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THE TETRON MODEL IN 6+1 DIMENSIONS

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Abstract

The possibility of a 6+1 dimensional spacetime model being the fundamental theory for elementary particle interactions is explored. The dynamical object is an (octonion) spinor defined over a spacetime lattice with S_8 permutation symmetry which gets broken to $S_4 \times S_4$. Electroweak parity violation is argued to arise from the interplay of the two permutation groups S_4 or eventually from the definition of the octonion product. It corresponds to a change in sign for odd permutation lattice transformations and is shown to suggest a form for the Hamiltonian.

Introduction

My grandfather, who as a young man had distributed anti-Hitler pamphlets even after the Nazis came to power, once asked me, why I would not engage into politics to help to make a better world, and I replied, I had discovered science to be more interesting and important than political issues.

My son, who is now studying physics and philosophy, has recently asked me, why I never tried to make a really fundamental contribution to my research fields. This note is an attempt to remedy the situation. It relies on the tetron ordering scheme of elementary particles which I introduced a few years ago [1, 2, 3]. That scheme elaborates on the one-to-one correspondence between the quarks and leptons and the elements of the permutation group S_4, made explicit in table 1. Furthermore, it does not only describe fermions but also leads to an ordering for the gauge bosons of the left-right symmetric Standard Model (SM) [4].

S_4 is isomorphic to the rotational symmetry group of a regular tetrahedron and is in fact a semi-direct product $S_4 = K \diamond Z_3 \diamond Z_2$ where K denotes the Kleinsche Vierergruppe and the Z_3 factor is the family symmetry [14]. Z_2 and K can be considered to be 'germs' of weak isospin and color symmetry, which give rise to the appearance of the full SM gauge group as a collective emergent phenomenon [2].

In refs. [1, 2, 3] a constituent picture was suggested where quarks and leptons are assumed to be built from 4 tetron 'flavors' a, b, c and d, whose interchanges generate the inner S_4 symmetry. In the present paper I follow a somewhat different approach which relies on the fact that S_4 is also the symmetry group of a tetrahedral lattice or of a fluctuating (quantum) lattice in 3 dimensions. In this approach the inner symmetry space is not continuous (with a continuous symmetry group) but has instead the discrete structure of

$\overline{1234}(id)$ ν_e γ_L	$\overline{2143}(k_1)$ u_1 W_{1L}	$\overline{3412}(k_2)$ u_2 W_{2L}	$\overline{4321}(k_3)$ u_3 W_{3L}
$\overline{2314}$ ν_μ g_{1L}	$\overline{3241}$ c_1 g_{3L}	$\overline{1423}$ c_2 g_{5L}	$\overline{4132}$ c_3 g_{7L}
$\overline{3124}$ ν_τ g_{2L}	$\overline{1342}$ t_1 g_{4L}	$\overline{2431}$ t_2 g_{6L}	$\overline{4213}$ t_3 g_{8L}
$\overline{3214}(1 \leftrightarrow 3)$ e γ_R	$\overline{2341}$ d_1 W_{1R}	$\overline{1432}(2 \leftrightarrow 4)$ d_2 W_{2R}	$\overline{4123}$ d_3 W_{3R}
$\overline{1324}(2 \leftrightarrow 3)$ μ g_{1R}	$\overline{3142}$ s_1 g_{3R}	$\overline{2413}$ s_2 g_{5R}	$\overline{4231}(1 \leftrightarrow 4)$ s_3 g_{7R}
$\overline{2134}(1 \leftrightarrow 2)$ τ g_{2R}	$\overline{1243}(3 \leftrightarrow 4)$ b_1 g_{4R}	$\overline{3421}$ b_2 g_{6R}	$\overline{4312}$ b_3 g_{8R}

Table 1: Quarks and leptons as well as vector bosons of the left-right symmetric SM as two S_4 multiplets. k_i denote the elements of the Kleinsche Vierergruppe and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. W_{iL} and W_{iR} are the gauge bosons of $SU(2)_L$ and $SU(2)_R$, respectively. The currents for photon and gluons were shown to be vectorlike in ref. [2], i.e. $\gamma_L = \gamma_R$ and $g_{iL} = g_{iR}$. It should be noted that this table is only a heuristic assignment. Actually one has to consider symmetry adapted linear combinations of permutation states transforming under representations of S_4, as discussed in refs. [1, 2, 3].

a tetrahedral or S_4-permutation lattice, and the original dynamics is governed by some unknown lattice interaction instead of by four tetron constituents. The observed quarks and leptons can then be interpreted as excitations on this lattice and characterized by representations $A_1 + A_2 + 2E + 3T_1 + 3T_2$ or $2G_1 + 2G_2 + 4H$ of the lattice symmetry group S_4 and its covering [5].

The lattice ansatz naturally explains the selection rule mentioned in ref. [1] that all physical states must be permutation states: just because the lattice excitations must transform under representations of S_4.

Advantages of a Planck Scale Quantum Lattice

The question then immediately arises, where the lattice and its discrete inner S_4 symmetry may come from. The most natural answer is obtained by considering a larger lattice in a 6+1 dimensional spacetime (with, for example, S_8 as symmetry group) and by assuming that the symmetry of this lattice is broken so that for each timestep one has

- a 3-dimensional inner symmetry lattice with symmetry group S_4^{in} (accounting for the tetron structure of elementary particles table 1) and
- a 3-dimensional spatial lattice with symmetry group S_4^{sp} (inducing a lattice structure on physical space)

The totality of excitations on the $S_4^{in} \times S_4^{sp}$ lattice can be classified according to [8]

$$(A_1 + A_2 + 2E + 3T_1 + 3T_2)^{in} \otimes (G_1 + G_2 + 2H)^{sp} \tag{1}$$

where the first factor contains the 24 inner symmetry d.o.f of quarks and leptons (table 1), and G_1^{sp} of the second factor describes their (spin 1/2) spatial transformation behavior.[1]

[1]The other components G_2^{sp} and H^{sp} of the second factor may serve as dark matter

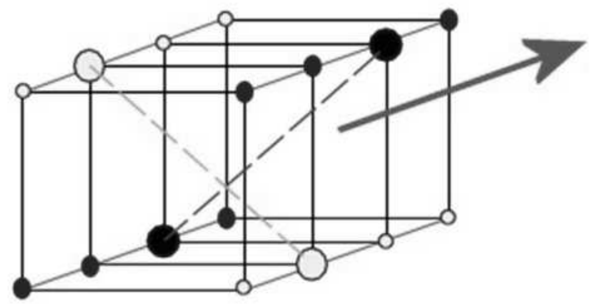

Figure 1: This picture shows part of a 3-dimensional S_4-symmetric lattice containing 1-dimensional sublattices with symmetries $S_4 \to S_2^{in} \times S_2^{sp}$ as visualization of $S_8 \to S_4^{in} \times S_4^{sp}$. The big fat arrow gives the direction of time (perpendicular to the paper plane). The 4 big circles correspond to one timestep t_0. At t_0, transitions between the big open circles correspond to spatial transformations (with symmetry S_2^{sp}), and transitions between the big black circles to inner transformations (with symmetry S_2^{in}). In other words, the dashed dark line defines 1-dimensional inner symmetry space at t_0, while the dashed pale line gives 1-dimensional physical space at t_0.

Using a discrete instead of a continuous spacetime has 2 advantages and 2 drawbacks. The advantages:

- Ultraviolet divergences do not appear; correspondingly there is no need for renormalization.
- No-go theorems like the Weinberg-Witten theorem [18] which in the continuum forbid the unification of spatial and inner symmetries do not apply.

On the other hand there are 2 drawbacks (which however are not relevant for a quantum lattice with lattice spacings of the order of the Planck scale):

- On a lattice Lorentz symmetry is broken to a discrete subgroup, which is of course in contradiction to everyday experience. However, for a classical observer Lorentz symmetry can be restored by assuming the lattice to be a fluctuating quantum lattice, where the lattice points move around randomly, with the fluctuations following some (quantum) stochastic process [6].[2]

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candidates, as discussed in [8]. If one takes serious the assumption that $S_4^{in} \times S_4^{sp}$ originally stems from a higher symmetry (like e.g. S_8), expression (1) should better be replaced by

$$(G_1 + G_2 + 2H)_{in} \otimes (G_1 + G_2 + 2H)_{sp} \tag{2}$$

i.e. one should work with projective representations of the tetrahedral group both in the inner symmetry and in the spatial sector, simply because there are no mixed bosonic and fermionic S_8 representations.

[2]There is some relation of this idea to other models which involve a fundamental length scale, like quantum foam models [15, 16, 17], which however assume gravity to play the central role in the dynamics, while in the present model gravitational interactions and cosmological phenomena appear only as byproducts of the tetron lattice interactions, as shown in ref. [8].

- It is usually difficult to define fermions on a lattice without getting problems with (micro)causality, because in contrast to bosons fermions 'know' about other fermions on neighbouring lattice sites and this induces nonlocal correlations and possible synchronisations beyond the event horizon. However, assuming the lattice spacings to be of the order of the Planck scale, quantum effects torpedo the concept of causality at those tiny distances anyhow. We shall see later, how this may be coined in a lattice Lagrangian model.

A 6+1 dimensional Spinor as the fundamental dynamical Field in the Tetron Model

I propose to use a 6+1 dimensional spinor as the fundamental dynamical field, whose lattice excitations are to describe the SM fields. Such a spinor corresponds to a SO(7) spinor by a Wick rotation[3] and reduces to two SO(6) spinors in the non-relativistic limit.

While the covering group of SO(6) is isomorphic to SU(4) and has 2 fundamental complex representations $\underline{\mathbf{4}}$ and $\underline{\mathbf{4}}^*$ which are conjugate to each other, the covering group of SO(7) is Spin(7) with one spinor representation of dimension $\underline{\mathbf{8}}$, which is closely related to the non-associative division algebra of octonions [9, 10, 11, 12]. Breaking Spin(7)→SU(4) there is a decomposition $\underline{\mathbf{8}} \rightarrow \underline{\mathbf{4}} + \underline{\mathbf{4}}^*$ which reveals the particle antiparticle content of the original SO(6,1) spinor.

Dividing SO(6) into a spatial and an inner symmetry part $SO(3)^{sp} \times SO(3)^{in}$ reduces its spinor representation $\underline{\mathbf{4}} \rightarrow (\underline{\mathbf{2}}, \underline{\mathbf{2}})$ further, and going to a lattice with $S_4^{sp} \times S_4^{in}$ symmetry one obtains states which transform as (G_1^{sp}, G_1^{in}) where G_1 is the 2-dimensional spinor representation of S_4 obtained by re-

[3] More generally, in SO(p,q) the spinor dimensions viewed over complex space coincide with the case of the (p+q)-dimensional Euclidean space.

relation to octonions	nonrelativistic limit	restriction to a lattice
$Spin(7) \leftrightarrow \widetilde{SO(6,1)}$	$\widetilde{SO(6,1)} \rightarrow SU(4)$	$SU(4) \rightarrow \widetilde{S_4^{sp}} \times \widetilde{S_4^{in}}$
$\underline{\mathbf{8}}$	$\underline{\mathbf{8}} \rightarrow \underline{\mathbf{4}} + \underline{\mathbf{4}}^*$	$\underline{\mathbf{4}} \rightarrow (G_1^{sp}, G_1^{in})$

Table 2: Group theoretic view on the fundamental dynamical field F in the tetron model. The tilde denotes covering groups.

stricting the Pauli representation of $SU(2) = \widetilde{SO(3)}$ to S_4.

The overall situation is summarized in table 2.

4-Fermion interactions of (6+1)-dimensional Dirac spinors F formally look similar like in (3+1)-dimensional Minkowski space, e.g.

$$\mathfrak{L} \propto (\bar{F} e_\mu F)(\bar{F} e^\mu F) \tag{3}$$

for vectorlike interactions where e_0, ..., e_6 are the seven 8×8 Dirac matrices in 6+1 dimensions. They are made up from building blocks of Dirac spinors in lower dimensions as described for example in [13] and have a close relationship to the octonion algebra [9], just as quaternions have to Pauli matrices. While the Pauli matrices can be identified more or less directly with the quaternion units, the situation in 7 or 6+1 dimensions is somewhat more subtle because the octonion algebra is not associative, i.e. the octonion units I, J, IJ, L, IL, JL and (IJ)L cannot be exactly represented by the 7 matrices e_μ. A detailed and explicit description of the relationship can be found, for example, in the book by Dixon [9].

Spin Models as an Approach to the Tetron Model Dynamics

It is hard to say, what the true dynamics on the tetron lattice may be. What seems to be clear is that a fermionic component as described in the last section is involved. In addition there could be a fundamental (6+1)-dimensional vector boson, for example on a supersymmetric basis, and other stuff. Candidates for a dynamical model would then be either a 4-fermion interaction like eq. (3) restricted to the S_8 or $S_4^{in} \times S_4^{sp}$ permutation lattices or a (6+1)-dimensional lattice Yang-Mills theory.

If one prefers effective theories, one should try out spin models. Spin models have been considered in statistical and solid state physics for a long time, and they have been used to describe magnetism and magnetic excitations as well as many other phenomena. The basic components in that model would be 'spin vectors' $\vec{S}(i)$ sitting on each lattice site i together with an interaction of Heisenberg type

$$H = g_{SS} \sum_{i,j} \vec{S}(i)\vec{S}(j) \tag{4}$$

where g_{SS} is the coupling strength and the sum runs over all neigbouring lattice sites i and j.[4] Fermionic excitations have been claimed to arise [7] when one decomposes these vectors in terms of a fermion degree of freedom as $\vec{S} = F^\dagger \vec{e} F$.

An essential requirement on the dynamics is that the parity violation (PV) of the weak interactions should be described correctly. Usually a *natural* explanation of weak parity violation in subquark models is not an easy task. In the present framework the situation is somewhat simpler. The point is that in the tetron model, as can be seen in table 1, weak isospin transfor-

[4] Note that an Euclidean notation is used here. In 6+1 dimensions ($x_0 \to ix_0$) the vector product used in eq. (6) is in fact imaginary.

mations are related to odd permutations, i.e. to rotoreflections in the inner symmetry lattice, and those rotoflections have something to do with parity transformations (see below).

There are in principle two possibilities to break parity: explicitly or spontaneously. In the 4-fermion interaction eq. (3), for example, an explicit parity violation can be introduced by adding a (6+1)-dimensional γ_5-matrix. In contrast, the Heisenberg ansatz (4) is parity even. So one needs a *spontaneous* breaking like in the left-right symmetric SM. This point of view was taken in earlier papers on the tetron idea, where the $SU(2)_L$ and $SU(2)_R$ bosons W_L and W_R together with photon and gluons were put into a single S_4^{in} multiplet (cf. table 1 and ref. [2]). In the framework of 6+1 dimensions this means one should start with a dynamics like (4) which in addition to S_4 is invariant under the product $P^{in}P^{sp}$, the parity operation in 6 dimensions. These operations simultaneously induce isospin transitions of fermions like $e \leftrightarrow \nu$ as well as from left- to righthanded states and of vector bosons $W_L \leftrightarrow W_R$ (cf. table 1), so that one ends up with the currents of the left-right symmetric SM

$$\begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \vec{W}_L^\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \begin{pmatrix} \bar{\nu}_R & \bar{e}_R \end{pmatrix} \vec{W}_R^\mu \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} \tag{5}$$

being parity invariant before spontaneous symmetry breaking.

An interesting example of a spin model with *explicit* PV and with a tetron excitation spectrum is given by

$$H = g_{SSS} \sum_t (\vec{S}_4 - \vec{S}_1)[(\vec{S}_3 - \vec{S}_1) \times (\vec{S}_2 - \vec{S}_1)] \tag{6}$$

which is defined on tetrahedra t with lattice points 1,2,3,4 and uses the existence of the cross product $\vec{A} \times \vec{B}$ in 7 dimensions.

The triple product $\vec{C}(\vec{A} \times \vec{B})$ is sometimes called 'associative calibration' in the literature [20]. In the continuum its invariance group is in fact not the full SO(7) but the exceptional Lie group $G_2 \subset$ SO(7) which comprises the $S_4^{in} \times S_4^{sp}$ symmetry group of the permutation lattice. The interaction (6) is antisymmetric under odd permutations in both factors of that group. This is precisely what is needed to describe the PV on the tetron lattice, simply because one will get negative energy contributions to the partition function, if an odd transformation is applied to the base points of the tetrahedron t.

With such an ansatz one actually connects weak PV to the definition of the octonion product. The point is that the 7-dimensional cross product is related to the imaginary part of the octonion product [10]. Namely, after identifying the axes of 7-dimensional space with the octonion units the cross product is given in terms of octonion multiplication by

$$\vec{A} \times \vec{B} = \frac{1}{2}(AB - BA) \tag{7}$$

for any two octonions $A = A_0 + \vec{A}$ and $B = B_0 + \vec{B}$. The failure of the cross product to satisfy the Jacobi identity is due to the nonassociativity of the octonions. A change in sign in the definition of the associator (IJ)L=-I(JL) would reverse all PV effects.

Conclusions

It is an old dream of theoretical physicists that inner symmetries may be obtained by extending ordinary space to higher dimensions (see for example ref. [19]). In the present paper a discrete (6+1)-dimensional spacetime has been proposed to unify spatial and inner symmetries with the additional benefit of having relations to the division algebra of octonions. A fundamental fermion field has been considered, and lattice spin models have been discussed as possible effective schemes for the implementation of the tetron idea.

At the present stage, the true nature of the underlying dynamics remains unclear. For example, it is possible that it turns out to be in some sense supersymmetric with a (vector) boson appearing in addition to the fundamental fermion. However, one should consider this option far from being compelling. In particular, the appearance of discrete lattice structures at Planck distances gives no indication that the fundamental Lagrangian will have anything to do with the nowadays popular superstring or M-theories.

The presented ideas are not only in opposition to superstrings. They also reduce the celebrated gauge theories and the Standard Model to what they probably are: a beautiful and logical theoretical framework which however holds true only on a certain level of matter and energy. The tetron model is an idea that goes beyond this level (just as quarks go beyond nuclear physics) and also offers explanations for outstanding cosmological problems [8].

In fact, such a situation is not unusual in the development of science. It is well known from the macroscopic world as well as from molecular and atomic physics that when going to a lower level of matter one has to give up the full understanding of some emergent phenomena known from the higher levels. In the present case we have given up continuus Lorentz invariance (which is restored at large distances / low energies) as well as the full gauge symmetry groups (whose appearance is considered to be a collective emergent effect). Instead of increasing the symmetry at higher energies like in GUT or SUSY ansätzen our starting point was a fluctuating quantum lattice picture in 6+1 dimensions with a permutation symmetry.

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