Fermat-Catalan Equations（1）

$d^2 = a^3 + c^5$ and $d^2 = a^3 + c^7$

Chun-Xuan Jiang
P. O. Box 3924, Beijing 100854. China
jcxuan@sina.com

Abstract

In this paper we prove that Fermat-Catalan equations $d^2 = a^3 + c^5$ and $d^2 = a^3 + c^7$ has infinitely many coprime integer solutions.

Theorem 1. The Diophantine equation

$$a^3 + mb^3 + m^2c^3 - 3mbc = d^n \quad (1)$$

has infinitely many integer solutions [1,2]

Define supercomplex number [3]

$$w = \begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix} = x + yJ + zJ^2, \quad (2)$$

where

$$J = \begin{pmatrix} 0 & 0 & m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = m.$$

Then from (2)

$$w^n = (x + yJ + zJ^2)^n = a + bJ + cJ^2 \quad (3)$$

Then from equation (3) circulant matrix

$$\begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix}^n = \begin{pmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{pmatrix} \quad (4)$$

Then from equation (4) circulant determinant

$$\begin{vmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{vmatrix}^n = \begin{vmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{vmatrix} \quad (5)$$

Then from equation (5)

$$d^n = a^3 + mb^3 + m^2c^3 - 3mabc \quad (6)$$

where

$$d = x^3 + my^3 + m^2 z^3 - 3mxyz \qquad (7)$$

We prove that (6) has infinitely many integer solutions.
Suppose $n = 2$. From (6) we have

$$d^2 = a^3 + mb^3 + m^2 c^3 - 3mabc \qquad (8)$$

when $n = 2$ from (3)

$$a = x^2 + 2myz, \quad b = 2xy + mz^2, \quad c = y^2 + 2xz \qquad (9)$$

Let $m = c$. From (8) and (9) we have

$$d^2 = a^3 + cb^3 + c^5 - 3abc^2 \qquad (10)$$

$$a = x^2 + 2cyz, \quad b = 2xy + cz^2, \quad c = m = y^2 + 2xz \qquad (11)$$

Suppose $a \neq 0, b = 0, c \neq 0$. From (10) we have

$$d^2 = a^3 + c^5 \qquad (12)$$

From (11) $b = 0$. We have

$$y^2 z^2 + 2xy + 2xz^3 = 0 \qquad (13)$$

From (13) we have

$$y = \frac{-x \pm \sqrt{x}\sqrt{x - 2z^5}}{z^2} \qquad (14)$$

Let $x = u^2$, we rewrite (14)

$$y = \frac{-u^2 \pm u\sqrt{u^2 - 2z^5}}{z^2} \qquad (15)$$

We take $z = 2^{2k+1}$. we have $x = (2^{10k+4} + 1)^2$.

$$y = \frac{-(2^{10k+4} + 1)^2 \pm (2^{20k+8} - 1)}{2^{4k+2}}, k = 0, 1, 2, \ldots$$

We prove that (12) has infinitely many coprime integer solutions. We have

$$(x, y, z) = (289, -136, 2), (a, c, d) = (-10607167, 19652, 41685581663)$$

Theorem 2. The Diophantine equation

$$d^2 = a^3 + m^2 b^3 + m^4 c^3 - 3m^2 abc \qquad (16)$$

where

$$d = x^3 + m^2 y^3 + m^4 z^3 - 3m^2 xyz \qquad (17)$$

$$a = x^2 + 2m^2 yz, \quad b = 2xy + m^2 z^2, \quad c = y^2 + 2xz \tag{18}$$

Suppose $m = c$. We rewrite (16)

$$d^2 = a^3 + c^2 b^3 + c^7 - 3abc^3 \tag{19}$$

Suppose $a \neq 0, b = 0, c \neq 0$. From (19) we have

$$d^2 = a^3 + c^7 \tag{20}$$

From (18) $b = 0$, we have

$$2xy + z^2 (y^2 + 2xz)^2 = 0 \tag{21}$$

From (21) we have

$$x = \frac{-y(1 + 2yz^3) \pm y\sqrt{1 + 4yz^3}}{4z^4} \tag{22}$$

Let

$$R^2 = 1 + 4yz^3 = (2k+1)^2, k = 1, 2, \ldots. \tag{23}$$

(21), (22) and (23) have infinitely many integer solutions. Hence we prove that (20) has infinitely many coprime integer solutions. We have

$$(x, y, z) = (-1, 2, 1), \quad (a, c, d) = (17, 2, 71)$$

Using our method [1-4] it is able to prove the Beal conjecture [5]. Using very complex methods they study these problems [6-8]

References

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