Chapter IV

The Proofs of Binary Goldbach's Theorem Using Only Partial Primes

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> When the answers to a mathematical problem cannot be found, then the reason is frequently the fact that we have not recognized the general idea, from which the given problem appears only as a single link in a chain of related problems.
>
> David Hilbert

In 1994 we discovered the new arithmetic function $J_2(\omega)$. Using it we proved the binary Goldbach's theorem [1]. In this chapter we yield the more detailed proofs of the binary Goldbach's theorem using only partial primes.

Definition. We define the arithmetic progressions [1-4]

$$E_{p_\alpha}(K) = \omega K + p_\alpha, \tag{1}$$

where $K = 0, 1, 2, ..., \omega = \prod_{2 \le p \le p_i} p, (p_\alpha, \omega) = 1, p_i < p_\alpha = p_1, p_2, ..., p_{\phi(\omega)} = \omega + 1, \phi(\omega) = \prod_{2 \le p \le p_i} (p-1)$ being Euler totient function.

$E_{p_\alpha}(K)$ can constitute all the primes and composites except the numbers of factors: $2, 3, ..., p_i$. We define the primes and the composites by K below.

Theorem 1. If there exist the infinitely manay primes p such that $ap+b$ is also a prime, then $ap+b$ must satisfy three necessary and sufficient conditions:

(I) Let $ap+b$ be an irreducible polynomial satisfying $ab \neq 0, (a,b) = 1, 2|ab$.

(II) There exists an arithmetic function $J_2(\omega)$ which denotes the number of subequations. It is also the number of solutions for

$$(aE_{p_\alpha}(K) + b, \omega) = (ap_\alpha + b, \omega) = 1. \qquad (2)$$

From (2) defining $J_2(\omega)$ can be written in the form

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{1}{(ap_\alpha + b, \omega)}\right] = \prod_{3\le p\le p_i} (p - 1 - \chi(p)). \qquad (3)$$

(III) t_α is independent of p_α [5], where t_α denotes the number of primes K_p less than n in $aE_{p_\alpha}(K_p) + b = p''$. Taking $t_1 = t_\alpha$, where $\alpha = 1, ..., J_2(\omega).t_\alpha$ seem to be equally distributed among the $J_2(\omega)$. We have

$$\pi_2(N, 2) = |\{p : p \le N, ap + b = p'\}| = \sum_{\alpha=1}^{J_2(\omega)} t_\alpha \sim J_2(\omega)t_1. \qquad (4)$$

First we deal with a subequation $aE_{p_1}(K) + b = p''$. We define the sequence

$$K = 0, 1, 2, ..., n. \qquad (5)$$

We take the average value

$$t_1 = |\{K_p : K_p \le n, aE_{p_1}(K_p) + b = p''\}| \sim \frac{(\pi_1(\omega n))^2}{n}, \qquad (6)$$

where $\pi_1(\omega n)$ denotes the number of primes K_p less than n in $E_{p_1}(K)$. We show that t_1 is independent of p_1, because $\pi_1(\omega n)$ is independent of p_1 [5].

Let $N = \omega n$ and $\pi_1(N) \sim \frac{N}{\phi(\omega)\log N}$. Substituting it into (6) and then (6) into (4) we have

$$\pi_2(N, 2) = |\{p : p \le N, ap + b = p'\}| \sim \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}. \qquad (7)$$

From (2) we have

$$ap_\alpha + b \equiv 0 (\mathrm{mod} \ \ p). \tag{8}$$

Every $p_\alpha > p$ can be expressed in the form

$$p_\alpha = pm + q, \tag{9}$$

where $q = 1, 2, ..., p-1$.

Substituting (9) into (8) we have

$$aq + b \equiv 0 (\mathrm{mod} \ \ p). \tag{10}$$

If $p|ab$, then (10) has no solutions. We define $\chi(p) = 0$. If $p \not| ab$, then (10) has a solution. We define $\chi(p) = 1$. Substituting it into (3) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \prod_{p|ab, 3 \le p \le p_i} \frac{p-1}{p-2} \neq 0, \tag{11}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist the infinitely many primes p such that $ap + b$ is also a prime. It is a generalization of Euler poof of the existence of the infinitely many primes [1-4].

Substituting (11) into (7) we have

$$\pi_2(N, 2) = |\{p : p \le N, ap+b = p'\}| \sim 2 \prod_{3 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|ab, 3 \le p \le p_i} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{12}$$

The Prime Twins Theorem. Let $a = 1$ and $b = 2$. From (11) we have

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p-2) \neq 0, \tag{13}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist the infinitely many primes p such that $p + 2$ is also a prime.

From (12) we have

$$\pi_2(N, 2) = |\{p : p \le N, p + 2 = p'\}| \sim 2 \prod_{3 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}. \tag{14}$$

(14) is the best asymptotic formula conjectured by Hardy and Littlewood [6].

The Binary Goldbach's Theorem [1-4]. Let $a=-1$ and $b=N$. From (11) we have

$$J_2(\omega)=\prod_{3\le p\le p_i}(p-2)\prod_{p|N,3\le p\le p_i}\frac{p-1}{p-2}\neq 0, \tag{15}$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, every even number N greater than 4 is the sum of two primes.

From (12) we have

$$\pi_2(N,2)=|\{p:p\le N, N-p=p'\}|\sim 2\prod_{3\le p\le p_i}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N}{\log^2 N}. \tag{16}$$

(16) is the best asymptotic formula conjectured by Hardy and Littlewood [6].

To understand the binary Goldbach's theorem, we yield the more detailed proofs below.

Corollary 1. Let $p_i=5$ and $\omega=30$. From (1) we have [1-2]

$$E_{p_\alpha}(K)=30K+p_\alpha, \tag{17}$$

where $K=0,1,2,\cdots;p_\alpha=7,11,13,17,19,23,29,31$.

All the even numbers N greater than 16 can be expressed as

$$N=30m+h, \tag{18}$$

where $m=0,1,2,\cdots;h=18,20,22,24,26,28,30,32,34,36,38,40,42,44,46$.

From (17) and (18) we have

$$N=30m+h=E_{p_1}(K_1)+E_{p_2}(K_2). \tag{19}$$

From (19) we have

$$m=K_1+K_2,\quad h\equiv p_1+p_2(\text{mod}\ \ 30). \tag{20}$$

$m=K_1+K_2$ is called Yu's mathematical problem, namely, integer m greater than 1 is the sum of primes K_1 and K_2. To prove $m=K_1+K_2$ is transformed into studying $N=E_{p_1}(K_1)+E_{p_2}(K_2)$.

If $3|N$ from (15) we have $J_2(30)=6$. From (17) and (19) we have six subequations: $6=2+2+2$

$$N = 30m + 18 = E_7(K_1) + E_{11}(K_2) = E_{17}(K_1) + E_{31}(K_2)$$
$$= E_{19}(K_1) + E_{29}(K_2), \tag{21}$$
$$N = 30m + 24 = E_7(K_1) + E_{17}(K_2) = E_{11}(K_1) + E_{13}(K_2)$$
$$= E_{23}(K_1) + E_{31}(K_2), \tag{22}$$
$$N = 30m + 36 = E_7(K_1) + E_{29}(K_2) = E_{13}(K_1) + E_{23}(K_2)$$
$$= E_{17}(K_1) + E_{19}(K_2), \tag{23}$$
$$N = 30m + 42 = E_{11}(K_1) + E_{31}(K_2) = E_{13}(K_1) + E_{29}(K_2)$$
$$= E_{19}(K_1) + E_{23}(K_2). \tag{24}$$

If $5|N$ from (15) we have $J_2(30) = 4$. From (17) and (19) we have four subequations: $4 = 2 + 2$

$$N = 30m+20 = E_7(K_1)+E_{13}(K_2) = E_{19}(K_1)+E_{31}(K_2), \tag{25}$$
$$N = 30m + 40 = E_{11}(K_1) + E_{29}(K_2) = E_{17}(K_1) + E_{23}(K_2). \tag{26}$$

If $3, 5 \not| N$ from (15) we have $J_2(30) = 3$. From (17) and (19) we have three subequations: $3 = 2 + 1$

$$N = 30m+22 = E_{23}(K_1)+E_{29}(K_2) = E_{11}(K_1)+E_{11}(K_2), \tag{27}$$
$$N = 30m+26 = E_7(K_1)+E_{19}(K_2) = E_{13}(K_1)+E_{13}(K_2), \tag{28}$$
$$N = 30m+28 = E_{11}(K_1)+E_{17}(K_2) = E_{29}(K_1)+E_{29}(K_2), \tag{29}$$
$$N = 30m+32 = E_{13}(K_1)+E_{19}(K_2) = E_{31}(K_1)+E_{31}(K_2), \tag{30}$$
$$N = 30m+34 = E_{11}(K_1)+E_{23}(K_2) = E_{17}(K_1)+E_{17}(K_2), \tag{31}$$
$$N = 30m+38 = E_7(K_1)+E_{31}(K_2) = E_{19}(K_1)+E_{19}(K_2), \tag{32}$$
$$N = 30m+44 = E_{13}(K_1)+E_{31}(K_2) = E_7(K_1)+E_7(K_2), \tag{33}$$
$$N = 30m+46 = E_{17}(K_1)+E_{29}(K_2) = E_{23}(K_1)+E_{23}(K_2). \tag{34}$$

If $3, 5|N$ from (15) we have $J_2(30) = 8$. From (17) and (19) we have eight subequations: $8 = 2 + 2 + 2 + 2$

$$N = 30m + 30 = E_7(K_1) + E_{23}(K_2) = E_{11}(K_1) + E_{19}(K_2)$$
$$= E_{13}(K_1) + E_{17}(K_2) = E_{29}(K_1) + E_{31}(K_2). \tag{35}$$

We can prove the binary Goldbach's theorem using only thirty subequations:

$$N = 30m + 18 = E_7(K_1) + E_{11}(K_2), \quad N = 30m + 20 = E_7(K_1) + E_{13}(K_2),$$
$$N = 30m + 22 = E_{23}(K_1) + E_{29}(K_2), \quad N = 30m + 24 = E_7(K_1) + E_{17}(K_2),$$
$$N = 30m + 26 = E_7(K_1) + E_{19}(K_2), \quad N = 30m + 28 = E_{11}(K_1) + E_{17}(K_2),$$
$$N = 30m + 30 = E_7(K_1) + E_{23}(K_2), \quad N = 30m + 32 = E_{13}(K_1) + E_{19}(K_2),$$
$$N = 30m + 34 = E_{11}(K_1) + E_{23}(K_2), \quad N = 30m + 36 = E_7(K_1) + E_{29}(K_2),$$
$$N = 30m + 38 = E_7(K_1) + E_{31}(K_2), \quad N = 30m + 40 = E_{11}(K_1) + E_{29}(K_2),$$
$$N = 30m + 42 = E_{19}(K_1) + E_{23}(K_2), \quad N = 30m + 44 = E_{13}(K_1) + E_{31}(K_2),$$
$$N = 30m + 46 = E_{17}(K_1) + E_{29}(K_2). \tag{36}$$

For every equation we have the arithmetic function

$$J_2(\omega>30)=\prod_{7\le p\le p_i}(p-2)\prod_{p|N,7\le p\le p_i}\frac{p-1}{p-2}\neq 0. \tag{37}$$

Since $J_2(\omega>30)\to\infty$ as $\omega\to\infty$, we prove Yu's mathematical problem. We prove also the binary Goldbach's theorem using the partial primes.

Substituting (37) into (7) we have the best asymptotic formula

$$\pi_2(N,2)=\sum_{m=K_1+K_2}1=\sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)}1\sim$$

$$\frac{15}{32}\prod_{7\le p\le p_i}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N}{\log^2 N}. \tag{38}$$

Corollary 2. Let $p_i=7$ and $\omega=210$. From (1) we have

$$E_{p_\alpha}(K)=210K+p_\alpha, \tag{39}$$

where $K=0,1,2,\cdots,(210,p_\alpha)=1,p_\alpha=11,41,71,101,131,191;$
$13,43,73,103,163,193;17,47,107,137,167,197;19,79,109,139,169,199;$
$23,53,83,113,143,173;29,59,89,149,179,209;31,61,121,151,181,211;$
$37,67,97,127,157,187.$

All the even numbers N greater than 38 can be expressed as

$$N=210m+h, \tag{40}$$

where $m=0,1,2,\cdots,h=40,42,\cdots,248.$

From (39) and (40) we have

$$N=210m+h=E_{p_1}(K_1)+E_{p_2}(K_2). \tag{41}$$

From (41) we have

$$m=K_1+K_2,\quad h\equiv p_1+p_2(\text{mod }\ 210). \tag{42}$$

From (39), (41) and (42) we have the 2304 subequations as follws.

$40\equiv 11+29\equiv 41+209\equiv 191+59\equiv 71+179\equiv 101+149;$
$40\equiv 17+23\equiv 197+53\equiv 167+83\equiv 107+143\equiv 137+113.$

$42 \equiv 11 + 31 \equiv 41 + 211 \equiv 191 + 61 \equiv 71 + 181 \equiv 101 + 151 \equiv 131 + 121;$
$42 \equiv 13 + 29 \equiv 43 + 209 \equiv 193 + 59 \equiv 73 + 179 \equiv 103 + 149 \equiv 163 + 89;$
$42 \equiv 19 + 23 \equiv 199 + 53 \equiv 79 + 173 \equiv 169 + 83 \equiv 109 + 143 \equiv 139 + 113.$
$44 \equiv 13 + 31 \equiv 43 + 211 \equiv 193 + 61 \equiv 73 + 181 \equiv 103 + 151;$
$44 \equiv 67 + 187 \equiv 97 + 157 \equiv 127 + 127.$
$46 \equiv 17 + 29 \equiv 47 + 209 \equiv 197 + 59 \equiv 167 + 89 \equiv 107 + 149;$
$46 \equiv 23 + 23 \equiv 83 + 173 \equiv 113 + 143.$
$48 \equiv 11 + 37 \equiv 191 + 67 \equiv 71 + 187 \equiv 101 + 157 \equiv 131 + 127;$
$48 \equiv 17 + 31 \equiv 47 + 211 \equiv 197 + 61 \equiv 107 + 151 \equiv 137 + 121;$
$48 \equiv 19 + 29 \equiv 199 + 59 \equiv 79 + 179 \equiv 169 + 89 \equiv 109 + 149.$
$50 \equiv 13 + 37 \equiv 193 + 67 \equiv 73 + 187 \equiv 163 + 97 \equiv 103 + 157;$
$50 \equiv 19 + 31 \equiv 199 + 61 \equiv 79 + 181 \equiv 109 + 151 \equiv 139 + 121.$
$52 \equiv 23 + 29 \equiv 53 + 209 \equiv 83 + 179 \equiv 173 + 89 \equiv 113 + 149.$
$52 \equiv 11 + 41 \equiv 71 + 191 \equiv 131 + 131.$
$54 \equiv 11 + 43 \equiv 41 + 13 \equiv 71 + 193 \equiv 191 + 73 \equiv 101 + 163;$
$54 \equiv 17 + 37 \equiv 197 + 67 \equiv 167 + 97 \equiv 107 + 157 \equiv 137 + 127;$
$54 \equiv 23 + 31 \equiv 53 + 211 \equiv 83 + 181 \equiv 113 + 151 \equiv 143 + 121.$
$56 \equiv 13 + 43 \equiv 73 + 193 \equiv 103 + 163;$
$56 \equiv 19 + 37 \equiv 199 + 67 \equiv 79 + 187 \equiv 169 + 97 \equiv 109 + 157 \equiv 139 + 127.$
$58 \equiv 11 + 47 \equiv 41 + 17 \equiv 71 + 197 \equiv 101 + 167 \equiv 131 + 137;$
$58 \equiv 29 + 29 \equiv 59 + 209 \equiv 89 + 179.$
$60 \equiv 13 + 47 \equiv 43 + 17 \equiv 73 + 197 \equiv 103 + 167 \equiv 163 + 107;$
$60 \equiv 19 + 41 \equiv 199 + 71 \equiv 79 + 191 \equiv 169 + 101 \equiv 139 + 131;$
$60 \equiv 23 + 37 \equiv 83 + 187 \equiv 173 + 97 \equiv 113 + 157 \equiv 143 + 127;$
$60 \equiv 29 + 31 \equiv 59 + 211 \equiv 209 + 61 \equiv 89 + 181 \equiv 149 + 121.$
$62 \equiv 19 + 43 \equiv 199 + 73 \equiv 79 + 193 \equiv 169 + 103 \equiv 109 + 163;$
$62 \equiv 31 + 31 \equiv 61 + 211 \equiv 121 + 151.$
$64 \equiv 11 + 53 \equiv 41 + 23 \equiv 191 + 83 \equiv 101 + 173 \equiv 131 + 143;$
$64 \equiv 17 + 47 \equiv 107 + 167 \equiv 137 + 137.$
$66 \equiv 13 + 53 \equiv 43 + 23 \equiv 193 + 83 \equiv 103 + 173 \equiv 163 + 113;$
$66 \equiv 19 + 47 \equiv 79 + 197 \equiv 169 + 107 \equiv 109 + 167 \equiv 139 + 137;$
$66 \equiv 29 + 37 \equiv 209 + 67 \equiv 89 + 187 \equiv 179 + 97 \equiv 149 + 127.$
$68 \equiv 31 + 37 \equiv 211 + 67 \equiv 181 + 97 \equiv 121 + 157 \equiv 151 + 127;$
$68 \equiv 79 + 199 \equiv 109 + 169 \equiv 139 + 139.$
$70 \equiv 11 + 59 \equiv 41 + 29 \equiv 71 + 209 \equiv 191 + 89 \equiv 101 + 179 \equiv 131 + 149;$
$70 \equiv 17 + 53 \equiv 47 + 23 \equiv 197 + 83 \equiv 107 + 173 \equiv 167 + 113 \equiv 137 + 143.$
$72 \equiv 11 + 61 \equiv 41 + 31 \equiv 71 + 211 \equiv 101 + 181 \equiv 131 + 151;$
$72 \equiv 13 + 59 \equiv 43 + 29 \equiv 73 + 209 \equiv 103 + 179 \equiv 193 + 89;$
$72 \equiv 19 + 53 \equiv 109 + 173 \equiv 169 + 113 \equiv 139 + 143 \equiv 199 + 83.$

$74 \equiv 13 + 61 \equiv 43 + 31 \equiv 73 + 211 \equiv 103 + 181 \equiv 163 + 121;$
$74 \equiv 37 + 37 \equiv 97 + 187 \equiv 127 + 157.$
$76 \equiv 17 + 59 \equiv 47 + 29 \equiv 197 + 89 \equiv 107 + 179 \equiv 137 + 149;$
$76 \equiv 23 + 53 \equiv 113 + 173 \equiv 143 + 143.$
$78 \equiv 11 + 67 \equiv 41 + 37 \equiv 191 + 97 \equiv 101 + 187 \equiv 131 + 157;$
$78 \equiv 17 + 61 \equiv 47 + 31 \equiv 107 + 181 \equiv 167 + 121 \equiv 137 + 151;$
$78 \equiv 19 + 59 \equiv 79 + 209 \equiv 199 + 89 \equiv 109 + 179 \equiv 139 + 149.$
$80 \equiv 13 + 67 \equiv 43 + 37 \equiv 193 + 97 \equiv 103 + 187 \equiv 163 + 127;$
$80 \equiv 19 + 61 \equiv 79 + 211 \equiv 109 + 181 \equiv 169 + 121 \equiv 139 + 151.$
$82 \equiv 11 + 71 \equiv 41 + 41 \equiv 101 + 191.$
$82 \equiv 23 + 59 \equiv 53 + 29 \equiv 83 + 209 \equiv 113 + 179 \equiv 143 + 149.$
$84 \equiv 11 + 73 \equiv 71 + 13 \equiv 41 + 43 \equiv 109 + 193 \equiv 191 + 103 \equiv 131 + 163;$
$84 \equiv 17 + 67 \equiv 47 + 37 \equiv 107 + 187 \equiv 167 + 127 \equiv 137 + 157 \equiv 197 + 97.$
$84 \equiv 23 + 61 \equiv 53 + 31 \equiv 83 + 211 \equiv 113 + 181 \equiv 173 + 121 \equiv 143 + 151.$
$86 \equiv 19 + 67 \equiv 109 + 187 \equiv 139 + 157 \equiv 169 + 127 \equiv 199 + 97;$
$86 \equiv 13 + 73 \equiv 43 + 43 \equiv 103 + 193.$
$88 \equiv 17 + 71 \equiv 47 + 41 \equiv 107 + 191 \equiv 167 + 131 \equiv 197 + 101;$
$88 \equiv 29 + 59 \equiv 89 + 209 \equiv 149 + 149.$
$90 \equiv 11 + 79 \equiv 71 + 19 \equiv 101 + 199 \equiv 131 + 169 \equiv 191 + 109;$
$90 \equiv 17 + 73 \equiv 47 + 43 \equiv 107 + 193 \equiv 137 + 163 \equiv 197 + 103;$
$90 \equiv 23 + 67 \equiv 53 + 37 \equiv 113 + 187 \equiv 143 + 157 \equiv 173 + 127;$
$90 \equiv 29 + 61 \equiv 59 + 31 \equiv 89 + 211 \equiv 149 + 151 \equiv 179 + 121.$
$92 \equiv 13 + 79 \equiv 73 + 19 \equiv 103 + 199 \equiv 163 + 139 \equiv 193 + 109;$
$92 \equiv 31 + 61 \equiv 121 + 181 \equiv 151 + 151.$
$94 \equiv 11 + 83 \equiv 71 + 23 \equiv 41 + 53 \equiv 131 + 173 \equiv 191 + 113;$
$94 \equiv 47 + 47 \equiv 107 + 197 \equiv 137 + 167.$
$96 \equiv 13 + 83 \equiv 43 + 53 \equiv 73 + 23 \equiv 163 + 143 \equiv 193 + 113;$
$96 \equiv 17 + 79 \equiv 107 + 199 \equiv 137 + 169 \equiv 167 + 139 \equiv 197 + 109;$
$96 \equiv 29 + 67 \equiv 59 + 37 \equiv 149 + 157 \equiv 179 + 127 \equiv 209 + 97.$
$98 \equiv 31 + 67 \equiv 61 + 37 \equiv 121 + 187 \equiv 151 + 157 \equiv 181 + 127 \equiv 211 + 97;$
$98 \equiv 19 + 79 \equiv 109 + 199 \equiv 139 + 169.$
$100 \equiv 11 + 89 \equiv 41 + 59 \equiv 71 + 29 \equiv 101 + 209 \equiv 131 + 179;$
$100 \equiv 17 + 83 \equiv 47 + 53 \equiv 137 + 173 \equiv 167 + 143 \equiv 197 + 113.$
$102 \equiv 13 + 89 \equiv 73 + 29 \equiv 43 + 59 \equiv 103 + 209 \equiv 163 + 149;$
$102 \equiv 19 + 83 \equiv 79 + 23 \equiv 139 + 173 \equiv 169 + 143 \equiv 199 + 113;$
$102 \equiv 31 + 71 \equiv 61 + 41 \equiv 121 + 191 \equiv 181 + 131 \equiv 211 + 101.$
$104 \equiv 31 + 73 \equiv 61 + 43 \equiv 121 + 193 \equiv 151 + 163 \equiv 211 + 103;$
$104 \equiv 37 + 67 \equiv 127 + 187 \equiv 157 + 157.$
$106 \equiv 17 + 89 \equiv 47 + 59 \equiv 107 + 209 \equiv 137 + 179 \equiv 167 + 149;$

$106 \equiv 23+83 \equiv 53+53 \equiv 143+173.$
$108 \equiv 11+97 \equiv 71+37 \equiv 41+67 \equiv 131+187 \equiv 191+127;$
$108 \equiv 19+89 \equiv 79+29 \equiv 109+209 \equiv 139+179 \equiv 169+149;$
$108 \equiv 47+61 \equiv 107+211 \equiv 137+181 \equiv 167+151 \equiv 197+121.$
$110 \equiv 13+97 \equiv 73+37 \equiv 43+67 \equiv 163+157 \equiv 193+127;$
$110 \equiv 31+79 \equiv 121+199 \equiv 151+169 \equiv 181+139 \equiv 211+109.$
$112 \equiv 23+89 \equiv 83+29 \equiv 53+59 \equiv 113+209 \equiv 143+179 \equiv 173+149;$
$112 \equiv 11+101 \equiv 41+71 \equiv 131+191.$
$114 \equiv 11+103 \equiv 41+73 \equiv 71+43 \equiv 101+13 \equiv 131+193;$
$114 \equiv 17+97 \equiv 47+67 \equiv 137+187 \equiv 167+157 \equiv 197+127;$
$114 \equiv 31+83 \equiv 61+53 \equiv 151+173 \equiv 181+143 \equiv 211+113.$
$116 \equiv 19+97 \equiv 79+37 \equiv 139+187 \equiv 169+157 \equiv 199+127;$
$116 \equiv 13+103 \equiv 43+73 \equiv 163+163.$
$118 \equiv 11+107 \equiv 71+47 \equiv 101+17 \equiv 131+197 \equiv 191+137;$
$118 \equiv 29+89 \equiv 59+59 \equiv 149+179.$
$120 \equiv 11+109 \equiv 41+79 \equiv 101+19 \equiv 131+199 \equiv 191+139;$
$120 \equiv 13+107 \equiv 73+47 \equiv 103+17 \equiv 163+167 \equiv 193+137;$
$120 \equiv 31+89 \equiv 61+59 \equiv 121+209 \equiv 151+179 \equiv 181+149;$
$120 \equiv 23+97 \equiv 53+67 \equiv 83+37 \equiv 143+187 \equiv 173+157.$
$122 \equiv 13+109 \equiv 43+79 \equiv 103+19 \equiv 163+169 \equiv 193+139;$
$122 \equiv 61+61 \equiv 121+211 \equiv 151+181.$
$124 \equiv 11+113 \equiv 41+83 \equiv 71+53 \equiv 101+23 \equiv 191+143;$
$124 \equiv 17+107 \equiv 137+197 \equiv 167+167.$
$126 \equiv 13+113 \equiv 43+83 \equiv 73+53 \equiv 103+23 \equiv 163+173 \equiv 193+143;$
$126 \equiv 17+109 \equiv 47+79 \equiv 107+19 \equiv 137+199 \equiv 167+169 \equiv 197+139;$
$126 \equiv 29+97 \equiv 59+67 \equiv 89+37 \equiv 149+187 \equiv 179+157 \equiv 209+127.$
$128 \equiv 31+97 \equiv 61+67 \equiv 151+187 \equiv 181+157 \equiv 211+127;$
$128 \equiv 19+109 \equiv 139+199 \equiv 169+169.$
$130 \equiv 17+113 \equiv 47+83 \equiv 107+23 \equiv 167+173 \equiv 197+143;$
$130 \equiv 29+101 \equiv 59+71 \equiv 89+41 \equiv 149+191 \equiv 209+131.$
$132 \equiv 11+121 \equiv 71+61 \equiv 101+31 \equiv 131+211 \equiv 191+151;$
$132 \equiv 19+113 \equiv 79+53 \equiv 109+23 \equiv 169+173 \equiv 199+143;$
$132 \equiv 29+103 \equiv 59+73 \equiv 89+43 \equiv 149+193 \equiv 179+163.$
$134 \equiv 13+121 \equiv 73+61 \equiv 103+31 \equiv 163+181 \equiv 193+151.$
$134 \equiv 37+97 \equiv 67+67 \equiv 157+187.$
$136 \equiv 29+107 \equiv 89+47 \equiv 149+197 \equiv 179+167 \equiv 209+137;$
$136 \equiv 23+113 \equiv 53+83 \equiv 173+173.$
$138 \equiv 11+127 \equiv 41+97 \equiv 71+67 \equiv 101+37 \equiv 191+157;$
$138 \equiv 17+121 \equiv 107+31 \equiv 137+211 \equiv 167+181 \equiv 197+151;$

$138 \equiv 29 + 109 \equiv 59 + 79 \equiv 149 + 199 \equiv 179 + 169 \equiv 209 + 139.$
$140 \equiv 13 + 127 \equiv 43 + 97 \equiv 73 + 67 \equiv 103 + 37 \equiv 163 + 187 \equiv 193 + 157;$
$140 \equiv 19 + 121 \equiv 79 + 61 \equiv 109 + 31 \equiv 139 + 211 \equiv 169 + 151 \equiv 199 + 151.$
$142 \equiv 29 + 113 \equiv 59 + 83 \equiv 89 + 53 \equiv 179 + 173 \equiv 209 + 143;$
$142 \equiv 11 + 131 \equiv 41 + 101 \equiv 71 + 71.$
$144 \equiv 13 + 131 \equiv 43 + 101 \equiv 73 + 71 \equiv 103 + 41 \equiv 163 + 191;$
$144 \equiv 17 + 127 \equiv 47 + 97 \equiv 107 + 37 \equiv 167 + 187 \equiv 197 + 157;$
$144 \equiv 23 + 121 \equiv 83 + 61 \equiv 113 + 31 \equiv 143 + 211 \equiv 173 + 181.$
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$146 \equiv 43 + 103 \equiv 73 + 73 \equiv 163 + 193.$
$148 \equiv 11 + 137 \equiv 41 + 107 \equiv 101 + 47 \equiv 131 + 17 \equiv 191 + 167;$
$148 \equiv 59 + 89 \equiv 149 + 209 \equiv 179 + 179.$
$150 \equiv 11 + 139 \equiv 41 + 109 \equiv 71 + 79 \equiv 131 + 19 \equiv 191 + 169;$
$150 \equiv 13 + 137 \equiv 43 + 107 \equiv 103 + 47 \equiv 163 + 197 \equiv 193 + 167;$
$150 \equiv 23 + 127 \equiv 53 + 97 \equiv 83 + 67 \equiv 113 + 37 \equiv 173 + 187;$
$150 \equiv 29 + 121 \equiv 89 + 61 \equiv 149 + 211 \equiv 179 + 181 \equiv 209 + 151.$
$152 \equiv 13 + 139 \equiv 43 + 109 \equiv 93 + 79 \equiv 163 + 199 \equiv 193 + 169;$
$152 \equiv 31 + 121 \equiv 151 + 211 \equiv 181 + 181.$
$154 \equiv 11 + 143 \equiv 41 + 113 \equiv 71 + 83 \equiv 101 + 53 \equiv 131 + 23 \equiv 191 + 173;$
$154 \equiv 17 + 137 \equiv 47 + 107 \equiv 167 + 197.$
$156 \equiv 13 + 143 \equiv 43 + 113 \equiv 73 + 83 \equiv 103 + 53 \equiv 193 + 173;$
$156 \equiv 17 + 139 \equiv 47 + 109 \equiv 137 + 19 \equiv 167 + 199 \equiv 197 + 169;$
$156 \equiv 29 + 127 \equiv 59 + 97 \equiv 89 + 67 \equiv 179 + 187 \equiv 209 + 157.$
$158 \equiv 31 + 127 \equiv 97 + 61 \equiv 121 + 37 \equiv 181 + 187 \equiv 211 + 157;$
$158 \equiv 19 + 139 \equiv 79 + 79 \equiv 169 + 199.$
$160 \equiv 11 + 149 \equiv 71 + 89 \equiv 101 + 59 \equiv 131 + 29 \equiv 191 + 179;$
$160 \equiv 17 + 143 \equiv 47 + 113 \equiv 107 + 53 \equiv 137 + 23 \equiv 197 + 173.$
$162 \equiv 11 + 151 \equiv 41 + 121 \equiv 101 + 61 \equiv 131 + 31 \equiv 191 + 181;$
$162 \equiv 13 + 149 \equiv 73 + 89 \equiv 103 + 59 \equiv 163 + 209 \equiv 193 + 179;$
$162 \equiv 19 + 143 \equiv 79 + 83 \equiv 109 + 53 \equiv 139 + 23 \equiv 199 + 173.$
$164 \equiv 13 + 151 \equiv 43 + 121 \equiv 103 + 61 \equiv 163 + 211 \equiv 193 + 181;$
$164 \equiv 37 + 127 \equiv 67 + 97 \equiv 187 + 187.$
$166 \equiv 17 + 149 \equiv 107 + 59 \equiv 137 + 29 \equiv 167 + 209 \equiv 197 + 179;$
$166 \equiv 23 + 143 \equiv 53 + 113 \equiv 83 + 83.$
$168 \equiv 11 + 157 \equiv 41 + 127 \equiv 71 + 97 \equiv 101 + 67 \equiv 131 + 37 \equiv 191 + 187;$
$168 \equiv 17 + 151 \equiv 47 + 121 \equiv 107 + 61 \equiv 137 + 31 \equiv 167 + 211 \equiv 197 + 181;$
$168 \equiv 19 + 149 \equiv 79 + 89 \equiv 109 + 59 \equiv 139 + 29 \equiv 169 + 209 \equiv 199 + 179.$
$170 \equiv 13 + 157 \equiv 43 + 127 \equiv 73 + 97 \equiv 103 + 67 \equiv 193 + 181;$
$170 \equiv 19 + 151 \equiv 109 + 61 \equiv 139 + 31 \equiv 169 + 211 \equiv 199 + 181.$

$172 \equiv 23 + 149 \equiv 83 + 89 \equiv 113 + 59 \equiv 143 + 29 \equiv 173 + 209;$
$172 \equiv 41 + 131 \equiv 71 + 101 \equiv 191 + 191.$
$174 \equiv 11 + 163 \equiv 71 + 103 \equiv 101 + 73 \equiv 131 + 43 \equiv 191 + 193;$
$174 \equiv 17 + 157 \equiv 47 + 127 \equiv 107 + 67 \equiv 137 + 37 \equiv 197 + 187;$
$174 \equiv 23 + 151 \equiv 53 + 121 \equiv 113 + 61 \equiv 143 + 31 \equiv 173 + 211.$
$176 \equiv 19 + 157 \equiv 79 + 97 \equiv 109 + 67 \equiv 139 + 37 \equiv 199 + 187;$
$176 \equiv 13 + 163 \equiv 73 + 103 \equiv 193 + 193.$
$178 \equiv 11 + 167 \equiv 41 + 137 \equiv 107 + 71 \equiv 131 + 47 \equiv 191 + 197;$
$178 \equiv 29 + 149 \equiv 89 + 89 \equiv 179 + 209.$
$180 \equiv 11 + 169 \equiv 41 + 139 \equiv 71 + 109 \equiv 101 + 79 \equiv 191 + 199;$
$180 \equiv 13 + 167 \equiv 43 + 137 \equiv 73 + 107 \equiv 163 + 17 \equiv 193 + 197;$
$180 \equiv 23 + 157 \equiv 53 + 127 \equiv 83 + 97 \equiv 113 + 67 \equiv 143 + 37;$
$180 \equiv 29 + 151 \equiv 59 + 121 \equiv 149 + 31 \equiv 179 + 211 \equiv 209 + 181.$
$182 \equiv 13 + 169 \equiv 43 + 139 \equiv 73 + 109 \equiv 103 + 79 \equiv 163 + 19 \equiv 193 + 199;$
$182 \equiv 31 + 151 \equiv 61 + 121 \equiv 181 + 211.$
$184 \equiv 11 + 173 \equiv 41 + 143 \equiv 71 + 113 \equiv 101 + 83 \equiv 131 + 53;$
$184 \equiv 17 + 167 \equiv 47 + 137 \equiv 197 + 197.$
$186 \equiv 13 + 173 \equiv 43 + 143 \equiv 73 + 113 \equiv 103 + 83 \equiv 163 + 23;$
$186 \equiv 17 + 169 \equiv 47 + 139 \equiv 107 + 79 \equiv 167 + 19 \equiv 197 + 199;$
$186 \equiv 29 + 157 \equiv 59 + 127 \equiv 89 + 97 \equiv 149 + 37 \equiv 209 + 187.$
$188 \equiv 31 + 157 \equiv 61 + 127 \equiv 121 + 67 \equiv 151 + 37 \equiv 211 + 187;$
$188 \equiv 19 + 169 \equiv 79 + 109 \equiv 199 + 199.$
$190 \equiv 11 + 179 \equiv 41 + 149 \equiv 101 + 89 \equiv 131 + 59 \equiv 191 + 209;$
$190 \equiv 17 + 173 \equiv 47 + 143 \equiv 107 + 83 \equiv 137 + 53 \equiv 167 + 23.$
$192 \equiv 11 + 181 \equiv 41 + 151 \equiv 71 + 121 \equiv 131 + 61 \equiv 191 + 211;$
$192 \equiv 13 + 179 \equiv 43 + 149 \equiv 103 + 89 \equiv 163 + 29 \equiv 193 + 209;$
$192 \equiv 23 + 169 \equiv 53 + 139 \equiv 83 + 109 \equiv 113 + 79 \equiv 173 + 19.$
$194 \equiv 13 + 181 \equiv 43 + 151 \equiv 73 + 121 \equiv 163 + 31 \equiv 193 + 211;$
$194 \equiv 37 + 157 \equiv 69 + 127 \equiv 97 + 97.$
$196 \equiv 17 + 179 \equiv 47 + 149 \equiv 107 + 89 \equiv 167 + 29 \equiv 197 + 209 \equiv 137 + 59;$
$196 \equiv 23 + 173 \equiv 53 + 143 \equiv 83 + 113.$
$198 \equiv 11 + 187 \equiv 41 + 157 \equiv 71 + 127 \equiv 101 + 97 \equiv 131 + 67;$
$198 \equiv 17 + 181 \equiv 47 + 151 \equiv 137 + 61 \equiv 167 + 31 \equiv 197 + 211;$
$198 \equiv 19 + 179 \equiv 109 + 89 \equiv 139 + 59 \equiv 169 + 29 \equiv 199 + 209.$
$200 \equiv 13 + 187 \equiv 43 + 157 \equiv 73 + 127 \equiv 103 + 97 \equiv 163 + 37;$
$200 \equiv 19 + 181 \equiv 79 + 121 \equiv 139 + 61 \equiv 169 + 31 \equiv 199 + 211.$
$202 \equiv 23 + 179 \equiv 53 + 149 \equiv 113 + 89 \equiv 143 + 59 \equiv 173 + 29;$
$202 \equiv 11 + 191 \equiv 71 + 131 \equiv 101 + 101.$
$204 \equiv 11 + 193 \equiv 41 + 163 \equiv 101 + 103 \equiv 131 + 73 \equiv 191 + 13;$

$204 \equiv 17 + 187 \equiv 47 + 157 \equiv 107 + 97 \equiv 137 + 67 \equiv 167 + 37;$
$204 \equiv 23 + 181 \equiv 53 + 151 \equiv 83 + 121 \equiv 143 + 61 \equiv 173 + 31.$
$206 \equiv 19 + 187 \equiv 79 + 127 \equiv 109 + 97 \equiv 139 + 67 \equiv 169 + 37;$
$206 \equiv 13 + 193 \equiv 43 + 163 \equiv 103 + 103.$
$208 \equiv 11 + 197 \equiv 41 + 167 \equiv 71 + 137 \equiv 101 + 107 \equiv 191 + 17;$
$208 \equiv 29 + 179 \equiv 59 + 149 \equiv 209 + 209.$
$210 \equiv 11 + 199 \equiv 41 + 169 \equiv 71 + 139 \equiv 101 + 109 \equiv 131 + 79 \equiv 191 + 19;$
$210 \equiv 13 + 197 \equiv 43 + 167 \equiv 73 + 137 \equiv 103 + 107 \equiv 163 + 47 \equiv 193 + 17;$
$210 \equiv 23 + 187 \equiv 53 + 157 \equiv 83 + 127 \equiv 113 + 97 \equiv 143 + 67 \equiv 173 + 37;$
$210 \equiv 29 + 181 \equiv 59 + 151 \equiv 89 + 121 \equiv 149 + 61 \equiv 179 + 31 \equiv 209 + 211.$
$212 \equiv 13 + 199 \equiv 43 + 169 \equiv 73 + 139 \equiv 103 + 109 \equiv 193 + 19;$
$212 \equiv 31 + 181 \equiv 61 + 151 \equiv 211 + 211.$
$214 \equiv 23 + 191 \equiv 83 + 131 \equiv 113 + 103 \equiv 143 + 71 \equiv 173 + 41;$
$214 \equiv 17 + 197 \equiv 47 + 167 \equiv 107 + 107.$
$216 \equiv 17 + 199 \equiv 47 + 169 \equiv 107 + 109 \equiv 137 + 79 \equiv 197 + 19;$
$216 \equiv 23 + 193 \equiv 53 + 163 \equiv 113 + 103 \equiv 143 + 73 \equiv 173 + 43;$
$216 \equiv 29 + 187 \equiv 59 + 157 \equiv 89 + 127 \equiv 149 + 67 \equiv 179 + 37.$
$218 \equiv 31 + 187 \equiv 61 + 157 \equiv 121 + 97 \equiv 151 + 67 \equiv 181 + 37;$
$218 \equiv 19 + 199 \equiv 79 + 139 \equiv 109 + 109.$
$220 \equiv 11 + 209 \equiv 41 + 179 \equiv 71 + 149 \equiv 131 + 89 \equiv 191 + 29;$
$220 \equiv 23 + 197 \equiv 53 + 167 \equiv 83 + 137 \equiv 113 + 107 \equiv 173 + 47.$
$222 \equiv 11 + 211 \equiv 41 + 181 \equiv 71 + 151 \equiv 101 + 121 \equiv 191 + 31;$
$222 \equiv 13 + 209 \equiv 43 + 179 \equiv 73 + 149 \equiv 163 + 59 \equiv 193 + 29;$
$222 \equiv 23 + 199 \equiv 53 + 169 \equiv 83 + 139 \equiv 113 + 109 \equiv 143 + 79.$
$224 \equiv 13 + 211 \equiv 43 + 181 \equiv 73 + 151 \equiv 103 + 121 \equiv 163 + 61 \equiv 193 + 31;$
$224 \equiv 37 + 187 \equiv 67 + 157 \equiv 97 + 127.$
$226 \equiv 17 + 209 \equiv 47 + 179 \equiv 137 + 89 \equiv 167 + 59 \equiv 197 + 29;$
$226 \equiv 53 + 173 \equiv 83 + 143 \equiv 113 + 113.$
$228 \equiv 17 + 211 \equiv 47 + 181 \equiv 107 + 121 \equiv 167 + 61 \equiv 197 + 31;$
$228 \equiv 19 + 209 \equiv 79 + 149 \equiv 139 + 89 \equiv 169 + 59 \equiv 199 + 29;$
$228 \equiv 37 + 191 \equiv 97 + 131 \equiv 127 + 101 \equiv 157 + 71 \equiv 187 + 41.$
$230 \equiv 19 + 211 \equiv 79 + 151 \equiv 109 + 121 \equiv 169 + 61 \equiv 199 + 31;$
$230 \equiv 37 + 193 \equiv 67 + 163 \equiv 127 + 103 \equiv 157 + 73 \equiv 187 + 43.$
$232 \equiv 23 + 209 \equiv 53 + 179 \equiv 83 + 149 \equiv 143 + 89 \equiv 173 + 59;$
$232 \equiv 41 + 191 \equiv 101 + 131 \equiv 11 + 11.$
$234 \equiv 23 + 211 \equiv 53 + 181 \equiv 83 + 151 \equiv 113 + 121 \equiv 173 + 61;$
$234 \equiv 37 + 197 \equiv 67 + 167 \equiv 97 + 137 \equiv 127 + 107 \equiv 187 + 47;$
$234 \equiv 41 + 193 \equiv 191 + 43 \equiv 71 + 163 \equiv 131 + 103 \equiv 11 + 23.$
$236 \equiv 37 + 199 \equiv 67 + 169 \equiv 97 + 139 \equiv 127 + 109 \equiv 157 + 79;$

$236 \equiv 43+193 \equiv 73+163 \equiv 13+13.$
$238 \equiv 41+197 \equiv 71+167 \equiv 101+137 \equiv 131+107 \equiv 191+47 \equiv 11+17;$
$238 \equiv 29+209 \equiv 59+179 \equiv 89+149.$
$240 \equiv 29+211 \equiv 59+181 \equiv 89+151 \equiv 179+61 \equiv 209+31;$
$240 \equiv 41+199 \equiv 71+169 \equiv 101+139 \equiv 131+109 \equiv 11+19;$
$240 \equiv 43+197 \equiv 73+167 \equiv 103+137 \equiv 193+47 \equiv 13+17;$
$240 \equiv 23+187 \equiv 173+67 \equiv 83+157 \equiv 143+87 \equiv 113+127.$
$242 \equiv 43+199 \equiv 73+169 \equiv 163+97 \equiv 103+139 \equiv 13+19;$
$242 \equiv 31+211 \equiv 61+181 \equiv 121+121.$
$244 \equiv 53+191 \equiv 173+71 \equiv 143+101 \equiv 113+131 \equiv 11+13;$
$244 \equiv 47+197 \equiv 107+137 \equiv 17+17.$
$246 \equiv 37+209 \equiv 181+59 \equiv 67+179 \equiv 157+89 \equiv 97+149;$
$246 \equiv 47+199 \equiv 167+79 \equiv 107+139 \equiv 137+109 \equiv 17+19;$
$246 \equiv 53+193 \equiv 173+73 \equiv 83+163 \equiv 143+103 \equiv 13+23.$
$248 \equiv 37+211 \equiv 187+61 \equiv 67+181 \equiv 97+151 \equiv 127+121;$
$248 \equiv 79+169 \equiv 109+139 \equiv 19+19.$

For studying the binary Goldbach's theorem we discuss only 210 subequations:

$$N=210m+40=E_{11}(K_1)+E_{29}(K_2),\cdots,N=210m+248=E_{79}(K_1)+E_{169}(K_2). \tag{43}$$

For every equation we have the arithmetic function

$$J_2(\omega>210)=\prod_{11\le p\le p_i}(p-2)\prod_{p|N}\frac{p-1}{p-2}\ne 0. \tag{44}$$

Since $J_2(\omega>210)\to\infty$ as $\omega\to\infty$ every even number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (44) into (7) we have the best asymptotic formula

$$\pi_2(N,2)=\sum_{m=K_1+K_2}1=\sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)}1\sim$$

$$\frac{35}{384}\prod_{11\le p\le p_i}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N}{\log^2 N}. \tag{45}$$

Corollary 3. Let $p_i=11$ and $\omega=2310$. From (1) we have

$$E_{p_\alpha}(K)=2310K+P_\alpha, \tag{46}$$

where $K = 0, 1, 2, ...;\quad (2310, p_\alpha) = 1;\quad p_\alpha = 13, 43, 73, ..., 2263, 2293; 17, 47, 107, ..., 2267, 2273; 19, 79, 109, ..., 2239, 2269; 23, 53, 83, ..., 2243, 2273; 29, 59, 89, ..., 2279, 2309;\ 31, 61, 151, ..., 2281, 2311; 37, 67, 97, ..., 2257, 2287; 41, 71, 101, ..., 2231, 2291.$

All the even numbers N greater than 126 can be expressed as

$$N = 2310m + h, \tag{47}$$

where $m = 0, 1, 2, ...; h = 128, 130, ..., 2436.$

From (46) and (47) we have

$$N = 2310m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{48}$$

From (48) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 2310). \tag{49}$$

From (48) we have the $(480)^2$ subequations as follows:

$$\begin{gathered} N = 2310m + 128 = E_{31}(K_1) + E_{97}(K_2) = E_{61}(K_1) + E_{67}(K_2) = \cdots, \\ \cdots \quad \cdots \\ N = 2310m + 2436 = E_{13}(K_1) + E_{113}(K_2) = E_{43}(K_1) + E_{83}(K_2) = \cdots. \end{gathered} \tag{50}$$

For studying the binary Goldbach's theorem we discuss only 2310 subequations:

$$N = 2310m+128 = E_{31}(K_1)+E_{97}(K_2), ..., N = 2310m+2436 = E_{13}(K_1)+E_{113}(K_2). \tag{51}$$

For every equation we have the arithmetic function

$$J_2(\omega > 2310) = \prod_{13 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{52}$$

Since $J_2(\omega > 210) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (52) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{77}{7680} \prod_{13 \le p \le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{53}$$

Corollary 4. Let $p_i = 13$ and $\omega = 30030$. From (1) we have

$$E_{p_\alpha}(K) = 30030K + P_\alpha, \tag{54}$$

where $K = 0, 1, 2, ...$; $(30030, p_\alpha) = 1$; $p_\alpha = 17, 47, , ..., 29987; 19, 79, ..., 29989;$ $23, 53, ..., 29993;$ $29, 59, ...30029; 31, 61, ..., 30031; 37, 67, ..., 30007;$ $41, 71, ...,$ $30011;$ $43, 73, ..., 30013.$

All the even numbers N greater than 254 can be expressed as

$$N = 30030m + h, \tag{55}$$

where $m = 0, 1, 2, ...; h = 256, 258, ..., 30284.$

From (54) and (55) we have

$$N = 30030m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{56}$$

From (56) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 30030). \tag{57}$$

From (56) we have the $(5760)^2$ subequations as follows:

$$\begin{gathered} N = 30030m + 256 = E_{17}(K_1) + E_{239}(K_2) = \cdots, \\ \cdots \quad \cdots \\ N = 30030m + 30284 = E_{31}(K_1) + E_{223}(K_2) = \cdots. \end{gathered} \tag{58}$$

For studying the binary Goldbach's theorem we discuss only 30030 subequations:

$$N = 30030m + 256 = E_{17}(K_1) + E_{239}(K_2),$$

$$N = 30030m + 258 = E_{17}(K_1) + E_{241}(K_2),$$

$$N = 30030m + 260 = E_{19}(K_1) + E_{241}(K_2), \cdots,$$

$$N = 30030m + 30282 = E_{19}(K_1) + E_{233}(K_2),$$

$$N = 30030m + 30284 = E_{31}(K_1) + E_{233}(K_2).$$

For every equation we have the arithmetic function

$$J_2(\omega > 30030) = \prod_{17 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{59}$$

Since $J_2(\omega > 30030) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (59) into (7) we have the best asymptotic formula

$$\pi_2(N,2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{1001}{1105920} \prod_{17\le p\le p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{60}$$

Corollary 5. Let $p_i = 17$ and $\omega = 510510$. From (1) we have

$$E_{p_\alpha}(K) = 510510K + P_\alpha, \tag{61}$$

where $K = 0, 1, 2, ...;$ $(510510, p_\alpha) = 1;$ $p_\alpha = 19, ..., 510469; 23, ..., 510473;$ $29, ..., 510509; 31, ..., 510511; 37, ..., 510487; 41, ..., 510491;$ $43, ..., 510463;$ $47, ..., 510467.$

All the even numbers N greater than 510 can be expressed as

$$N = 510510m + h, \tag{62}$$

where $m = 0, 1, 2, ...; h = 512, 514, ..., 511020.$

From (61) and (62) we have

$$N = 510510m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{63}$$

From (63) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } 510510). \tag{64}$$

From (63) we have the $(92160)^2$ subequations as follows:

$$N = 510510m + 512 = E_{73}(K_1) + E_{439}(K_2) = E_{103}(K_1) + E_{409}(K_2) = \cdots,$$

$$\cdots \quad \cdots \tag{65}$$

$$N = 510510m + 511020 = E_{19}(K_1) + E_{491}(K_2) = E_{79}(K_1) + E_{431}(K_2) = \cdots.$$

For studying the binary Goldbach's theorem we discuss only 510510 subequations:

$$N = 510510m + 512 = E_{73}(K_1) + E_{439}(K_2), ...,$$

$$N = 510510m + 511020 = E_{19}(K_1) + E_{491}(K_2). \tag{66}$$

For every equation we have the arithmetic function

$$J_2(\omega > 510510) = \prod_{19 \leq p \leq p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{67}$$

Since $J_2(\omega > 510510) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (67) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{17017}{283115520} \prod_{19 \leq p \leq p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \tag{68}$$

Corollary 6. Let $p_i = p_g$ and $\omega_g = \prod_{2 \leq p \leq p_g} p$. From (1) we have

$$E_{p_\alpha}(K) = \omega_g K + P_\alpha, \tag{69}$$

where $K = 0, 1, 2, ...; (\omega_g, p_\alpha) = 1,\ p_g < p_\alpha = p_1, \cdots, p_{\phi(\omega_g)} = \omega_g + 1$. All the even numbers N greater than H can be expressed as

$$N = \omega_g m + h, \tag{70}$$

where $m = 0, 1, 2, ...; h = H+2, H+4, \cdots, \omega_g + H, H$ being an even number.

From (69) and (70) we have

$$N = \omega_g m + h = E_{p_1}(K_1) + E_{p_2}(K_2). \tag{71}$$

From (71) we have

$$m = K_1 + K_2, \quad h \equiv p_1 + p_2 (\text{mod } \omega_g). \tag{72}$$

From (71) we have the $\phi^2(\omega_g)$ subequations. For studying the binary Goldbach's theorem we discuss only ω_g subequations among them. For every equation we have the arithmetic function

$$J_2(\omega > \omega_g) = \prod_{p_g < p \leq p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \neq 0. \tag{73}$$

Since $J_2(\omega > \omega_g) \to \infty$ as $\omega \to \infty$ every even natural number N from some point onward can be expressed as the sum of two primes using only partial primes.

Substituting (73) into (7) we have the best asymptotic formula

$$\pi_2(N, 2) = \sum_{m=K_1+K_2} 1 = \sum_{N=E_{p_1}(K_1)+E_{p_2}(K_2)} 1 \sim$$

$$\frac{\omega_g}{\phi^2(\omega_g)} \prod_{p_g<p\le p_i} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}. \quad (74)$$

From $\omega_g = 6$ and (74) we have

$$m_0 = \exp\Big(\frac{3\sqrt{c}\phi(\omega_g)\log\omega_g - \omega_g \log 6}{\omega_g - 3\sqrt{c}\phi(\omega_g)}\Big), \quad (75)$$

where

$$c = \prod_{3<p\le p_g} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

From (75) we have $\omega_g = 30, m_0 = 42; \omega_g = 210, m_0 = 141; \omega_g = 2310, m_0 = 946$. The integer m greater than m_0 is the sum of primes K_1 and K_2, that is every even number N greater than $\omega_g m_0$ can be expressed as the sum of two primes using only partial primes. It is $\omega_g/\phi^2(\omega_g)$ of the the total primes. In the same way we can prove the prime twins theorem and other problems using only partial primes. We will establish the additive prime theory with partial primes.

Theorem 2. $p_1 = p + 6,\ p_2 = N - p.$

We have the arithmetic function

$$J_2(\omega) = \prod_{3|N}(p-1) \prod_{5\le p\le p_i}(p-3) \prod_{p|N, p|(N+6)} \frac{p-2}{p-3} \neq 0. \quad (76)$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$ every even number N from some point onward can be expressed as the sum of two primes satisfying that $p + 6$ is a prime.

We have exact asymptotic formula

$$\pi_3(N, 2) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}. \quad (77)$$

Theorem 3. $p_1 = p + 6,\ p_2 = p + 12,\ p_3 = N - p.$

We have the arithmetic function

$$J_2(\omega)=\prod_{3|N}(p-1)\prod_{5\le p\le p_i}(p-4)\prod_{p|N,p|(N+6),p|(N+12)}\frac{p-3}{p-4}\neq 0. \qquad (78)$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$ every even number N from some point onward can be expressed as the sum of two primes satisfying that $p+6$ and $p+12$ are primes.

We have exact asymptotic formula

$$\pi_4(N,2)\sim\frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}. \qquad (79)$$

Theorem 4. $p_1=p+2,\quad p_2=N-p.$
We have that $J_2(\omega)\neq 0$ if $3\nmid(N-2)$; $J_2(3)=0$ if $3|(N-2)$.

Theorem 5. $p_1=p+4,\quad p_2=N-p.$
We have that $J_2(\omega)\neq 0$ if $3\nmid(N-1)$; $J_2(3)=0$ if $3|(N-1)$.

Theorem 6. $p_1=p+30,\ p_2=N-p.$
We have the arithmetic function

$$J_2(\omega)=\prod_{3,5|N}(p-1)\prod_{5\nmid N}(p-2)\prod_{7\le p\le p_i}(p-3)\prod_{p|N,p|(N+30)}\frac{p-2}{p-3}\neq 0. \qquad (80)$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist primes p such that $p+30$ and $N-p$ are primes for every even number N from some point onward.

We have exact asymptotic formula

$$\pi_3(N,2)\sim\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}. \qquad (81)$$

Theorem 7. $p_1=p+30,\ p_2=p+60,\quad p_3=N-p.$
We have the arithmetic function

$$J_2(\omega)=\prod_{3,5|N}(p-1)\prod_{5\nmid N}(p-2)\prod_{7\le p\le p_i}(p-4)\prod_{p|N,p|(N+30),p|(N+60)}\frac{p-3}{p-4}\neq 0. \qquad (82)$$

Since $J_2(\omega)\to\infty$ as $\omega\to\infty$, there exist primes p such that $p+30,\ p+60$ and $N-p$ are primes for every even number N from some point onward.

We have exact asymptotic formula

$$\pi_4(N,2) \sim \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}. \tag{83}$$

Theorem 8. $p_1 = p^2 + 30,\ p_2 = N - p.$

We have the arithmetic function

$$J_2(\omega) = \prod_{3,5|N} (p-1) \prod_{3,5\nmid N} (p-2) \prod_{7 \le p \le p_i} \Big(p - 3 - (\frac{-30}{p}) - \chi(p) \Big) \ne 0. \tag{84}$$

where $\chi(p) = -1$ if $p|N$; $\chi(p) = 0$ if $p \nmid N$.

We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{85}$$

Theorem 9. $p_1 = p^2 + 210,\ p_2 = N - p.$

We have the arithmetic function

$$J_2(\omega) = \prod_{3,5,7|N} (p-1) \prod_{3,5,7\nmid N} (p-2) \prod_{11 \le p \le p_i} \Big(p - 3 - (\frac{-210}{p}) - \chi(p) \Big) \ne 0. \tag{86}$$

where $\chi(p) = -1$ if $p|N$; $\chi(p) = 0$ if $p \nmid N$.

We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{87}$$

Theorem 10. $p_1 = p^2 + p + 41,\ p_2 = N - p.$

We have the arithmetic function

$$J_2(\omega) = \prod_{3 \le p \le p_i} (p - 3 - (\frac{-163}{p}) - \chi(p)) \ne 0, \tag{88}$$

where $\chi(p) = -1$ if $p|N$; $\chi(p) = 0$ if $p \nmid N$; $(\frac{-163}{163}) = 0$.

We have exact asymptotic formula

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}. \tag{89}$$

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