i\;\delta_c^b\;\mathcal{E}_a^d\;+i\;\delta_a^d\;\mathcal{E}_c^b;\quad[\mathcal{E}_c^d,\;\mathcal{E}_a^5]\;=\;-i\;\delta_a^d\;\mathcal{E}_c^5;\quad[\mathcal{E}_c^d,\;\mathcal{E}_5^a]\;=\;i\;\delta_c^a\;\mathcal{E}_5^d. \tag{3.4}$$

and $[\mathcal{E}_5^5,\;\mathcal{E}_5^a] = -i\;\delta_5^5\;\mathcal{E}_5^a$ .... such that now $\mathcal{I}(=2Z_{55})$ *no* longer commutes with $Z_a, \bar{Z}_a$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz-subalgebra generators L_{ab} and the "shear"-like generators M_{ab} as

$$Z_{ab} \equiv \frac{1}{2}\,(M_{ab}-iL_{ab});\quad L_{ab} = L_{[ab]}\;=\;i\,(Z_{ab}-Z_{ba});\quad M_{ab} = M_{(ab)} =\;(Z_{ab}+Z_{ba}), \tag{3.5}$$

one can see that the "shear"-like generators M_{ab} are $Hermitian$ and the Lorentz generators L_{ab} are $anti - Hermitian$ with respect to the fiber internal space indices. The explicit commutation relations of the Hermitian generators Z_{ab} can be rewritten as

$$[L_{ab},\ L_{cd}] \ = \ (\eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \qquad (3.6a)$$

$$[M_{ab},\ M_{cd}] \ = \ -(\eta_{bc}L_{ad} + \eta_{ac}L_{bd} + \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \qquad (3.6b)$$

$$[L_{ab},\ M_{cd}] \ = \ (\eta_{bc}M_{ad} - \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc}). \qquad (3.6c)$$

Defining $Z_{ab} = \frac{1}{2}(M_{ab} - iL_{ab}),\ Z_{cd} = \frac{1}{2}(M_{cd} - iL_{cd})$ after straightforward algebra it leads to the $U(3,1)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -i\ (\ \eta_{bc}\ Z_{ad} \ - \ \eta_{ad}\ Z_{cb}\ ). \qquad (3.6d)$$

as expected, and which requires that the commutators $[M, M] \sim L$ otherwise one would not obtain the $U(3,1)$ commutation relations (3.9d) nor the Jacobi identities will be satisfied. The commutators of the (anti-Hermitian) Lorentz boosts generators L_{ab} with the $X_c,\ P_c$ generators are

$$[L_{ab},\ X_c] \ = \ (\ \eta_{bc}\ X_a \ - \ \eta_{ac}\ X_b\ ); \quad [L_{ab},\ P_c] \ = \ (\ \eta_{bc}\ P_a \ - \ \eta_{ac}\ P_b\ ). \qquad (3.7a)$$

Since the Hermitian M_{ab} generators are the *reciprocal* boosts transformations which *exchange* X for P, in addition to boosting (rotating) those variables, one has in

$$[M_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \quad [M_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \qquad (3.7b)$$

such that upon recurring to the above equations after lowering indices it leads to [1]

$$[\ Z_{ab},\ Z_c\ ] \ = \ -\ \frac{i}{2}\ \eta_{bc}\ Z_a \ + \ \frac{i}{2}\ \eta_{ac}\ Z_b \ - \frac{1}{2}\ \eta_{bc}\ \bar{Z}_a \ - \ \frac{1}{2}\ \eta_{ac}\ \bar{Z}_b$$

$$[\ Z_{ab},\ \bar{Z}_c\ ] \ = \ -\ \frac{i}{2}\ \eta_{bc}\ \bar{Z}_a \ + \ \frac{i}{2}\ \eta_{ac}\ \bar{Z}_b \ + \ \frac{1}{2}\ \eta_{bc}\ Z_a \ + \ \frac{1}{2}\ \eta_{ac}\ Z_b\ . \qquad (3.7c)$$

In the noncommutative Yang's phase-space algebra case [20], associated with a noncommutative phase space involving noncommuting spacetime coordinates and momentum x^μ, p^μ, the generator $\mathcal{N}$ which appears in the modified $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}\mathcal{N}$ commutator is the *exchange* operator $x \leftrightarrow p$, $\ [p^\mu, \mathcal{N}] = i\hbar x^\mu / R_H^2$ and $[x^\mu, \mathcal{N}] = iL_P^2 p^\mu/\hbar$. L_P, R_H are taken to be the minimal Planck and maximal

[1]These commutators $differ$ from those in [2] because he chose all generators X, P, M, L to be anti-Hermitian so there are no i terms in the commutators in the r.h.s of eq-(3.7b) and there are also sign changes

Hubble length scales, respectively. The Hubble upper scale R_H corresponds to a *minimal* momentum $\hbar/R_H$, because by "duality" if there is a minimal length there should be a minimal momentum also.

Yang's [20] noncommutative phase space algebra is isomorphic to the conformal algebra $so(4,2) \sim su(2,2)$ after the correspondence $x^\mu \leftrightarrow L^{\mu 5}$, $p^\mu \leftrightarrow L^{\mu 6}$, and $\mathcal{N} \leftrightarrow L^{56}$. In the $deformed$ Quaplectic algebra case, it is in addition to the $\mathcal{I}$ generator, the M_{ab} generator which plays the role of the exchange operator of X with P and which also appears in the $deformed$ Weyl-Heisenberg algebra leading to a matrix-valued generalized Planck-constant, and noncommutative fiber coordinates, as follows

$$[\frac{X_a}{\lambda_l}, \frac{P_b}{\lambda_p}] = i\, \alpha_\hbar\, (\eta_{ab}\, \mathcal{I} + M_{ab});\ [X_a, X_b] = -(\lambda_l)^2\, L_{[ab]};\ [P_a, P_b] = (\lambda_p)^2\, L_{[ab]}; \tag{3.8}$$

One could interpret the term $\eta_{ab}\, \mathcal{I} + M_{ab}$ as a matrix-valued Planck constant $\hbar_{ab}$ (in units of $\hbar$). The $deformed$ (noncommutative) Weyl-Heisenberg algebra can also be rewritten as

$$[Z_a,\ \bar{Z}_b] = -\alpha_\hbar\, (\ \eta_{ab}\, \mathcal{I} + M_{ab}\ );\quad [Z_a,\ Z_b] = [\bar{Z}_a,\ \bar{Z}_b] = -i\, Z_{[ab]} = -L_{ab}.$$

$$[Z_a,\ \mathcal{I}] = 2\, \bar{Z}_a;\quad [\bar{Z}_a,\ \mathcal{I}] = -2\, Z_a;\quad [Z_{ab},\ \mathcal{I}] = 0.\quad \mathcal{I} = 2\, Z_{55}. \tag{3.9}$$

where $[\frac{X_a}{\lambda_l}, \mathcal{I}] = 2i\frac{P_a}{\lambda_p}$; $[\frac{P_a}{\lambda_p}, \mathcal{I}] = 2i\frac{X_a}{\lambda_l}$ and the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices . The $deformed$ Quaplectic algebra obeys the Jacobi identities. No longer $\mathcal{I}$ commutes with $Z_a, \bar{Z}_a$, it *exchanges* them, as one can see from eq-(3.9) since $Z_{55} = \mathcal{I}/2$.

The *complex* tetrad E^a_μ which transforms under the fundamental representation of $U(1,3)$ is defined as

$$E^a_\mu = \frac{1}{\sqrt{2}}\, (\ e^a_\mu + i f^a_\mu\ );\quad \bar{E}^a_\mu = \frac{1}{\sqrt{2}}\, (\ e^a_\mu - i f^a_\mu\ ). \tag{3.10}$$

The complex Hermitian metric is given by

$$G_{\mu\nu} = \bar{E}^a_\mu\, E^b_\nu\, \eta_{ab} = g_{(\mu\nu)} + i g_{[\mu\nu]} = g_{(\mu\nu)} + i B_{\mu\nu}. \tag{3.11}$$

such that

$$(G_{\mu\nu})^\dagger = \bar{G}_{\nu\mu} = G_{\mu\nu};\quad \bar{G}_{\mu\nu} = G_{\nu\mu}. \tag{3.12}$$

where the bar denotes complex conjugation. Despite that the metric is complex the infinitesimal line element is *real*

$$ds^2 = G_{\mu\nu}\, dx^\mu\, dx^\nu = g_{(\mu\nu)}\, dx^\mu\, dx^\nu,\ \ because\ \ i\, g_{[\mu\nu]}\, dx^\mu\, dx^\nu = 0. \tag{3.13}$$

The (deformed) Quaplectic-algebra-valued anti-Hermitian gauge field $(\mathbf{A}_\mu)^\dagger = -\mathbf{A}_\mu$ is given by

$$\mathbf{A}_\mu = \Omega^{ab}_\mu\, Z_{ab} + \frac{i}{L_p}\, (\ E^a_\mu\, Z_a + \bar{E}^a_\mu\, \bar{Z}_a\ ) + i\, \Omega_\mu\, I\ . \tag{3.14}$$

where a length scale that we chose to coincide with the the Planck length scale L_P has been introduced in the second terms in the r.h.s since the connection $\mathbf{A}_\mu$ must have units of $(length)^{-1}$. In natural units of $\hbar = c = 1$ the gravitational coupling in $4D$ is $G = L_P^2$. Decomposing the anti-Hermitian components of the connection Ω_μ^{ab} into anti-symmetric $[ab]$ and symmetric (ab) pieces with respect to the internal indices

$$\Omega_\mu^{ab} \ = \ \Omega_\mu^{[ab]} \ + i \ \ \Omega_\mu^{(ab)}. \tag{3.15}$$

gives the anti-Hermitian $U(1,3)$-valued connection

$$\Omega_\mu^{ab} \ Z_{ab} \ = \ (\Omega_\mu^{[ab]} \ + i \ \ \Omega_\mu^{(ab)}) \ \frac{1}{2}(M_{ab} - i \ L_{ab}) \ =$$

$$- \frac{i}{2} \ \Omega_\mu^{[ab]} \ L_{ab} \ + \ \frac{i}{2} \ \Omega_\mu^{(ab)} \ M_{ab} \ \Rightarrow \ (\Omega_\mu^{ab} \ Z_{ab})^\dagger \ = \ - \ \Omega_\mu^{ab} \ Z_{ab}. \tag{3.16}$$

since $(Z_{ab})^\dagger \ = \ Z_{ab}$

The *deformed* Quaplectic algebra-valued (anti-Hermitian) field strength is given by

$$\mathbf{F}_{\mu\nu} \ = \ \partial_\mu \mathbf{A}_\nu \ - \ \partial_\nu \mathbf{A}_\mu \ + \ [\mathbf{A}_\mu, \ \mathbf{A}_\nu] \ =$$

$$F_{\mu\nu}^{ab} \ Z_{ab} \ + i \ (F_{\mu\nu}^a \ Z_a \ + \ \bar{F}_{\mu\nu}^a \ \bar{Z}_a) \ + \ F_{\mu\nu} \ I \ =$$

$$\frac{i}{2} \ F_{\mu\nu}^{(ab)} \ M_{ab} \ - \frac{i}{2} \ F_{\mu\nu}^{[ab]} \ L_{ab} \ + i \ (F_{\mu\nu}^a \ Z_a \ + \ \bar{F}_{\mu\nu}^a \ \bar{Z}_a) \ + \ F_{\mu\nu} \ I \tag{3.17}$$

after decomposing $Z_{ab} \ = \ \frac{1}{2}(M_{ab} - iL_{ab})$. The components of the curvature two-form associated with the anti-Hermitian connection $\Omega_\mu^{ab} \ = \ \Omega_\mu^{[ab]} + i\Omega_\mu^{(ab)}$ are

$$-i \ F_{\mu\nu}^{[ab]} \ = \ \partial_\mu \Omega_\nu^{[ab]} \ - \ \partial_\nu \Omega_\mu^{[ab]} \ + \ \Omega_{[\mu}^{[ac]} \ \Omega_{\nu]}^{[cb]} \ -$$

$$\Omega_{[\mu}^{(ac)} \ \Omega_{\nu]}^{(cb)} \ + \frac{1}{L_P^2} \ E_{[\mu}^a \ E_{\nu]}^b \ + \frac{1}{L_P^2} \ \bar{E}_{[\mu}^a \ \bar{E}_{\nu]}^b \ . \tag{3.18}$$

$$i \ F_{\mu\nu}^{(ab)} \ = \ \partial_\mu \Omega_\nu^{(ab)} \ - \ \partial_\nu \Omega_\mu^{(ab)} \ + \ \Omega_{[\mu}^{(ac)} \ \Omega_{\nu]}^{[cb]} \ + \ \Omega_{[\mu}^{(bc)} \ \Omega_{\nu]}^{[ca]} \ +$$

$$\frac{1}{L_P^2} \ E_{[\mu}^a \ \bar{E}_{\nu]}^b \ + \frac{1}{L_P^2} \ E_{[\mu}^b \ \bar{E}_{\nu]}^a \tag{3.19}$$

where a summation over the repeated c indices is implied and $[\mu\nu]$ denotes the anti-symmetrization of indices with weight one. Notice the presence of the *extra* terms EE in the above expressions for the *deformed* field strengths due to the *noncommutative* $[Z_a, Z_b] \neq 0$, and which in turn, *modifies* the Weyl-Heisenberg algebra due to the Jacobi identities. In the *undeformed* ordinary Quaplectic-algebra case these terms are *absent* because $[Z_a, Z_b] = 0, ...$ and, furthermore, there is *no* M_{ab} term in the ordinary Weyl-Heisenberg algebra. These extra terms $E^a \wedge E^b, ...$ in eqs-(3.18,3.19) are one of the hallmarks of the *deformed* Quaplectic gauge field theory formulation of the *deformed* Born's Reciprocal Complex Gravity.

The components of the torsion two-form are

$$F^a_{\mu\nu} \ = \ \partial_\mu E^a_\nu \ - \ \partial_\nu E^a_\mu \ - i \ \Omega^{[ac]}_{[\mu} \ E^c_{\nu]} \ + i \ \Omega^{(ac)}_{[\mu} \ \bar{E}^c_{\nu]} \ - 2i \ \bar{E}^a_{[\mu} \ \Omega_{\nu]}. \quad (3.20a)$$

$$\bar{F}^a_{\mu\nu} \ = \ \partial_\mu \bar{E}^a_\nu \ - \ \partial_\nu \bar{E}^a_\mu \ + i \ \Omega^{[ac]}_{[\mu} \ \bar{E}^c_{\nu]} \ - i \ \Omega^{(ac)}_{[\mu} \ E^c_{\nu]} \ + 2i \ E^a_{[\mu} \ \Omega_{\nu]}. \quad (3.20b)$$

The remaining field strength has roughly the same form as a $U(1)$ field strength in noncommutative spaces due to the additional contribution of $B_{\mu\nu}$ resulting from the nonabelian nature of the Weyl-Heisenberg algebra in the internal space (fibers) and which is reminiscent of the noncommutativity of the coordinates with the momentum :

$$F_{\mu\nu} \ = \ i \ \partial_\mu \Omega_\nu \ - \ i \ \partial_\nu \Omega_\mu \ + \ \frac{1}{L_P^2} \ E^a_\mu \ \bar{E}^b_\nu \ \eta_{ab} \ - \ \frac{1}{L_P^2} \ \bar{E}^a_\mu \ E^b_\nu \ \eta_{ab} \ =$$

$$i \ \partial_\mu \Omega_\nu \ - i \ \partial_\nu \Omega_\mu \ + \ \frac{1}{L_P^2} \ (\ G_{\mu\nu} \ - \ G_{\nu\mu} \ ) \ = \ i \ \Omega_{[\mu\nu]} \ + i \ \frac{2}{L_P^2} \ G_{[\mu\nu]} \quad (3.21)$$

after recurring to the commutation relations (for $\alpha_h = 1$) in eqs-(3.8,3.9) and the Hermitian property of the metric

$$G_{\mu\nu} \ = \ \bar{E}^a_\mu \ E^b_\nu \ \eta_{ab} \ = \ [\ \eta_{ab} \ \bar{E}^b_\nu \ E^a_\mu \ ]^* \ = \ (G_{\nu\mu})^* \ \Rightarrow \ (G_{\mu\nu})^* \ = \ G_{\nu\mu}. \quad (3.22)$$

where $*$ stands for (bar) complex conjugation.

The curvature tensor is defined in terms of the anti-Hermitian connection $\Omega^{[ab]}_\mu + i \ \Omega^{(ab)}_\mu$ as

$$\mathcal{R}^\rho_{\mu\nu\lambda} \ \equiv \ (\ F^{[ab]}_{\mu\nu} \ + \ i \ F^{(ab)}_{\mu\nu} \ ) \ (E^\rho_a \ E_{b\lambda} \ + \ \bar{E}^\rho_a \ \bar{E}_{b\lambda} \ + \ E^\rho_a \ \bar{E}_{b\lambda} \ + \ \bar{E}^\rho_a \ E_{b\lambda}). \quad (3.23)$$

where the explicit components $F^{[ab]}_{\mu\nu}$ and $F^{(ab)}_{\mu\nu}$ can be read from the defining relations (3.18, 3.19). Note that both values of values of $F^{[ab]}_{\mu\nu}$ and $F^{(ab)}_{\mu\nu}$ are purely *imaginary* such that one may rewrite the complex-valued $F^{ab}_{\mu\nu}$ field strength as $(\mathcal{F}^{(ab)}_{\mu\nu} + i\mathcal{F}^{[ab]}_{\mu\nu})$ for real valued $\mathcal{F}^{(ab)}_{\mu\nu}$, $\mathcal{F}^{[ab]}_{\mu\nu}$ expressions. The contraction of indices yields *two* different complex-valued (Hermitian) Ricci tensors.

$$\mathcal{R}_{\mu\lambda} \ = \ g^{\sigma\nu} \ g_{\rho\sigma} \ R^\rho_{\mu\nu\lambda} \ = \ \delta^\nu_\rho \ R^\rho_{\mu\nu\lambda} \ = \ R_{(\mu\lambda)} + i \ R_{[\mu\lambda]}; \quad (\mathcal{R}_{\mu\lambda})^* \ = \ \mathcal{R}_{\lambda\mu} \quad (3.24)$$

and

$$\mathcal{S}_{\mu\lambda} \ = \ g^{\sigma\nu} \ g_{\sigma\rho} \ R^\rho_{\mu\nu\lambda} \ = \ \mathcal{S}_{(\mu\lambda)} \ + \ i \ \mathcal{S}_{[\mu\lambda]}; \quad (\mathcal{S}_{\mu\lambda})^* \ = \ \mathcal{S}_{\lambda\mu} \quad (3.25)$$

due to the fact that

$$g^{\sigma\nu} \ g_{\rho\sigma} \ = \ \delta^\nu_\rho \ \ and \ \ g^{\sigma\nu} \ g_{\sigma\rho} \ \neq \ \delta^\nu_\rho. \quad (3.26)$$

because $g_{\sigma\rho} \neq g_{\rho\sigma}$. The position of the indices is crucial. There is a third Ricci tensor $Q_{[\mu\nu]} = \mathcal{R}^{\rho}_{\mu\nu\lambda}\delta^{\lambda}_{\rho}$ related to the curl of the *nonmetricity* Weyl vector Q_μ [31] which one may set to zero. However, in the most general case one should include *nonmetricity*. Nonmetricity was essential in the recent findings by [25].

A further contraction yields the generalized (real-valued) Ricci scalars

$$\mathcal{R} \;=\; (g^{(\mu\lambda)} \;+ i\; g^{[\mu\lambda]})\;(\;R_{(\mu\lambda)} \;+\; i\; R_{[\mu\lambda]}\;) =$$

$$\mathcal{R} \;=\; g^{(\mu\lambda)}\; R_{(\mu\lambda)} \;-\; B^{\mu\lambda}\; R_{[\mu\lambda]}; \quad g^{[\mu\lambda]} \equiv B^{\mu\lambda}. \tag{3.27a}$$

$$\mathcal{S} \;=\; (g^{(\mu\lambda)} \;+ i\; g^{[\mu\lambda]})\;(\;S_{(\mu\lambda)} \;+\; i\; S_{[\mu\lambda]}\;) =$$

$$\mathcal{S} \;=\; g^{(\mu\lambda)}\; \mathcal{S}_{(\mu\lambda)} \;-\; B^{\mu\lambda}\; \mathcal{S}_{[\mu\lambda]}. \tag{3.27b}$$

The first term $g^{(\mu\lambda)}\; R_{(\mu\lambda)}$ corresponds to the usual scalar curvature of the ordinary Riemannian geometry. The presence of the extra terms $B^{\mu\lambda}\; R_{[\mu\lambda]}$ and $B^{\mu\lambda}\; \mathcal{S}_{[\mu\lambda]}$ due to the anti-symmetric components of the metric and the two different types of Ricci tensors are one of the hallmarks of the deformed Born complex gravity. We should notice that the inverse complex metric is

$$g^{(\mu\lambda)} \;+\; ig^{[\mu\lambda]} \;=\; [\; g_{(\mu\nu)} \;+\; ig_{[\mu\nu]}\;]^{-1} \;\neq\; (g_{(\mu\nu)})^{-1} \;+\; (ig_{[\mu\nu]})^{-1}. \tag{3.28}$$

so $g^{(\mu\nu)}$ is now a complicated expression of both $g_{\mu\nu}$ *and* $g_{[\mu\nu]} = B_{\mu\nu}$. The same occurs with $g^{[\mu\nu]} = B^{\mu\nu}$. Rigorously we should have used a different notation for the inverse metric $\tilde{g}^{(\mu\lambda)} \;+\; i\tilde{B}^{[\mu\lambda]}$, but for notational simplicity we chose to drop the tilde symbol.

One could add an *extra* contribution to the complex-gravity real-valued action stemming from the terms $iB^{\mu\nu}F_{\mu\nu}$ which is very reminiscent of the BF terms in Schwarz Topological field theory and in Plebanksi's formulation of gravity. In the most general case, one must include both the contributions from the torsion and the $i\; B^{\mu\nu}F_{\mu\nu}$ terms. The contractions involving $G^{\mu\nu} = g^{(\mu\nu)} + iB^{\mu\nu}$ with the components $F_{\mu\nu}$ (due to the antisymmetry property of $F_{\mu\nu} = -F_{\nu\mu}$) lead to

$$i\, B^{\mu\nu}\; F_{\mu\nu} \;=\; -\, B^{\mu\nu}\,(\; \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu\;) - 2\, B^{\mu\nu}\; B_{\mu\nu} \;=\; -\, B^{\mu\nu}\; \Omega_{\mu\nu} - 2\, B^{\mu\nu}\; B_{\mu\nu}\;. \tag{3.29}$$

where we have set the length scale $L_P = 1$ for convenience. These BF terms contain a mass-like term for the $B_{\mu\nu}$ field. Mass terms for the $B_{\mu\nu}$ and a *massive* graviton formulation of bi-gravity (in addition to a massles graviton) based on a $SL(2,C)$ gauge formulation have been studied by [31], [32], [30],. When the torsion is not constrained to vanish one must include those contributions as well. The real-valued torsion two-form is $(F^a_{\mu\nu}Z_a + \bar{F}^a_{\mu\nu}\bar{Z}_a)dx^\mu \wedge dx^\nu$ and the torsion tensor and torsion vector are

$$T^{\rho}_{\mu\nu} \;=\; F^a_{\mu\nu}\; E^{\rho}_a; \;\; \bar{T}^{\rho}_{\mu\nu} \;=\; \bar{E}^{\rho}_a\; \bar{F}^a_{\mu\nu}\;; \;\; T_{\mu\nu\rho} \;=\; g_{\rho\sigma}\; T^{\sigma}_{\mu\nu}; \;\; \bar{T}_{\mu\nu\rho} \;=\; \bar{T}^{\sigma}_{\mu\nu}\;(g_{\rho\sigma})^* \;=$$

$$\bar{T}^{\sigma}_{\mu\nu}\; g_{\sigma\rho}; \quad T_\mu \;=\; \delta^{\nu}_{\rho}\; T^{\rho}_{\mu\nu}; \quad \bar{T}_\mu \;=\; \bar{T}^{\rho}_{\mu\nu}\; \delta^{\nu}_{\rho}. \tag{3.30}$$

The (real-valued) action, linear in the two (real-valued) Ricci curvature scalars and quadratic in the torsion is of the form

$$\frac{1}{2\kappa^2} \int_{M_4} d^4x \sqrt{|\ det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ |}\ (\ a_1\ \mathcal{R} + a_2\ \mathcal{S} + a_3\ T_{\mu\nu\rho}\ T^{\mu\nu\rho} + a_4\ T_\mu\ T^\mu + c.c). \tag{3.31}$$

$$|\ det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ |\ =\ \sqrt{det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ det\ (g_{(\mu\nu)} - iB_{\mu\nu})} \tag{3.32}$$

where one must add the complex conjugate (cc) terms in order to render the action real-valued. $\kappa^2 = 8\pi G$ is the gravitational coupling and in natural units $\hbar = c = 1$ one has $G = L^2_{Planck}$. We may add the BF terms (3.29) to the action (3.31) as well as Yang-Mills terms $F \wedge^* F$. In the most general case one should include *nonmetricity* terms as well. The action (3.31) is invariant under infinitesimal $U(1,3)$ gauge transformations of the complex tetrad $\delta E^a_\mu = (\xi^a_{b(1)} + i\xi^a_{b(2)})E^b_\mu$ where the real $\xi^{(1)}_{[ab]}$ and imaginary $\xi^{(2)}_{(ab)}$ components of the complex parameter are anti-symmetric and symmetric, respectively, with respect to the indices a, b for anti-Hermitian infinitesimal $U(1,3)$ gauge transformations.

The a_1, a_2, a_3, a_4 are suitable numerical coefficients that will be constrained to have certain values if one wishes to avoid the presence of ghosts, tachyons and higher order poles in the propagator, not unlike it occurs in Moffat's nonsymmetric gravity theory [31]. The instabilities of Moffat's nonsymmetric gravity found by [32] are bypassed when one extends the theory to spacetimes with *complex* coordinates [30].

The action (3.31) defined in $4D$ can be extended to a $4D$ *complex* spacetime; i.e. an action in $8D$ real-dimensional Phase Space associated with the cotangent bundle of spacetime. The geometry of curved Phase spaces and bounded complex homogeneous domains has been studied by [12]. The presence of matter sources can be incorporated, for example, by recurring to the invariant action for a point-particle in Born's Reciprocal Relativity involving Casimir group invariant quantities associated with the world-line of the particle. The quantization of a point-particle corresponding to the undeformed Quaplectic group is far richer than the ordinary Poincare case since acceleration boosts can change the spin of the particle. The spectrum contains towers of integer massive spin states, as well as unconventional massless representations [2].

To conclude, we should emphasize that the complex deformed Born Reciprocal Gravitational theory advanced here $differs$ from the modified gravitational theories in the literature [31], [30], [33], and it is mainly due to the fact that we have constructed a deformed complex Born's reciprocal gravitational theory in $4D$ as a gauge theory of the $deformed$ Quaplectic group given by the semidirect product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg algebra of eqs-(3.8, 3.9). The deformed Weyl-Heisenberg algebra already encodes the noncommutativity of the fiber coordinates such that $Z_\mu(w^i) = E^a_\mu(w^i)\ Z_a$ and $\bar{Z}_\mu(w^i) = \bar{E}^a_\mu(w^i)\ \bar{Z}_a$ could be interpreted as the p-brane *noncommutative* target complex-spacetime background embedding functions $Z_\mu(w^i), \bar{Z}_\mu(w^i)$ in terms of the $p+1$ world-volume coordinates w^i ($i = 1, 2, ..., p+1$).

Since the vielbein E_μ^a is required in the definition of the embedding coordinates $Z_\mu, \bar{Z}_\mu$, it is not surprising to see why string-theory (p-branes) encodes gravity. For plausible relations between nonsymmetric gravity and string theory see [31], [30]. Finally, gravitational theories based on Born's reciprocal relativity principle involving a maximal speed limit and a maximal proper force, is a very promising avenue to quantize gravity that does *not* rely in *breaking* the Lorentz symmetry at the Planck scale, in contrast to other approaches based on deformations of the Poincare algebra, Hopf algebras, quantum groups, etc...

Acknowledgements

We thank M. Bowers for very kind assistance.

References

[1] M. Born, Proc. Royal Society **A 165** (1938) 291. Rev. Mod. Physics **21** (1949) 463.

[2] S. Low, Jour. Phys **A** Math. Gen **35** (2002) 5711; Il Nuovo Cimento **B 108** (1993) 841; Found. Phys. **36** (2007) 1036. J. Math. Phys **38** (1997) 2197.

[3] P. Jarvis and S. Morgan, Found. Phys. Lett **19** (2006) 501.

[4] R. Delbourgo and D. Lashmar, "Born reciprocity and the $1/r$ potential" hep-th/0709.0776.

[5] E. Caianiello, "Is there a maximal acceleration?", Lett. Nuovo Cimento **32** (1981) 65.

[6] V. Nesterenko, Class. Quant. Grav. **9** (1992) 1101; Phys. Lett. **B 327** (1994) 50;

[7] V. Bozza, A. Feoli, G. Lambiase, G. Papini and G. Scarpetta, Phys. Let **A 283** (2001) 53. V. Nesterenko, A. Feoli, G. Lambiase and G. Scarpetta, Phys. Rev **D 60**, 065001 (1999).

[8] K. Rama, "Classical velocity in kappa-deformed Poincare algebra and amaximal acceleration" [arXiv: hep-th/0209129].

[9] H. Brandt: Contemporary Mathematics **196** (1996) 273. Chaos, Solitons and Fractals **10** (2-3) (1999) 267.

[10] G.Amelino-Camelia, Phys. Lett **B 510** (2001) 255. Int. J. Mod. Phys **D 11** (2002) 35. Int. J. Mod. Phys **D 11** (2002) 1643.

[11] J. Lukierski, A. Nowicki, H. Ruegg and V. Tolstoy, Phys. Letts **B 264** (1991) 331.

[12] R. Coquereaux and A. Jadczyk, Rev. in Math. Phys **2** , No. 1 (1990) 1-44.

[13] G. Gibbons and S. M. Gielen, "Deformed General Relativity and Torsion" Class. Quant. Grav. **26** (2009) 135005. S. M. Gielen, " Geometric Aspects of Gauge and Spacetime Symmetries" (Ph.D Thesis, Univ. of Cambridge) arXiv.org : 1103.0731.

[14] R. Aldrovandi, J. Beltran Almeida and J. Pereira, Class. Quan. Grav **24** (2007) 1385; R. Aldrovandi and J. Pereira, "de Sitter Relativity : a new road to quantum gravity" arXiv.org : 0711.2274.

[15] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman and L. Smolin, " The principle of relative locality" arXiv.org : 1101.0931.

[16] K. Greisen, Phys. Rev. Lett **16** (1966) 748. G.T. Zatsepin, V. Kurmin, Sov. Phys. JETP Lett **4** (1966) 78.

[17] J. Ellis, N. Mavromatos and D. V. Nanopolous, Chaos, Solitons and Fractals, **10** (1999) 345.

[18] M. Toller, "The Geometry of Maximal Acceleration" [ArXiv: hep-th/0312016].

[19] M. Calixto and E. Perez-Romero, "Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in $U(2,2)$ and Born's Reciprocity" arXiv.org : 1006.5958. M. Calixto , E. Perez-Romero and V. Aldaya, "Unruh Effect and the Spontaneous Breakdown of the Conformal Symmetry" arXiv.org : 1012.4758.

[20] C.N Yang, Phys. Rev **72** (1947) 874. Proceedings of the International Conference on Elementary Particles, (1965) Kyoto, pp. 322-323.

[21] S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonta, *Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity* (Balkan Press, 2006).

[22] F. Schuller, Ann. of Phys. **299** (2002) 174.

[23] C. Castro, Prog. in Phys. **1** (April 2005) 20; Int. J. Mod. Phys **A 18** (2003) 5445; Int. J. Mod. Phys **A 23**, (2008) 1487; J. Phys **A** : Math. Gen **39** (2006) 14205.

[24] C. Castro and M. Pavsic, Prog. in Phys. **1** (April 2005) 31.

[25] L. Freidel and L. Smolin, "Gamma ray burst delay times probe the geometry of momentum space" arXiv.org : 1103.5626

[26] C. Castro, Phys. Letts **B 668** (2008) 442; Int. J. Mod. Phys. **A 23** (2008) 1487.

[27] J. Govaerts, P. Jarvis, S. Morgan and S. Low, " Worldline Quantization of a Reciprocally Invariant System" arXiv : 0706.3736. P. Jarvis and S. Morgan, "Constraint Quantization of a worldline system invariant under reciprocal Relativity, II " arXiv.org : 0806.4794.

[28] M. Kaku, P. Townsend and P. van Nieuwenhuizen, Phys. Lett **B 69** (1977) 304; C. Shi, G. Hanying, L. Gendao and Z. Yuanzhong, Scientia Sinica, **23**, no. 3 (1980) 299.

[29] P. Achieri, M. Dimitrijevic, F. Meyer and J. Wess, "Noncommutative Geometry and Gravity" [arXiv : hep-th/0510059] and references therein.

[30] A. Chamseddine, Comm. Math. Phys **218**, 283 (2001). " Gravity in Complex Hermitian Spacetime" arXiv : hep-th/0610099.

[31] J. Moffat, J. Math. Phys **36**, no. 10 (1995) 5897. J. Moffat and D. Boal, Phys. Rev **D 11**, 1375 (1975). J. Moffat, "Noncommutative Quantum Gravity" arXiv : hep-th/0007181. N. Poplawski, " On Nonsymmetric purely affine Gravity" gr-qc/0610132,

[32] T. Damour, S. Deser and J. McCarthy, Phys. Rev. **D 47** (1993) 1541.

[33] A. Einstein, Ann. Math **46**, 578 (1945). A. Einstein and E. Strauss, Ann. Math **47**, 731 (1946).