# The Geometry of Large Rotating Systems

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**Abstract**

This paper presents an analytical solution to the geometry of large rotating systems which reconciles the peculiar rotation profiles of distant galaxies with Einstein's principle of General Relativity. The resulting mathematical solution shows that large rotating systems are distorted in the space of a non-rotating observer into a spiral pattern with tangential velocities that behave in agreement with those observed in distant galaxies. This paper also demonstrates how the scale of the spiral structure of rotating systems can be used to determine its distance from the observer. The authors' proposed equations for the rotation profile and the distance measure are compared with the observed rotation profiles and Cepheid distance measurements of several galaxies with strong agreement. A formal error analysis is not included however the authors suggest a method for better qualifying the accuracy of the theorums.

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# 1 Derivation

## 1.1 Introduction

In previous papers submitted by Bruce Rout, an analytical model of spiral galaxies was presented along with various consequences. After receiving useful feedback from the membership and leadership of the Society of Amateur Radio Astronomers, the authors herein present the derivation and validation of the most recent and complete version of this model in a single paper.

The model is a geometric solution to the consideration of general relativity in a large rotating system. For the purpose of this paper we define a system as large when the communicability of the reality of one part of the system to another is limited by the speed of light such that it can no longer be considered rigid when the system is subject to acceleration. When such systems accelerate, describing the geometry requires the application of Einsteins principle of general relativity (GR).

It is common to consider the earths gravitational field using GR since the gravitational field of the earth is prevalent for terrestrial systems. So, in order to find other accelerating systems which require the application of GR it is practical to look at objects much farther away. One type of system which is very large and displays GR due to rotation is a galactic system. Galaxies make a good candidate for the application of relativistic considerations because of the high velocities, large dimensions, and abundance of observational data.

Provided below is Einsteins initial description of GR in a rotating system which is used in this paper to determine equations for predicting the geometry of large rotating systems. Equations for the rotational velocity profile and observational distance measure are then compared with observational data from a variety of galaxies.

> In a space which is free of gravitational fields we introduce a Galilean system of reference $K(x, y, z, t)$, and also a system of co-ordinates $K'(x', y', z', t')$ in uniform rotation relatively to $K$. Let the origins of both systems, as well as their axes of $Z$ permanently coincide... For reasons of symmetry it is clear that a circle around the origin in the $X$, $Y$ plane of $K$ may at the same time be regarded as a circle in the $X'$, $Y'$ plane of $K'$. We suppose that the circumference and diameter of this circle have been measured with a unit measure infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with a measuring-rod at rest relatively to the Galilean system $K$ then the quotient would be $\pi$. With a measuring rod at rest relatively to $K'$, the quotient would be greater than $\pi$. This is readily understood if we envisage the whole process of measuring from the "stationary" system $K$, and take into consideration that the measuring-rod applied to the periphery undergoes a Lorentzian contraction , while the one applied along the radius does not.[1]
> *-A. Einstein*

## 1.2 Units

The units for space and time used in this derivation are light years (ly) and years (y). The resulting units for other measurements are expressed in terms of these units, for example velocity is measured in light years per year (ly/y).

## 1.3 The Measurement of Proportionality

According to Einsteins words above, the measurement of proportionality between the radius and the circumference of a circle is not a constant in a rotating reference system. In order to determine the relationship that accurately describes proportionality in a rotating system we first describe proportionality as a function of the radius.

$$\Pi = \frac{s'_c}{2r} \tag{1}$$

Where,

$\Pi$ = the factor of proportionality between the locally measured circumference and the diameter
$s'_c$ = the locally measured length of the circumference of a circle
r = the radius of a circle

Since any circle around the origin in $K'(x', y', z', t')$ is in rotational motion, the measurement of the circumference is altered by Lorentzs transformation

$$s'_c = 2\pi r \cdot \gamma \tag{2}$$

Where,
$\gamma$ = the factor by which the measurement of the circumference has lengthened by the Lorentz transformation. Substituting (2) into (1) and simplifying,

$$\Pi = \pi\gamma \tag{3}$$

For the stationary system $K(x, y, z, t)$ the factor of proportionality is then equal to $\pi$. The appropriate Lorentz transformation is as follows,

$$\gamma = \sqrt{\frac{1}{1 - v^2}} \tag{4}$$

Where,
$v$ = the translational velocity at the point of measurement in units of light years per year (ly/yr)
Since the tangential velocity of a rotating system is,

$$v = \omega r \tag{5}$$

And the angular velocity is itself affected by the Lorentz transformation,

$$\omega = \frac{\omega_0}{\gamma} \tag{6}$$

Where,
$\omega_0$ = the uniform rotation of $K'(x', y', z', t')$ with respect to $K(x, y, z, t)$
Substituting (6) into (4) and solving for $\gamma$, the Lorentz factor can be determined for a rotating reference system with angular velocity $\omega_0$,

$$\gamma = \sqrt{1 + (\omega_0 r)^2} \tag{7}$$

and we can express $\Pi$ in terms of the angular velocity for a rotating reference system,

$$\Pi = \pi\sqrt{1 + (\omega_0 r)^2} \tag{8}$$

For circles about the origin in $K'(x', y', z', t')$ with greater radii, the value of $\Pi$ approaches the relationship

$$\Pi \approx \pi\omega_0 r \tag{9}$$

### 1.4 Rotational Velocity

For the rotating system $K'(x', y', z', t')$, the factor of proportionality is a function of the radius from the origin. The circumference of a rotating circle at large distances from the origin must be measured using the proportionality factor described in (9).

$$s'_c \to 2\Pi r = 2\pi\omega_0 r^2 \tag{10}$$

The measure of radians in a circle at sufficient distance from the origin in the rotating reference

$$\frac{s'_c}{r} \to 2\Pi = 2\pi\omega_0 r \tag{11}$$

also at large distance from the origin, the angular velocity converges to

$$\omega = \frac{\omega_0}{\gamma} \to \frac{1}{r} \tag{12}$$

The velocity, in light years per year, is given by the measure of radians multiplied by the angular velocity,

$$v' = \omega\frac{s'_c}{r} \tag{13}$$

Where,
$v'$ = the tangential velocity of $K'(x', y', z', t')$ as measured by an observer in $K(x, y, z, t)$

Ultimately $v'$ approaches a maximum as shown in (14) by substituting (11) and (12) into (13).

$$v' \to v_{max} = 2\pi\omega_0 \tag{14}$$

The behaviour of $v'$ as it approaches $v_{max}$ is given by combining (14) and (11) and substituting into (13)

$$\begin{aligned} \frac{s'_c}{r} &= rv_{max} \\ v' &= \omega \frac{s'_c}{r} \\ &= v_{max} \cdot \frac{\omega_0 r}{\gamma} \end{aligned} \tag{15}$$

Or,

$$v' = v_{max} \cdot \frac{\omega_o r}{\sqrt{1 + \omega_0^2 r^2}} \tag{16}$$

## 1.5 Spiral Geometry

The geometry of $K'(x', y', z', t')$ as experienced by an observer in $K(x, y, z, t)$ can be determined using parametric equations to conserve the space-time metric. We have for the radial direction,

$$T = \frac{r}{c} \tag{17}$$

and for the tangential direction using the constant tangential velocity determined in (14)

$$T = \frac{2\pi\theta}{v_{max}} \tag{18}$$

Resulting in the equation of the spiral in $K(x, y)$ formed by a radial geodesic in the $K'(x', y')$ plane

$$r = \frac{2\pi\theta}{v_{max}/c} \tag{19}$$

Since $v_{max}$ is in units of light years per year,

$$r = \frac{2\pi\theta}{v_{max}} \tag{20}$$

Thus, for large $r$, the result is an Archimedes spiral in the form of

$$r = \kappa\theta \tag{21}$$

Where,

$$\kappa = \frac{2\pi}{v_{max}} \tag{22}$$

Keeping in mind that (20) describes the mirror image of the radial geodesic for $\theta < 0$, let us consider the radial geodesic to be defined for $-\infty < \theta < \infty$.

## 1.6 Observed Scale

For an Archimedes spiral, the distance between two spiral arms is determined as follows,

$$\begin{aligned}\Delta r_{spiral} &= \pi\kappa \\ &= \frac{2\pi^2}{v_{max}}\end{aligned} \tag{23}$$

Where,
$\Delta r_{spiral}$ = the spacing between two adjacent spiral arms[1]

let $\Delta\phi$ represent the angular distance between the spiral arms of any radial geodesic in $K'(x', y', z', t')$ as measured by an observer in the static system $K(x, y, z, t)$. If the observer is not positioned on the axis of rotation then the viewing angle is oblique to the spiral the spacing of the spiral arms is measured along the major axis.

Figures 1 and 2 below illustrate how the properties of an Archimedes spiral allow the measurement of $\Delta\phi$ for an Archimedes spiral regardless of the orientation of the viewer.

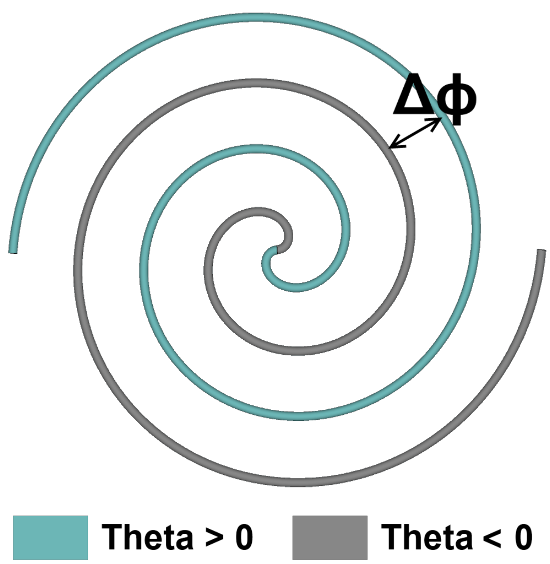



Figure 1: Illustration of Radial Geodesic

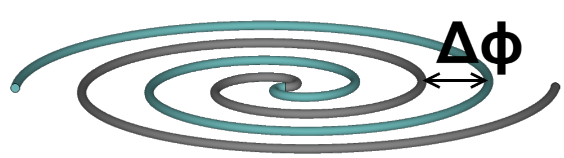



Figure 2: Illustration of Angular Arm Separation from Oblique Viewing Angle

[1]Note: Adjacent arms of the same spiral are on different rays emanating from the center. The arm defined over the domain $\theta > 0$ is adjacent to the arm defined over the domain $\theta < 0$. The result is that the separation is proportional to $2\pi^2$ instead of $4\pi^2$ (see Figure 1)

Let,

$$\alpha = \frac{\Delta r}{\Delta \phi} \tag{24}$$

So that $\alpha$ may be used to express the scale of $K'(x', y', z', t')$ in units of radial distance per angular separation as measured by the observer in $K(x, y, z, t)$.

Substituting,

$$\alpha = \frac{2\pi^2}{v_{max} \cdot \Delta\phi} \tag{25}$$

Thus the velocity profile for a rotating system as measured between $K$ and $K'$ can be expressed in terms of the angular distance measured by the observer using the scaling factor $\alpha$, Let,

$$\Phi = \frac{r}{\alpha} \tag{26}$$

Where,
$\Phi$ = the angular separation from the center rotation in $K'(x', y', z', t')$ as measured by an observer in $K(x, y, z, t)$.

$$\begin{aligned} r &= \Phi\alpha \\ &= \Phi\frac{2\pi^2}{v_{max} \cdot \Delta\phi} \end{aligned} \tag{27}$$

Thus, by substituting (27) into (20), the spiral geometry of a radial geodesic can then be described in terms of the angular distance from the center of rotation as measured by an observer in $K(x, y, z, t)$.

$$\Phi = \frac{\Delta\phi}{\pi}\theta \tag{28}$$

## 1.7 Rotation Profile

An observer in $K(x, y, z, t)$ cannot measure absolute distances from the origin in $K'(x', y', z', t')$ without knowing how far away the origin is from the observer. The observer can, however, measure a distance in units of the angle it subtends. And, if the points of $K'(x', y', z', t')$ are illuminated, then redshift can be used to measure instantaneous translational velocities.

By using the scaling factor described in (25) it is possible to describe the tangential velocity of $K'(x', y', z', t')$ using only properties and measures that can be made by the observer.

Subsituting (26) into (16),

$$\begin{aligned} v' &= v_{max} \cdot \frac{\omega_0 \Phi \alpha}{\sqrt{1+\omega_0^2 \Phi^2 \alpha^2}} \\ &= v_{max} \cdot \frac{\omega_0 \Phi \Delta r}{\Delta\phi \sqrt{1+\omega_0^2 \Phi^2 \left(\frac{\Delta r}{\Delta \phi}\right)^2}} \\ &= v_{max} \cdot \frac{\omega_0 \Phi \Delta r}{\sqrt{\Delta\phi^2 + \omega_0^2 \Phi^2 \Delta r^2}} \end{aligned} \tag{29}$$

Using (23) and (14) it can be shown that

$$\begin{aligned} \omega_0 \Delta r &= \omega_0 \frac{2\pi^2}{v_{max}} \\ &= \pi \end{aligned} \tag{30}$$

substituing (30) into (29),

$$v' = v_{max} \frac{\pi \Phi}{\sqrt{\Delta\phi^2 + \pi^2 \Phi^2}} \tag{31}$$

It is evident that the tangential velocity of $K'(x', y', z', t')$ as measured by an observer in $K(x, y, z)$ is transformed by a factor dependent on the distance from the center of rotation and the separation of the spiral arms of the geodesic.

$$v' = \xi v_{max} \tag{32}$$

Where $\xi$ can be shown in terms of absolute distances or in terms of angular distances measured by an observer in $K(x, y, z)$,

$$\begin{aligned} \xi &= \frac{\pi \Phi}{\sqrt{\Delta\phi^2 + \pi^2 \Phi^2}} \\ &= \frac{\omega_0 r}{\sqrt{1+\omega_0^2 r^2}} \end{aligned}$$

## 1.8 Distance Measure

The scaling factor $\alpha$ may be used to determine the distance of the observer to the origin. If $\Delta\phi$ is measured in arcminutes then.

$$d = \alpha \frac{60 \cdot 360}{4\pi} \tag{33}$$

Where,
$d$ =the distance between the observer in $K(x, y, z)$ and the radial geodesic of $K'(x', y', z', t')$ in lightyears
Substituting (25) into (33),

$$d = \frac{\pi\,(60 \cdot 360)}{2\left(\frac{v_{max}}{c}\right)\Delta\phi} \tag{34}$$

Where the speed of light ($c$) is included to ensure units of light years per year for velocity.

# 2 Validation

Here we shall compare the results of the above formulae for the Rotation Profile and Distance Measure of several galaxies.

## 2.1 Rotation Profiles

Equation (31) describes the rotational velocity of a rotating system using the observed separation of spiral arms and the maximum observed tangential velocity of the galaxy.

$$v' = v_{max} \frac{\pi\Phi}{\sqrt{\Delta\phi^2 + \pi^2\Phi^2}}$$

**Equation** (31)**:** *The Rotational Velocity Profile of a Rotating System as Measured by an External Observer*

Where,

$v'$ = the tangential velocity of a rotating system
$v_{max}$ = the maximum tangential velocity of a rotating system
$\Phi$ = the angular distance from the center of rotation
$\Delta\phi$ = the angular separation of two adjacent spiral arms of a radial geodesic

Distant galaxies often have clear spiral morphologies wherein the dust lanes, stars, and empty spaces form a spiral vector field which may suggest the shape of the geodesic for the rotating system.

Physical parameters are taken from the sky survey used for Matthewsons work[2] on the rotation profiles of spiral galaxies and entered into the above equation. The results are then compared with the corresponding rotation profiles published in Matthewson's article titled "A Southern Sky Survey of the Peculiar Velocities of 1355 Spiral Galaxies"[3]. In addition to the work by Matthewson, the rotation profile of NGC 3198 as measured by Begeman [4] is used for comparison. Begeman's work was dedicated to the measurement of the velocity profile of only one galaxy and consequently are made with comparatvely high resolotion.

### Sample Selection

The galaxies were selected for inspection based on the criterion that a clear spiral pattern must be apparent in order to model a radial geodesic and measure $\Delta\phi$. The first ten galaxies with discernible Archimedes spiral patterns were used in order of appearance in the reference.

**Error**

The calculated profiles and values are based on the application of spirals as shown in the diagram accompanying each measurement. The accuracy of the fit of each spiral to the galaxy is not considered and a full error analysis is not included.

**Description of Figures**

The process of determining the pitch of the spiral geometry associated with spiral galaxies is illustrated with three examples (Figures 3-8).

1. The first image is the image provided by the The Second Palomar Observatory Sky Survey (POSS-II)[2].

2. The second image shows the major and minor axes selected for correcting the oblique viewing angle superimposed on a negative version of the first image.

3. The third image is the image corrected for the viewing angle, rotated so that the major axis is horizontal and stretched by a proportion equal to the ratio of the minor and major axes.

4. The fourth image shows the superimposition of the spiral geometry in the form of 2 radial geodesics (1 white,1 black) which are each plotted using (28).

The value of $\Delta\phi$ is determined from the spiral arm spacing in the fourth image and used in (31) to calculate the velocity profile. Below the 4 illustrative images, the measurements made by Matthewson et al. are superimposed to show the measured values of the rotational velocity for each galaxy in comparison to the rotational velocity profile as determined by (31).

[2]The Second Palomar Observatory Sky Survey (POSS-II) was made by the California Institute of Technology with funds from the National Science Foundation, the National Geographic Society, the Sloan Foundation, the Samuel Oschin Foundation, and the Eastman Kodak Corporation.

**Example 1: ESO 349-G32 (253 km/s)**

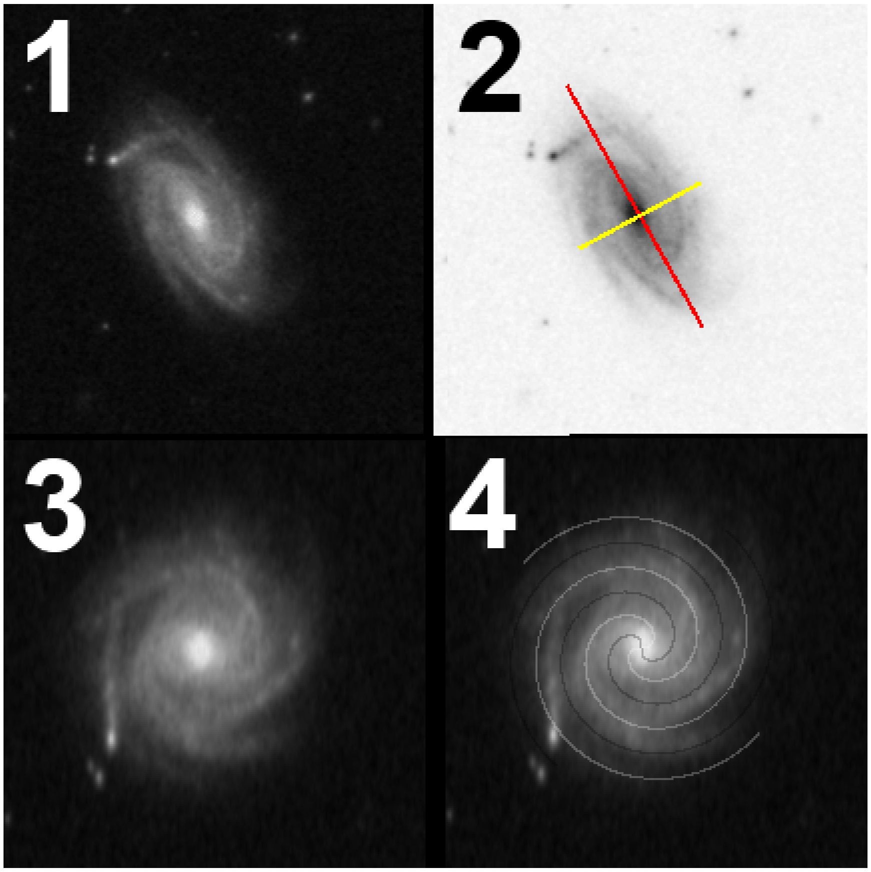



Figure 3: Illustration of Method for Determining Spiral Pattern of ESO 349-G32

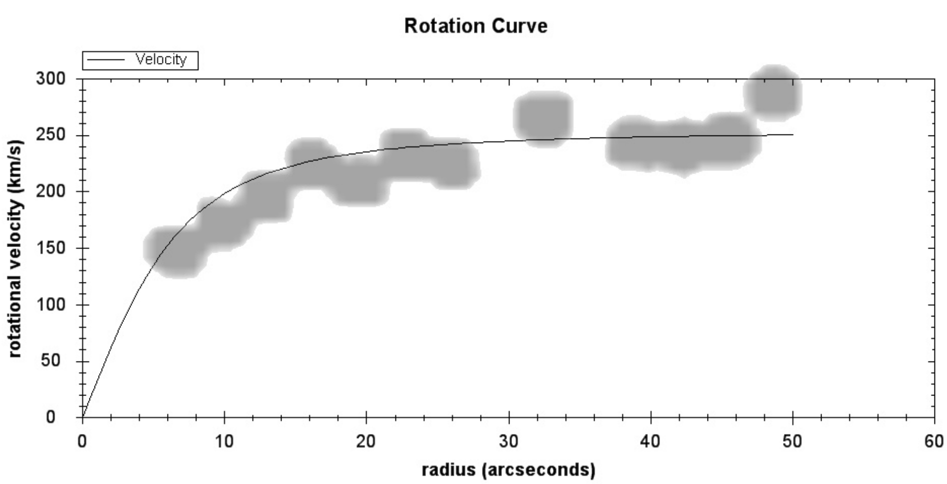



Figure 4: Calculated and Measured Rotation Profile ESO 349-G32 (Matthewson 1992)

**Example 2:MGC 02-02-051 (116 km/s)**

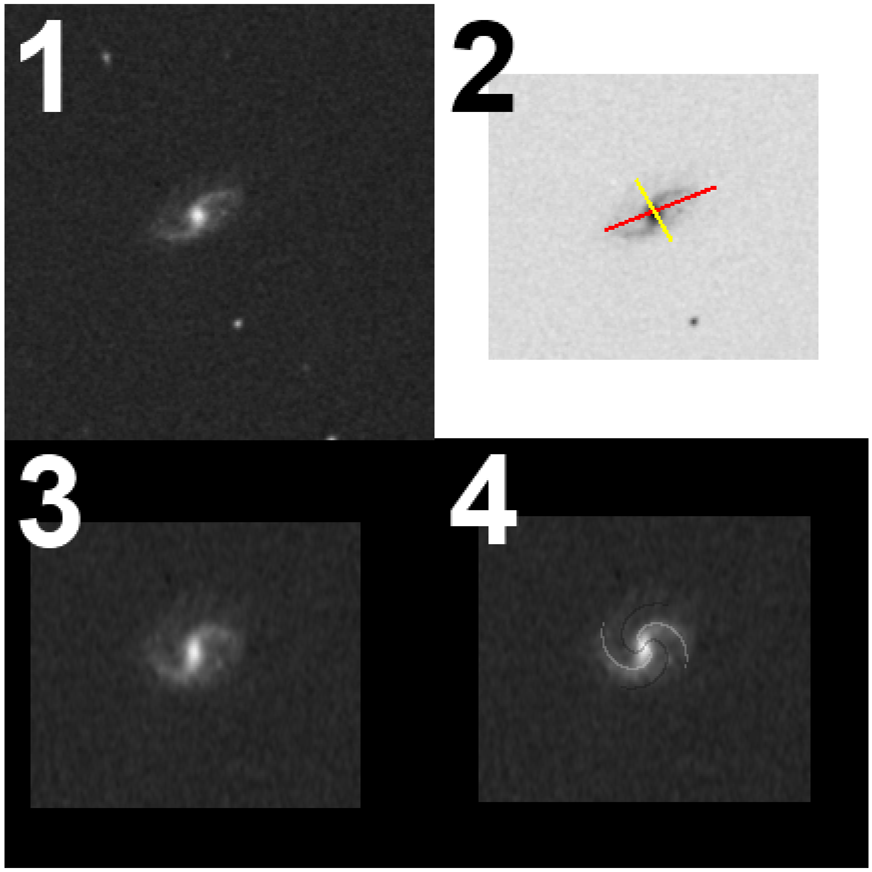



Figure 5: Illustration of Method for Determining Spiral Pattern of MGC 02-02-051

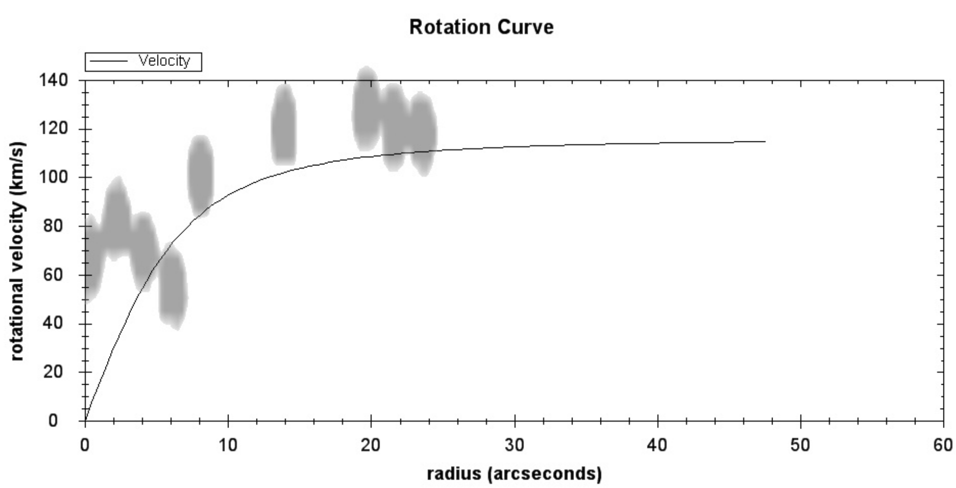



Figure 6: Calculated and Measured Rotation Profile MGC 02-02-051 (Matthewson 1992)

**Example 3:ESO 541-G1 (211 km/s)**

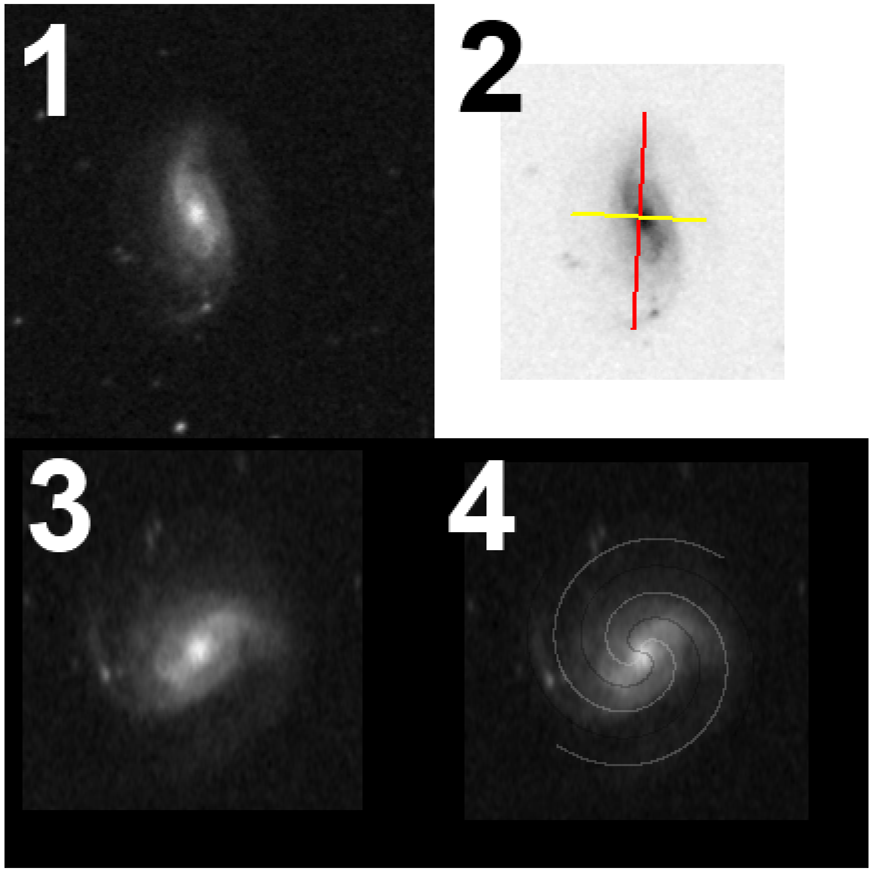



Figure 7: Illustration of Method for Determining Spiral Pattern of ESO 541-G1

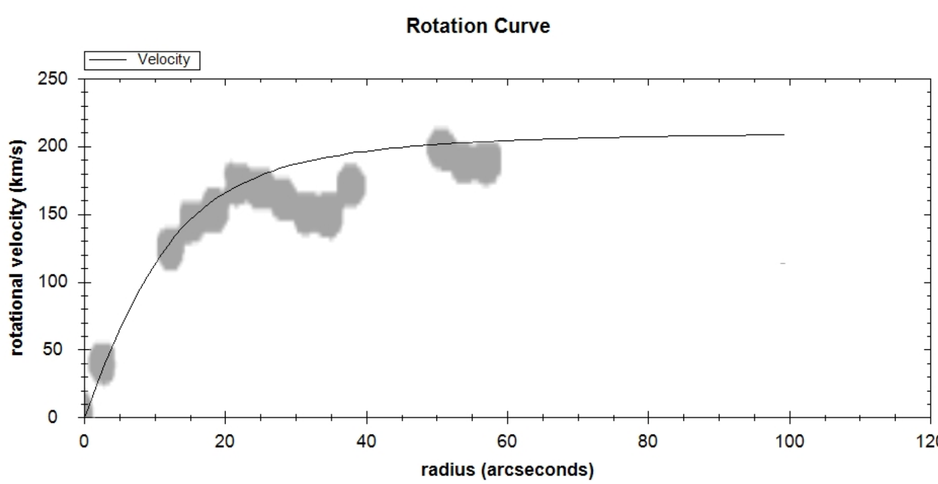
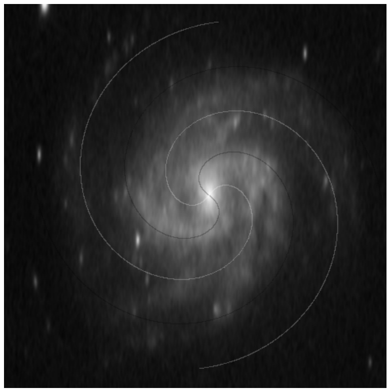
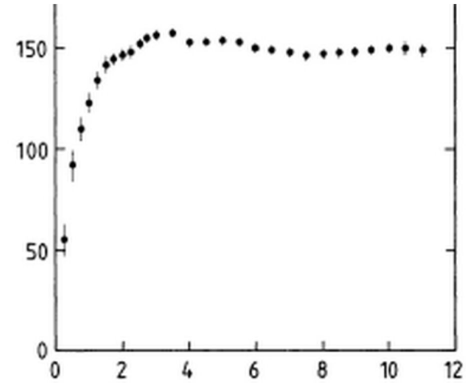



Figure 8: Calculated and Measured Rotation Profile ESO 541-G1 (Matthewson 1992)

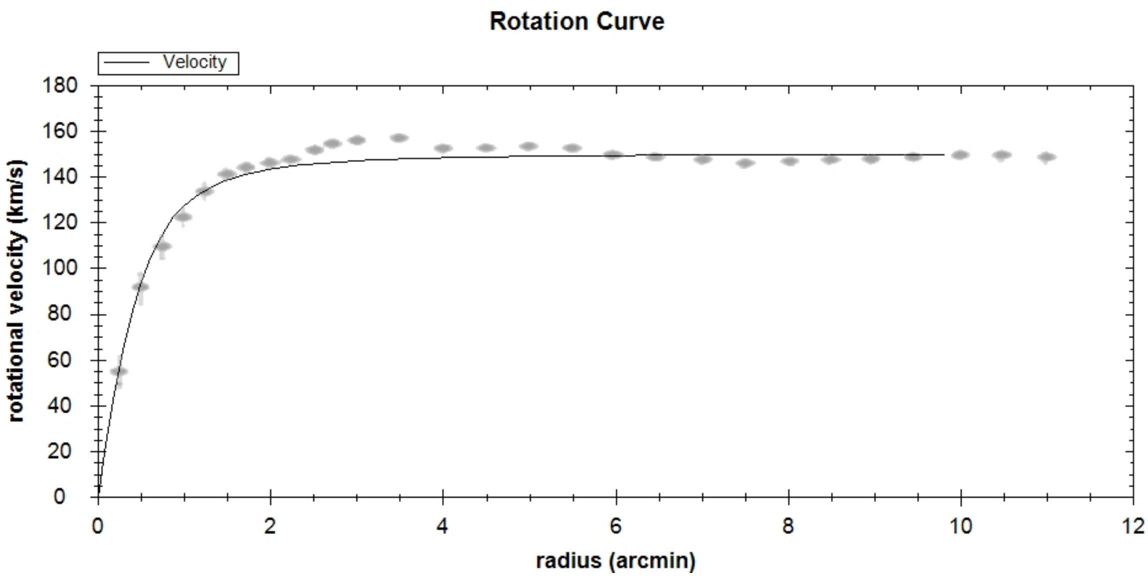
Figure 9: NGC 3198



(a) Spiral Fit



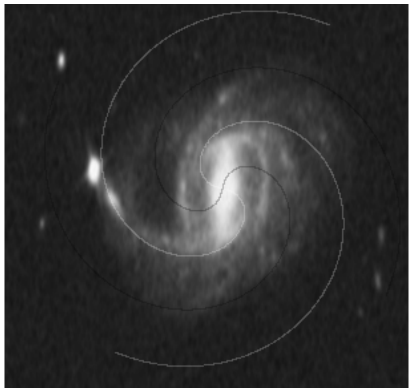

(b) Raw Velocity Profile



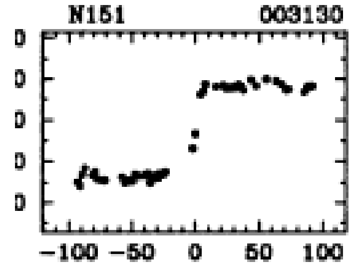

(c) Calculated Velocity Profile

$v_{max}$: 151 km/s $\Delta\phi$: 90 asec $\Delta r$: 39000 ly Distance: 13.7 Mpc

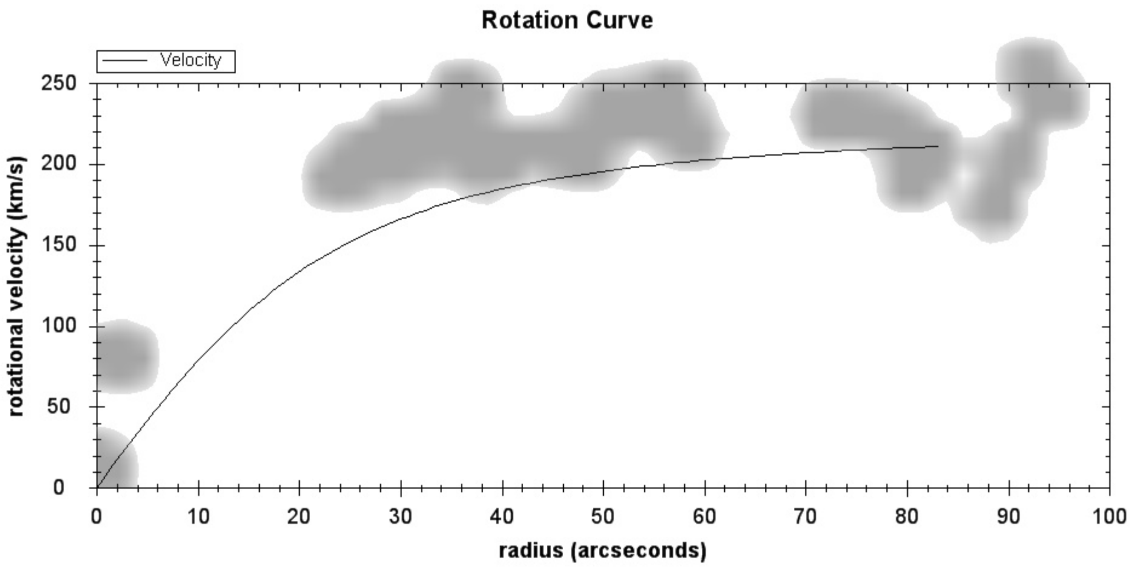
Figure 10: NGC 151



(a) Spiral Fit



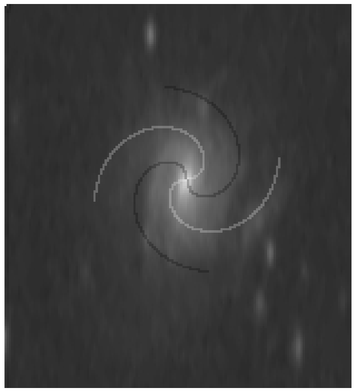

(b) Raw Velocity Profile



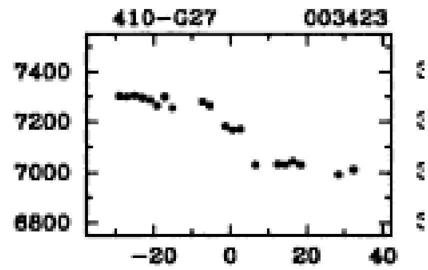

(c) Calculated Velocity Profile

$v_{max}$: 221 km/s $\Delta\phi$: 83 asec $\Delta r$: 26800 ly Distance: 10.2 Mpc

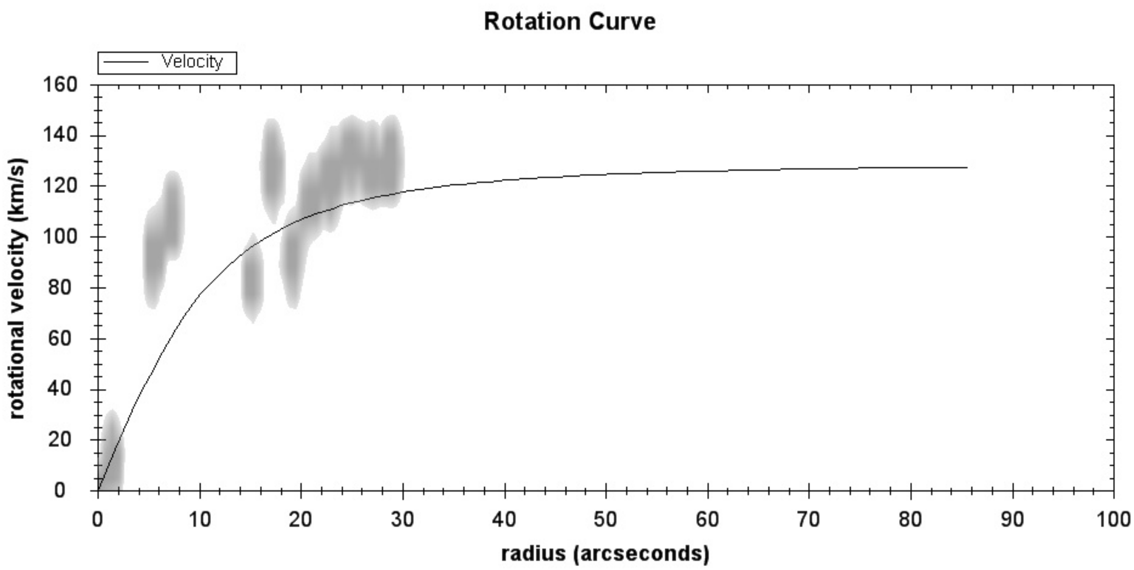
Figure 11: ESO 410-G27



(a) Spiral Fit




(b) Raw Velocity Profile

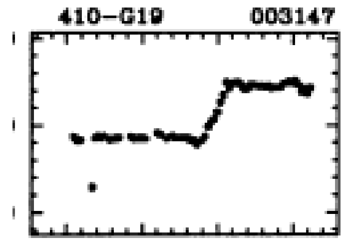



(c) Calculated Velocity Profile

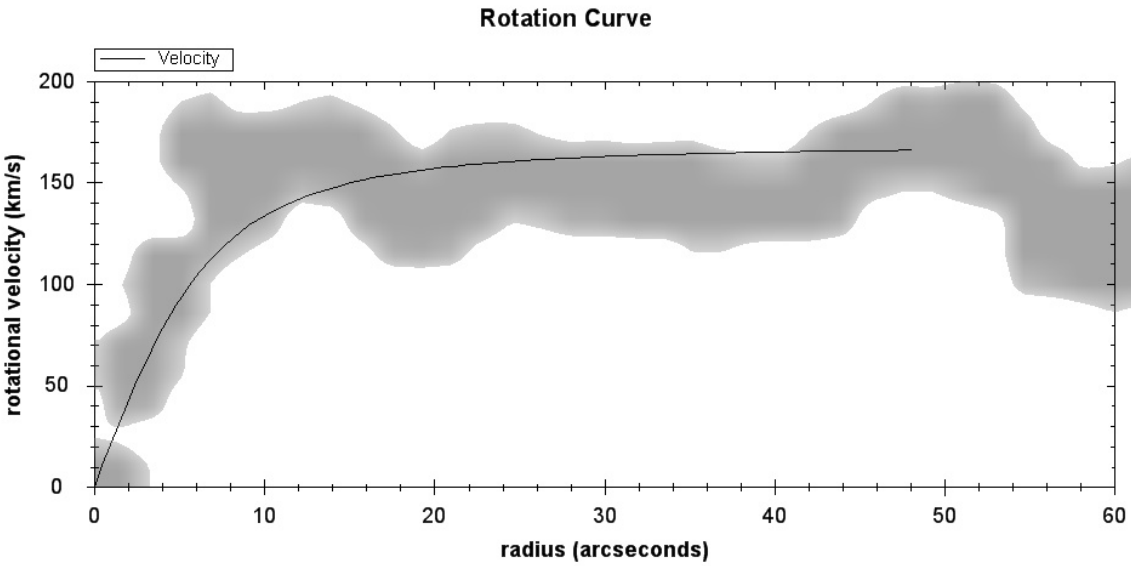
$v_{max}$: 129 km/s   $\Delta\phi$: 43 asec   $\Delta r$: 45900 ly   Distance: 33.9 Mpc

Figure 12: ESO 410-G19

(a) Spiral Fit



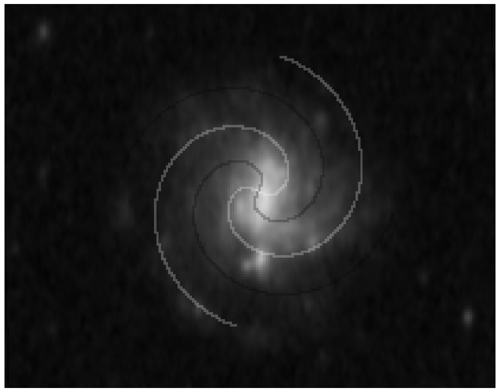

(b) Raw Velocity Profile



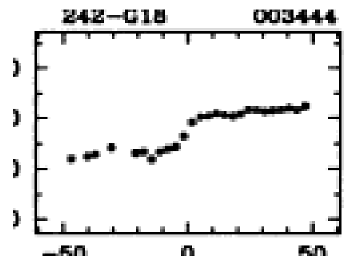

(c) Calculated Velocity Profile

$v_{max}$: 168 km/s $\Delta\phi$: 24 asec $\Delta r$: 35250 ly Distance: 46.4 Mpc

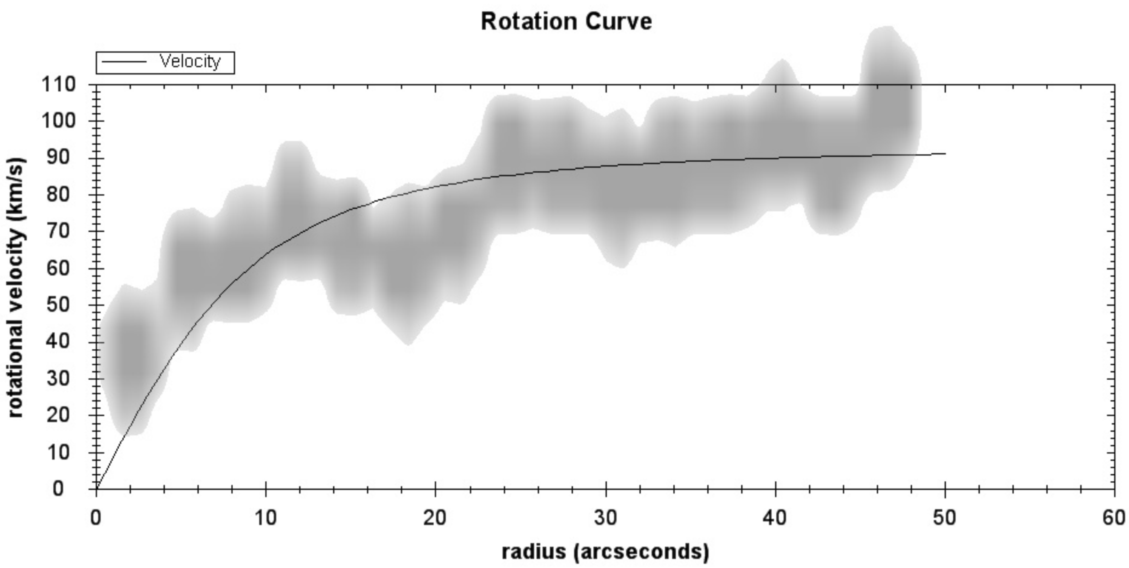
Figure 13: ESO 242-G18



(a) Spiral Fit



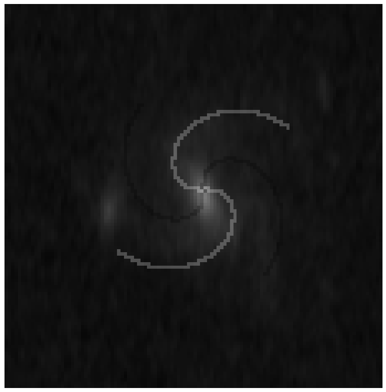

(b) Raw Velocity Profile



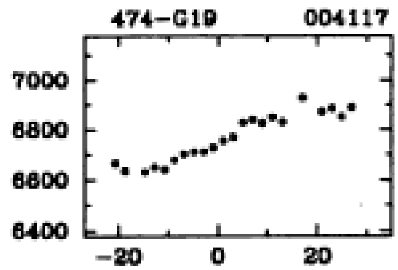

(c) Calculated Velocity Profile

$v_{max}$: 93 km/s $\Delta\phi$: 34 asec $\Delta r$: 63675 ly Distance: 60.0 Mpc

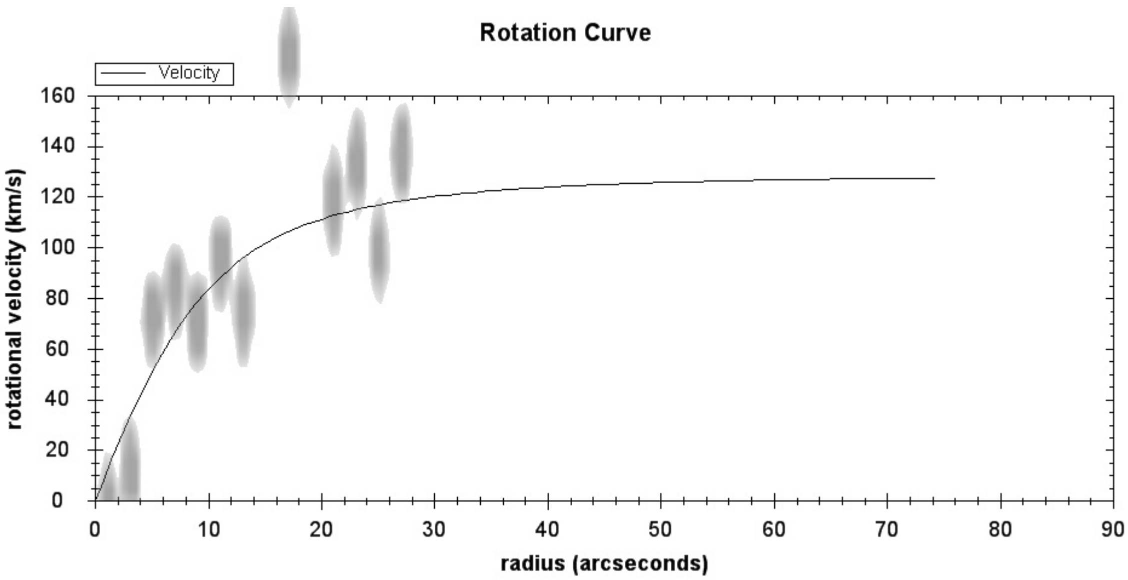
Figure 14: ESO 474-G19



(a) Spiral Fit



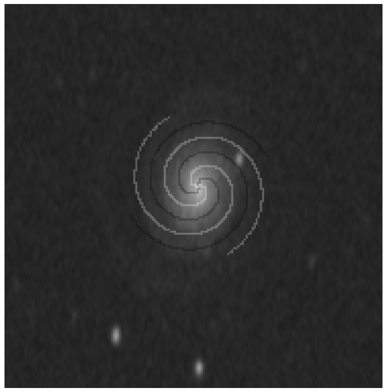

(b) Raw Velocity Profile



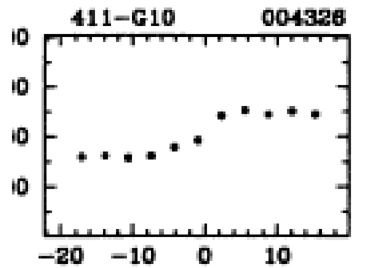

(c) Calculated Velocity Profile

$v_{max}$: 129 km/s $\Delta\phi$: 37 asec $\Delta r$: 45905 ly Distance: 39.2 Mpc

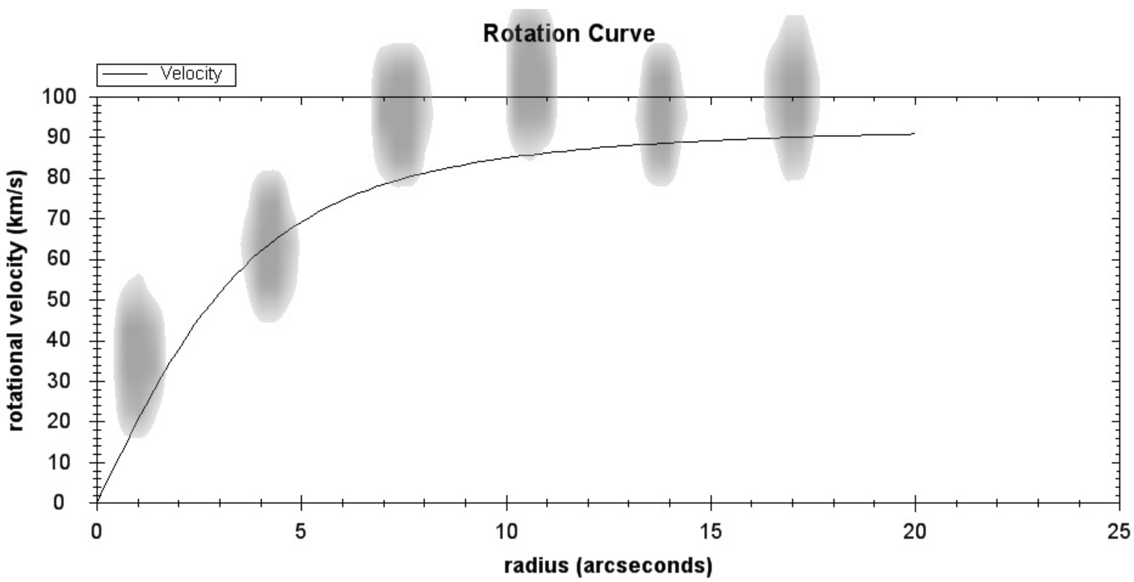
Figure 15: ESO 411-G10



(a) Spiral Fit



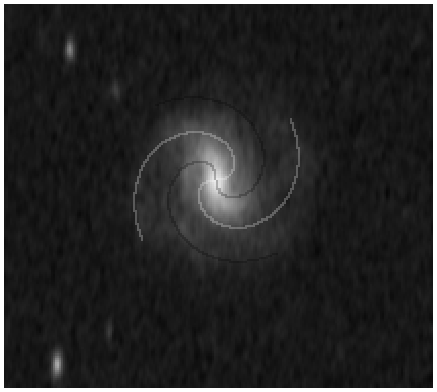

(b) Raw Velocity Profile



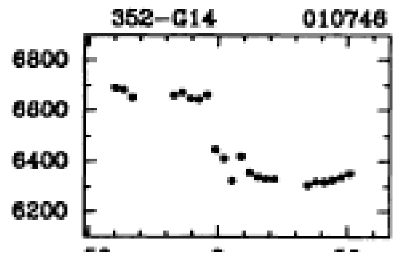

(c) Calculated Velocity Profile

$v_{max}$: 93 km/s $\Delta\phi$: 14 asec $\Delta r$: 63700 ly Distance: 142 Mpc

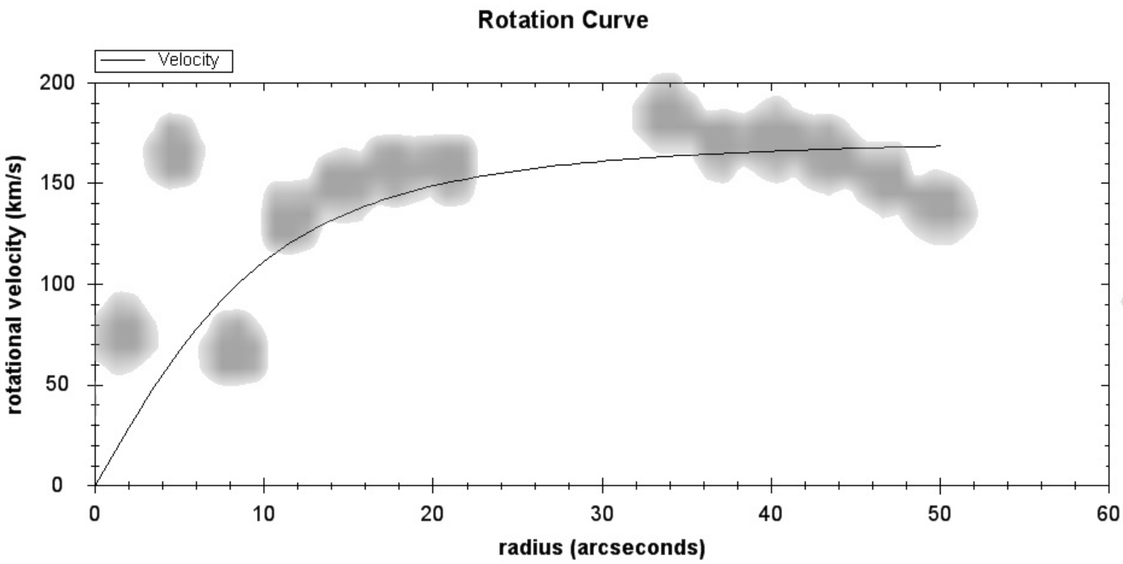
Figure 16: ESO 352-G14



(a) Spiral Fit




(b) Raw Velocity Profile




(c) Calculated Velocity Profile

$v_{max}$: 173 km/s $\Delta\phi$: 38 asec $\Delta r$: 34200 ly Distance: 28.8 Mpc

## 2.2 Distance Measure

Equation 34 describes the distance between an observer and the spiral formed by radial geodesic of a rotating system using the observed separation of spiral arms and the maximum observed tangential velocity. The results of this equation are compared with Cepheid variable measurements to several galaxies below.

$$d = \frac{\pi\,(60 \cdot 360)}{2\left(\frac{v_{max}}{c}\right)\Delta\phi}$$

**Equation** (34)**:** *The distance measure between an observer and a rotating system*

Where,

| | | |
|---|---|---|
| $d$ | $=$ | the distance between the observer in $K(x, y, z)$ and the radial geodesic of $K'(x', y', z', t')$ in lightyears |
| $v_{max}$ | $=$ | the maximum tangential velocity of a rotating system |
| $\Delta\phi$ | $=$ | the angular separation of two adjacent spiral arms of a radial geodesic |

The methodology used for measuring $\Delta\phi$ is the same as that employed in the calculation of rotation profiles above.

### Sample Selection

In Figures 17-22 below the Cepheid distance measurements of several galaxies are compared to calculated distance values using Equation 34. The galaxies were selected for inspection based on the criterion that a clear spiral pattern must be apparent in order to model a radial geodesic and measure $\Delta\phi$.

Table 1: Calculated Distances to 6 Cepheid Galaxies

| Galaxy | $v_{max}$[5] | $\Delta\phi$ | Calculated Distance(Mpc) |
|---|---|---|---|
| NGC 598 | 135±13 | 13.95 | 1.66 |
| NGC 7331 | 239±5 | 1.33 | 9.8 |
| NGC 3198 | 150±3 | 1.5 | 13.7 |
| NGC 4321 | 216±6 | 1.35 | 10.7 |
| NGC 2841 | 281±10 | 0.78 | 14.2 |
| NGC 4258 | 210±20 | 1.83 | 8.1 |

**Various Galactic Distance Measures**

The measurements from Table 1 are compared to Cepheid and Tulley-Fisher measurements from the NASA/IPAC Extragalactic Database.

Figure 17: NGC 598

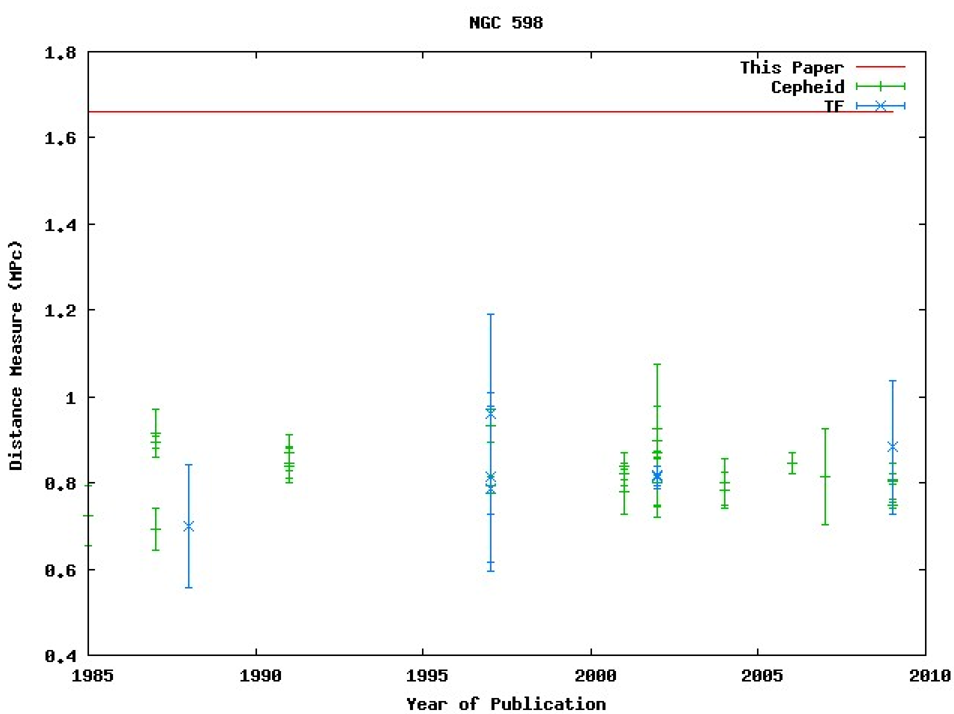


Figure 18: NGC 7331

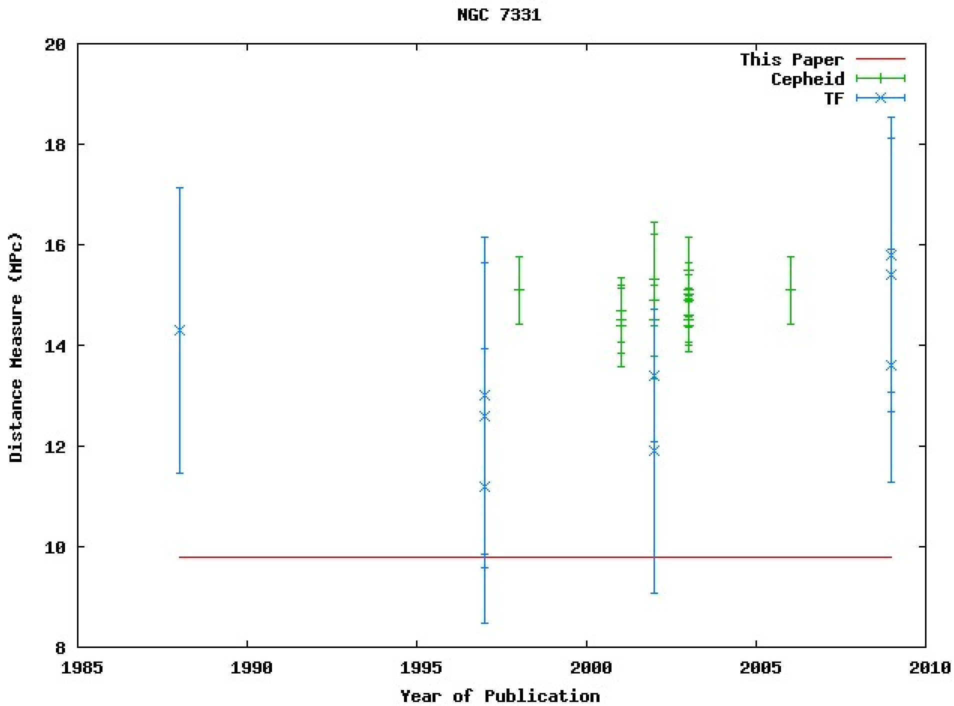



Figure 19: NGC 3198

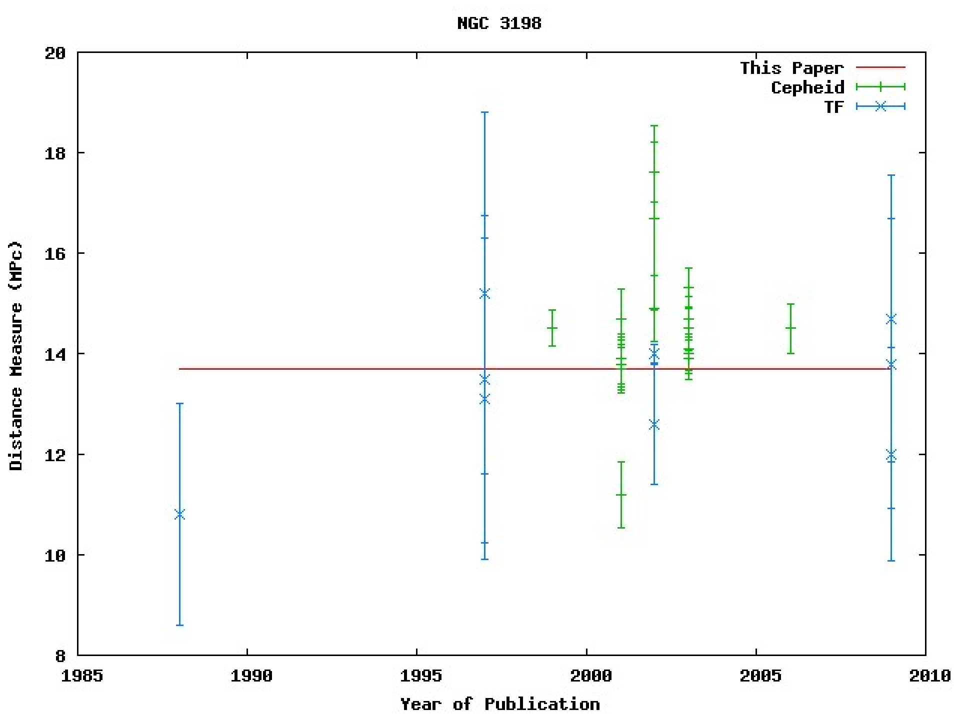


Figure 20: NGC 4321

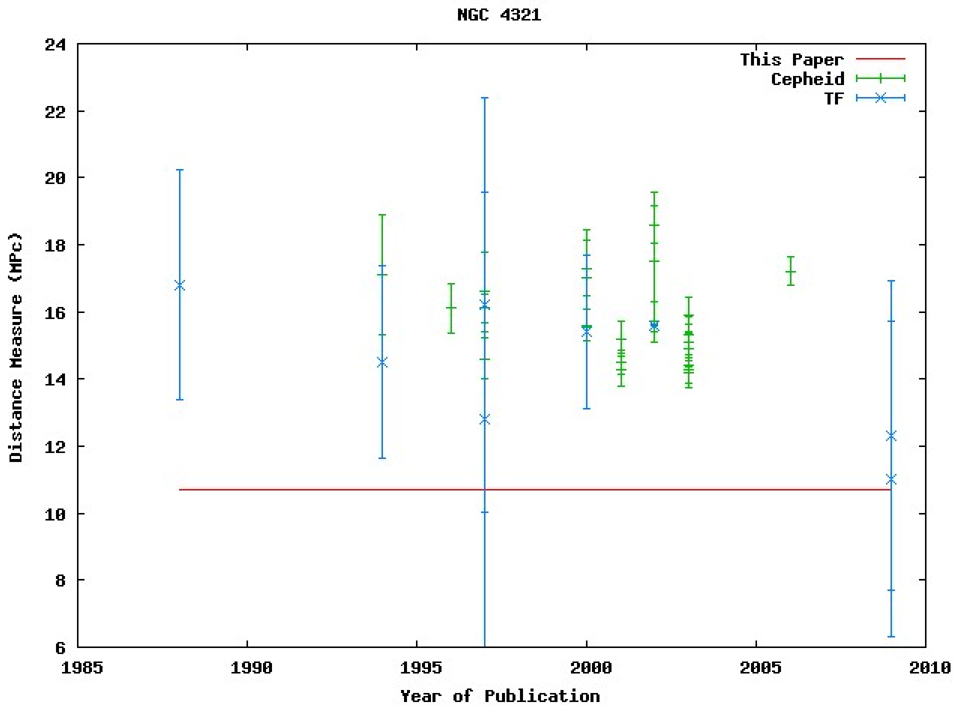



Figure 21: NGC 2841

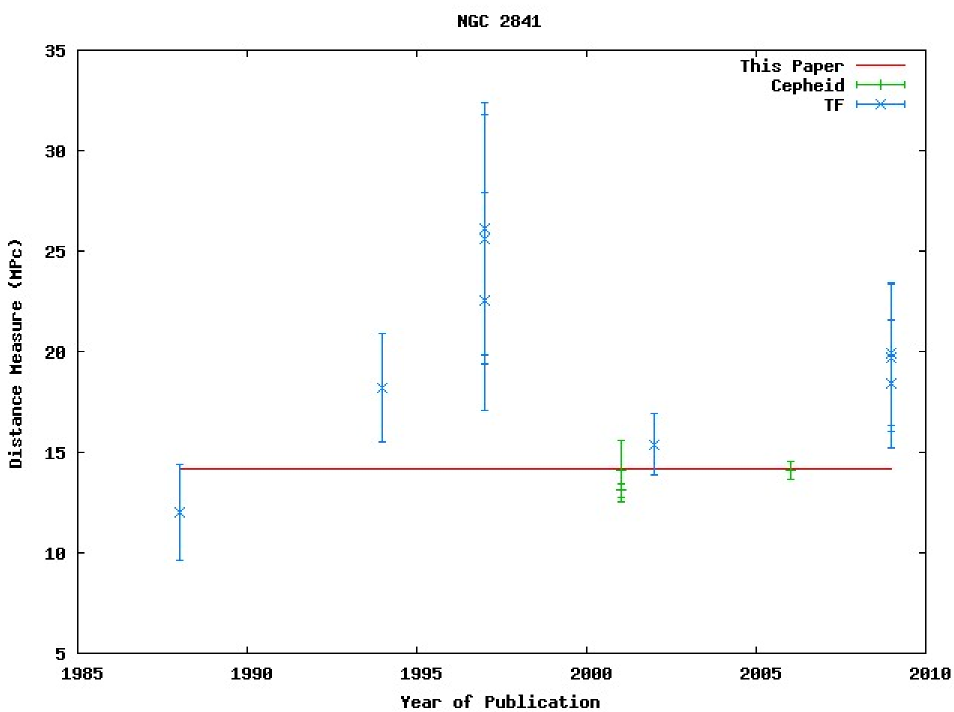
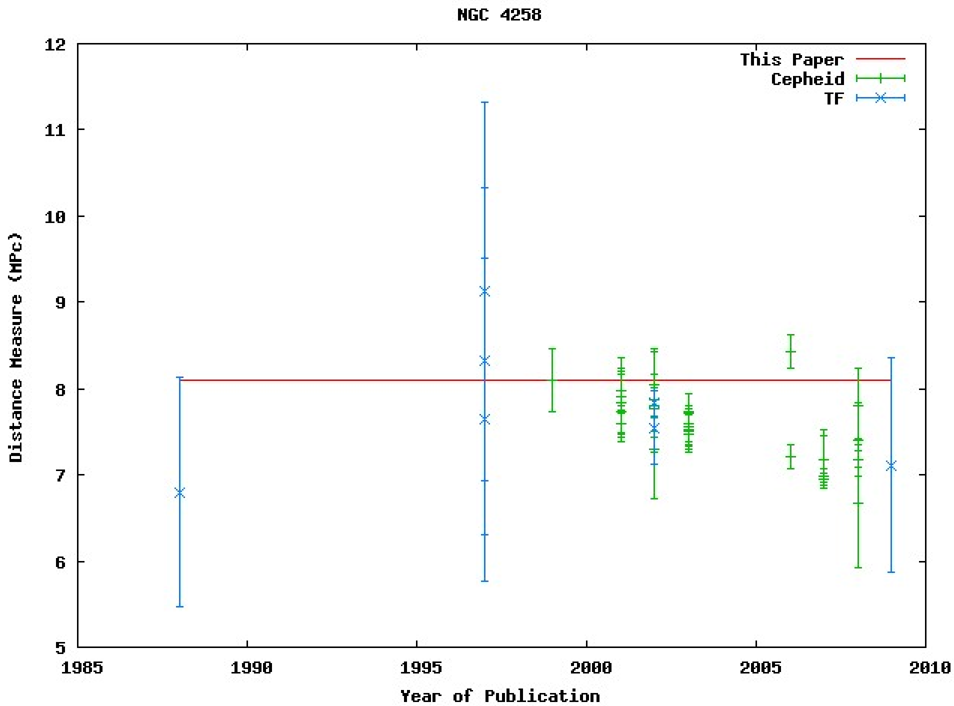


Figure 22: NGC 4258




# 3 Conclusions and Future Work

The authors are proposing that their derivation of the geometry of large rotating systems results in the description of a spiral-shaped space with a constant tangential velocity or 'flat rotation profile'. Since spiral galaxies have a structure congruent to the geometry derived in this paper, the rotation profile, scale, and distance of galaxies can be determined from two measurable parameters: the angular spacing of the arms of the spiral shaped radial geodesic ($\Delta\phi$), and the tangential velocity associated with the portion of the rotation profile that is constant ($v_{max}$).

This paper used manual measurements of $\Delta\phi$ from optical telescopic imagery. The authors suggest an approach that uses measurements of rotational velocity or Cepheid distances to determine $\Delta\phi$ and then use this parameter to verify the calculated values for the other galactic parameters. This would reduce measurement error associated with accurately measuring the spacing of spiral arms and allow measurements to be made for galaxies that do not have a visible spiral structure. For example, curve fitting (31) to the recorded values of rotational velocity would result in a value for $\Delta\phi$, this value could then be used in the calculation of the galactic distance and compared with measurements using Cepheid variables. The opposite approach could be used, using the Cepheid distance to validate the rotation curve of a galaxy.

In this manner, the authors hope to better quantify the error and degree of agreement held by their postulates.

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