THE TRISECTION OF ANY ANGLE

This article has been written during last month of 2010 and is solely for my own amusement and private study and also to all those readers that believe to Euclid Geometry as the model of all nature . A meter also for testing sufficiency of Geometries .

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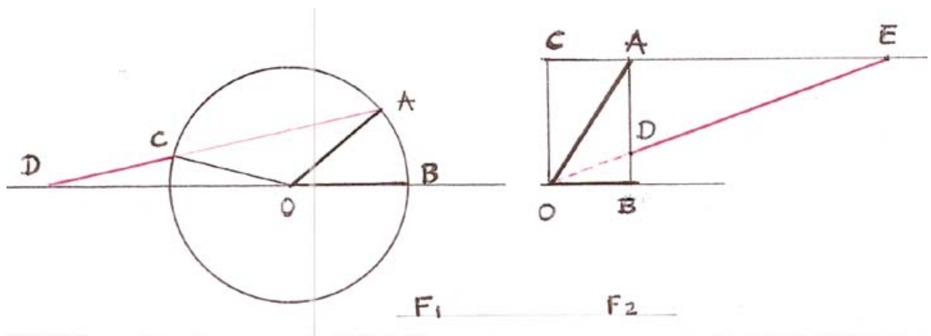
1. **Archimedes method** 2. **Pappus method**



Consider the angle < AOB .

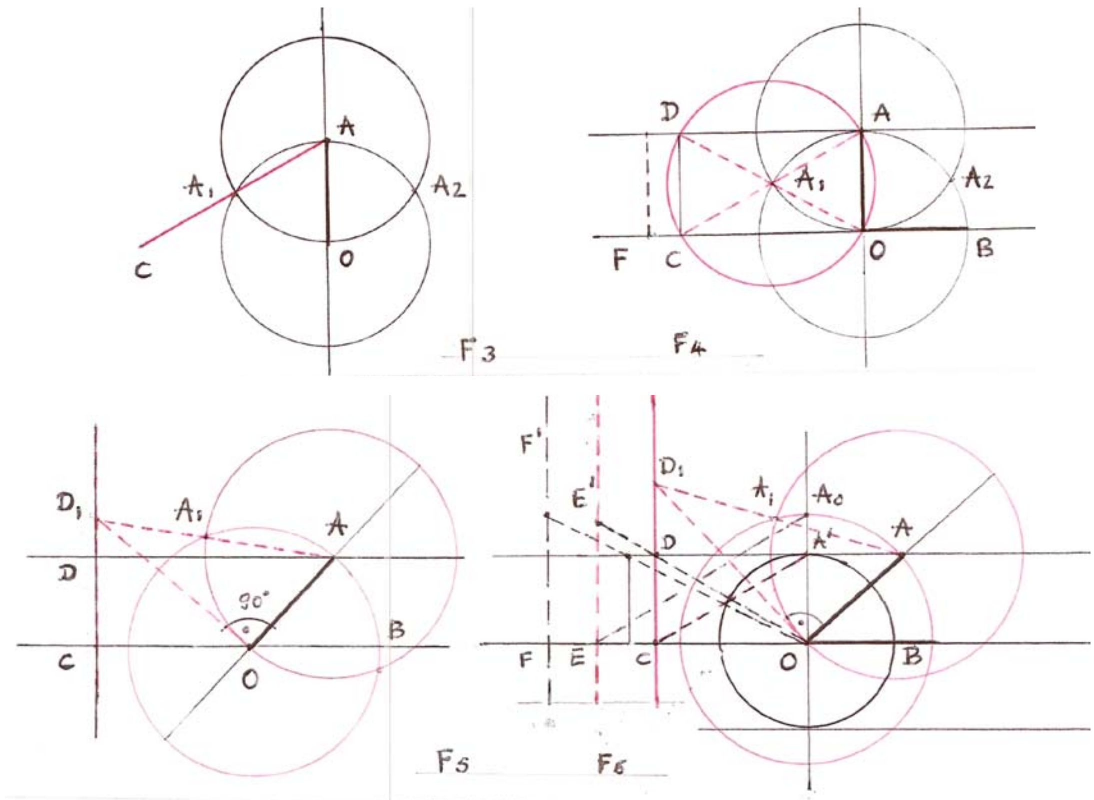
Draw circle (O , OA) with its center at the vertex O and produce side BO to D . Insert a straight line AD so point C is on the circle and point D on line BO and length DC **such that** it is equal to the radius of the circle .

Proof :
Since CD = CO then triangle CDO is isosceles and angle < CDO = COD
The external angle OCA of triangle CDO is < OCA = CDO + COD = 2. CDO
and equal to angle ADO and since angle < OAC = OCA then < OAC = 2.ODA
The external angle AOB of triangle OAD is < AOB = OAD + ODA =
2. ODA + ODA = 3 . ODA

2. **Pappus method** :

It is a slightly different of Archimedes method can be reduced to a neusis as follows
Consider the angle < AOB .
Draw AB perpendicular to OB .
Complete rectangle ABOC .
Produce the side CA to E .
Insert a straight line ED of given length 2 .OA between AE and AB
in such a way that ED verges towards O . Then angle < AOB = 3 . DOB

3. **The Present method :**



We extend Archimedes method as follows :

a . (F. 3 - 4) Given an angle < AOB = AOC = 90°

1. Draw circle (A , AO = OA) with its center at the vertex A intersecting circle (O , OA = AO) at the points A_1 , A_2 respectively .
2. Produce line AA_1 at C so that A_1C = A_1 A = AO and draw AD // OB .
3. Draw CD perpendicular to AD and complete rectangle AOCD .
4. Point F is such that OF = 2 . OA

b . (F. 5 - 6 -7a) Given an angle < AOB < 90°

1. Draw AD parallel to OB .
2. Draw circle (A , AO = OA) with its center at the vertex A intersecting circle (O , OA = AO) at the points A_1 , A_2 .
3. Produce line A A_1 at D_1 so that A_1D_1 = A_1 A = OA .
4. Point F is such that OF = 2 . OA = 2 . OAo .
5. Draw CD perpendicular to AD and complete rectangle A′OCD .
6. Draw Ao E Parallel to A′ C at point E (or sliding E on OC) .
7. Draw AoE′ parallel to OB and complete rectangle AoOEE′ .
8. Draw AF intersecting circle (O , OA) at point F_1 and insert on AF segment F_1 F_2 equal to OA → F_1 F_2 = OA .
9. Draw AE intersecting circle (O , OA) at point E_1 and insert on AE segment E_1 E_2 equal to OA → E_1 E_2 = OA = F_1 F_2 .

Show that :

a) For all angles equal to 90▫ Points C and E are at a constant distance
$OC = OA . \sqrt{3}$ and $OE = OAo . \sqrt{3}$, from vertices O , and also A′C //AoE .

b) The geometrical locus of points C , E is the perpendicular CD , EE′ on AB.

c) All equal circles with their center at the vertices O , A and radius OA = AO have the same geometrical locus EE′ ⊥ OE for all points A on AD , or All radius of equal circles drawn at the points of intersection with its Centers at the vertices O , A and radius OA = AO lie on CD , EE′ .

d) Angle < D_1OA is always equal to 90▫ and angle AOB is created by rotation of the right-angled triangle AOD_1 through vertex O .

e) Angle < AOB is created in two ways , By constructing circle (O , OA = OAo) and by sliding of point A′ on line A′ D Parallel to OB from point A′ to A .

f) The rotation of lines AE , AF on circle (O , OA = OAo) from point E to point F which lines intersect circle (O , OA) at the points E_1 , F_1 respectively , **fixes a point** G on line EF and a point G_1 common to line AG and to the circle (O , OA) **such that** $GG_1 = OA$.

Proof :

a) .. (F3 , F4)

Let OA be one-dimentional Unit perpendicular to OB such that angle
< AOB = AOC = 90▫
Draw the equal circles (O,OA) , (A , AO) and let points A_1 , A_2 be the points of intersection . Produce AA_1 to C .
Since triangle AOA_1 has all sides equal to OA ($AA_1 = AO = OA_1$) then it is an equilateral triangle and angle < A_1AO = 60 ▫
Since Angle < CAO = 60 ▫ and AC = 2. OA then triangle ACO is right-angled and angle < AOC = 90▫ , and so the angle ACO = 30 ▫ .
Complete rectangle AOCD

Angle < ADO = 180 – 90 – 60 = 30 ▫ = ACO = 90 ▫ / 3 = 30 ▫
From Pythagoras theorem $AC^2 = AO^2 + OC^2$ or $OC^2 = 4.OA^2 - OA^2 = 3. OA^2$

and ***$OC = OA . \sqrt{3}$.***
For OA = OAo then AoE = 2. OAo and ***$OE = OAo . \sqrt{3}$.***
Since OC / OE = OA / OAo → then line CA′ is parallel to EAo

b) .. (F5 , F6)

Triangle OAA_1 is isosceles , therefore angle < A_1AO = 60 ▫ . Since $A_1D_1 = A_1O$, triangle D_1A_1O is isosceles and since angle < OA_1A = 60 ▫ , therefore angle < OD_1A = 30 ▫ or , Since $A_1A = A_1D_1$ and angle < A_1AO = 60 ▫ then triangle AOD_1 is also right-angle triangle and angles < D_1OA = 90 ▫ , angle < OD_1A = 30 ▫ .

Since the circle of diameter D_1A passes through point O and also through the foot of the perpendicular from point D_1 to AD , and since also ODA = ODA′ = 30 ▫ , then this foot point coincides with point D , therefore the locus of point C is the perpendicular CD_1 on OC . For $AA_1 > A_1D_1$, D'_1 is on the perpendicular D'_1E on OC.

c) .. (F5 , F6)

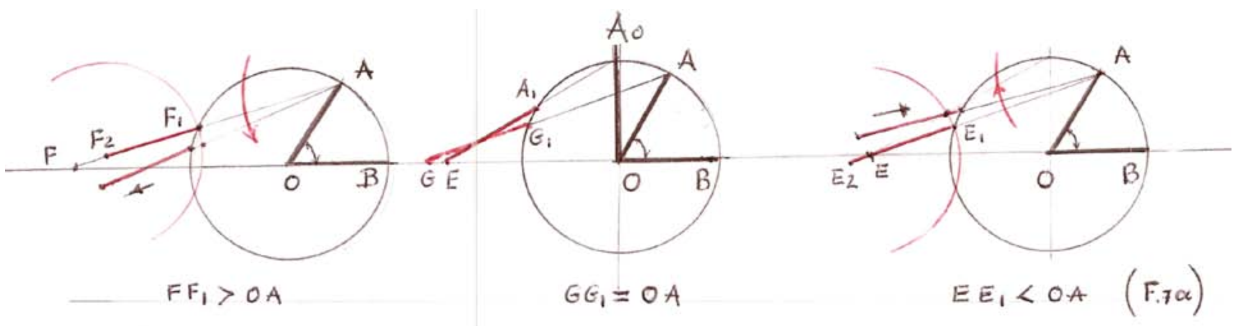
Since the Parallel from point A_1 to OA passes through the middle of OD_1 , *and in case where AOB = AOC = 90 ▫ through the middle of AD* , then the circle with diameter D_1A passes through point D which is the base of the perpendicular , **i.e.** ***The geometrical locus of points C or E is the perpendicular CD , EE′ on OB .***

d) .. (F5 , F6)

Since A_1A = A_1D_1 and angle < A_1AO = 60 ▫ then triangle AOD_1 is right-angle triangle and ***angle < D_1OA = 90 ▫*** .

Since angle < AD_1O is always equal to 30 ▫ and angle D_1OA is always equal to 90▫ , therefore angle < AOB is created by the rotation of the right - angled triangle AOD_1 through vertex O .

Since tangent through Ao to circle (O , OA′) lies on the circle of half radius OA then this is perpendicular to OA and equal to A′A .



F_2F_1 = OA A_1E = OAo ,,, GA_1 = OA E_2E_1 = OA

e) .. (F5 , F6 , F.7a)

Let point **G** be sliding on OB between points **E** and **F** where lines AE , AG , AF intersect circle (O , OA) at the points E_1 , G_1 , F_1 respectively where then exists FF_1 > OA , GG_1 = OA , EE_1 < OA .

Points E , F are the limiting points of rotation of lines AE , AF (because then for angle < AOB = 90 ▫ → $A_1C = A_1A$ = OA , $A_1Ao = A_1E$ = OAo and for angle < AOB = 0▫ → OF = 2 . OA) . Exists also E_1E_2 = OA , F_1F_2 = OA and point G_1 common to circle (O , OA) and on line AG such that GG_1 = OA .

AE_2 oscillating to AF_2 passes through AG so that GG_1 = OA and point G on EF. When point G_1 of line AG is moving (rotated) ***on circle (E_2 , E_2E_1 = OA***) and Point G_1 of ***G_1G is stretched on circle (O , OA)*** then $G_1G \neq$ OA .

A position of point G_1 is such that , when $GG_1 = OA$ point G lies on line EF.
When point G_1 of line AG is moving (rotated) ***on circle (F_2 , F_2F_1 = OA***) and point G_1 of ***G_1G is stretched on circle (O , OA)*** then $G_1G \neq OA$.
A position of point G_1 is such that , when $GG_1 = OA$ point G lies on line EF.
For both opposite motions there is only one position where point G lies on line OB and is not needed point G_1 of GA to be stretched on circle (O , OA) .

This position happens at the common point P of the two circles which is their point of intersection . At this point P exists only rotation and is not needed G_1 of GA to be stretched on circle (O , OA) so that point G to lie on line EF. This means that point P lies on the circle (G , GG_1 = OA) , or GP = OA .

Point A of angle < BOA is verged through two different and opposite motions , i.e.

1 . From point A′ to point Ao where *is done a parallel translation* of CA′ to the new position EAo , *this is for all angles equal to 90*▫ , and from this position to the new position EA by rotating EAo to the new position EA having always the distance $E_1 E_2$ = OA .
This motion is taking place on a circle of centre E_1 and radius $E_1 E_2$.
2. From point F , *where OF = 2. OA , is done a parallel translation of A′F” to FAo,* and from this position to the new position FA by rotating FAo to FA having always the distance $F_1 F_2$ = OA .
The two motions coexist again on a point **P** which is the point of intersection of the circles (E_2 , E_2E_1 = OA) and (F_2 , F_2F_1 = OA) .

f) .. (F5 , F6 , F.7a) Remarks – Conclusions .

1 . Point E_1 is common of line AE and circle (O, OA) and point E_2 is on line AE such that $E_1 E_2$ = OA and exists $E E_1 < E_2 E_1$ **.** $E_1 E_2$ = OA is stretched *,moves* on EA so that point E_2 is on EF . Circle (E , $E E_1 < E_2 E_1$ = OA) cuts circle (E_2 , $E_2 E_1$ = OA) at point E_1 . There is a point G_1 on circle (O ,OA) such that G_1G = OA , *where point G is on EF* , ***and is not needed G_1G to be stretched*** on GA where then , circle (G , GG_1= OA) cuts circle (E_2 , $E_2 E_1$ = OA) at a point P .

2. Point F_1 is common of line AF and circle (O,OA) and point F_2 is on line AF such that $F_1 F_2$ = OA and exists $F F_1 > F_2 F_1$ **.** $F_1 F_2$ = OA is stretched *,moves* on FA so that point F_2 is on F . Circle (F , $F F_1 > F_2 F_1$ = OA) cuts circle (F_2 , $F_2 F_1$ = OA) at point F_1 . There is a point G_1 on circle (O ,OA) such that G_1G = OA , *where point G is on FE* , ***and is not needed G_1G to be stretched*** on OB where then , circle (G , GG_1= OA) cuts circle (F_2 , $F_2 F_1$ = OA) at a point .

3. ***When point G is at such position on EF that GG_1= OA , then point G must be at* A COMMON , *to the three lines EE_1 , FF_1 , GG_1 , and also to the three circles (E_2 , $E_2 E_1$ = OA) , (G , GG_1= OA) , (F_2 , $F_2 F_1$ = OA) . This is possible at the common point P of Intersection of circle (E_2 , E_2E_1 = OA) and (F_2 , F_2F_1 = OA) and since GG_1 is equal to OA without G_1G be stretched on GA , then also GP= OA***

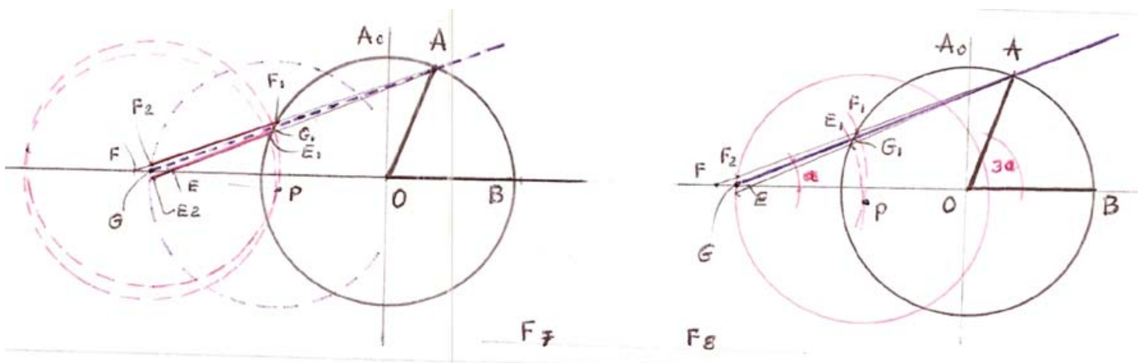
4. On the contrary , for point G_1 :

a. Point G_1 ,*from point* E_1*,* moving on circle (E_2 , $E_2 E_1$ = OA) formulates $GG_1 <$ OA on line GA . There is a point on circle (E_2 , $E_2 E_1$ = OA) such that GG_1 = OA .

b. Point G_1 ,*from point* F_1*,* moving on circle (F_2 , $F_2 F_1$ = OA) formulates $GG_1 >$ OA on line GA . There is a point on circle (F_2 , $F_2 F_1$ = OA) such that GG_1 = OA

c. Since for both opposite motions there is a point on the two circles that makes GG_1= OA then this point say P , is common to the two circles .

d. Since for both motions at point P exists GG_1 = OA then circle (G , GG_1= OA) passes through point P , and since point P is common to the three circles , then fixing point P as common to the two circles (E_2 , $E_2 E_1$ = OA) ,(F_2 , $F_2 F_1$ = OA) , point G is found as the point of intersection of circle (P , PG = OA) and line EF. This means that the common point P of the three circles is constant to this motion



5. ***The steps of Trisection of any angle < AOB*** **=** 90 ▫ → 0 ▫ (F6 , F7 , F8)

1. Draw circle (O , OA) and line AD parallel to OB .
2. Draw OAo ⊥ OB where point Ao is on the circle (O , OA) and the circle (Ao , AoE = 2.OA) which intersects line OB at the point E .
3. Fix point F on line OB such that OF = 2 . OA
4. Draw lines AF , AE intersecting circle (O , OA) at points F_1 , E_1 respectively .
5. On lines F_1A , E_1A fix points F_2 , E_2 such that F_2F_1 = OA and $E_2 E_1$ = OA
6. Draw circles (F_2 , $F_2 F_1$ = OA) , (E_2 , $E_2 E_1$ = OA) and fix point P as the common point of intersection .
7. Draw circle (P , PG = OA) intersecting line OB at point G and draw line GA intersecting circle (O , OA) at point G_1 . ***Segment GG1 = OA*** .

Proof :

1 . Since point P is common to circles (F_2 , $F_2 F_1$ = OA) , (E_2 , $E_2 E_1$ = OA) , then PG = PF_2 = PE_2 = OA and line AG between AE , AF intersects circle (O ,OA) at the point G_1 such that GG_1 = OA . (**e - f.3 - f.4**)

2 . Since point G_1 is on the circle (O , OA) and since GG_1 = OA then triangle GG_1O is isosceles and angle < AGO = G_1OG .

3 . The external angle of triangle GG_1O is < AG_1O = AGO + G_1OG = 2. AGO .

4 . The external angle of triangle GOA is < AOB = AGO + OAG = 3.AGO .
Therefore angle < AGB = (1 / 3) . (AOB) ………….. (ο.ε.δ.)

Conclusions:

1. Following the dialectic logic of ancient Greeks (Αναξίμανδρος)
« τό μή Ον , Ον Γίγνεσθαι » ‘ The Non-existent , Exists when is done ’ , ‘ The Non - existent becomes and never is ’ and the Structure of Euclidean geometry [6] in a Compact Logic Space Layer , as this exists in a known Unit (***case of 90 ▫ angle***) , then we may find a new machine that produces the 1/3 of angles . Since Non-Existent is found everywhere , then Existence is found and is done everywhere .
In Euclidean geometry points do not exist , but their position and correlation is doing geometry . The universe cannot be created , because becomes and never is . According to Euclidean geometry , and since the position of points (*empty Space*) creates geometry and Spaces , the trisection of any angle exists in this Space & way.

2. It has been proved [5] that two equal and perpendicular one-dimentional Units OA , OB formulate a machine which produces squares and one of them is equal to the area of the circle (O , OA = OB)

3. It has been proved [6] that three points formulate a Plane and from the one point passes only one Parallel to the other straight line (three points only) .

4. It has been proved [7] that all Subspaces in a unit circle of radius the one-Dimensional unit OA are the Regular Polygons in the unit circle .

5. Now is proved [8] that one-dimentional Unit OA lying on two parallel lines OB , AD formulate all angles < AOB = 90 ▫ → 0 and a new geometrical machine exists which divides angle < AOB to three equal angles .

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