Reducing the Quantumness of Composite Quantum Systems to Two Classical Compositions

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Dedicated to Marie-Louise Nykamp

Abstract

A family of *quantumness spaces* is identified and precisely defined. They are spaces which characterize the difference between states given by classical compositions of systems, and on the other hand, states corresponding to their quantum compositions. Consequently, the quantum composition of systems is *reduced* to two classical compositions. A family of rankings is also defined for the respective family of quantumness.

"History is written with the feet ..."

Ex-Chairman Mao,
of the Long March fame

"Science nowadays is not done
scientifically, since it is mostly done
by non-scientists ... "

Anonymous

1. Introduction

As seen in [1-3], there is a natural division of physical systems into Cartesian ones, and on the other hand, non-Cartesian ones, according to the way their state spaces compose.
Namely, given two Cartesian systems X and Y with the respective state spaces E and F, then the composite system "$X \, and \, Y$" has the state space given by the Cartesian product $E \times F$. Classical physical systems are in this sense Cartesian.
On the other hand, quantum systems, for instance, have considerably larger state spaces for their composites. Namely, if X and Y are two such systems and their state spaces are the complex Hilbert spaces E and F, respectively, then the state space of the composite quantum system "$X \, and \, Y$" is the tensor product $E \bigotimes F$. And indeed, this is a considerably larger space than the Cartesian product $E \times F$, since we have the injective mapping, which for convenience we shall consider to be an *embedding*

$$(1.1) \qquad E \times F \ni (x, y) \longmapsto x \otimes y \in E \bigotimes F$$

thus is seen according to the *inclusion*

$$(1.2) \qquad E \times F \subset E \bigotimes F$$

and the difference between the two sets, in this case both complex Hilbert spaces is clearly illustrated already in the finite dimensional case when, if m, n are the dimensions of E and F, respectively, then $m+n$ is the dimension of $E \times F$, while $E \bigotimes F$ will have the dimension mn. Thus in general, the set of *entangled* elements

$$(1.3) \qquad (E \bigotimes F) \setminus (E \times F)$$

is considerably larger than the set $E \times F$ of non-entangled elements.

Here it is important to note the following related to (1.2). Let E and F be abelian groups. Then the sets $E \times F$ and $E \bigotimes F$ obtain cor-

responding structures of abelian group. However, $E \times F$ it is *not* a subgroup of the abelian group $E \bigotimes F$. Instead, $E \times F$ is a *generator* of $E \bigotimes F$. Consequently, the mapping in (1.1) is *not* a group homomorphism.
Clearly, similar situations happen when E and F are vector or Hilbert spaces.

An essential difference, therefore, between Cartesian and non-Cartesian physical systems is that in the state spaces of the composites of two of the latter kind there are states which can be seen as *entangled* in some appropriate sense, namely, those states which *cannot* be expressed simply in terms of a pair of states, with each state in the pair taken from one of the two component systems.
So far, physics happens to know only about two ways of composing systems, namely, the classical Cartesian one, and the quantum one. However, there is no known reason to expect that these two ways would be the only ones physical systems are indeed possible to be composed. In this sense, being an entangled state of the composite of two physical system may eventually have a more general meaning than being an element in the set (1.3).

As far as quantum systems are concerned, it is well known that entangled composite states are most important, for instance, in quantum computation, and in general, quantum information technology.

As mentioned, so far, it appears that the only known non-Cartesian physical systems are the quantum ones.
In this regard, in [1-3], the problem was formulated to find physical systems other than the quantum ones and which are non-Cartesian.
Needless to say, there may be various applicative advantages in such systems. Among others, they may be used to build computers which - due to the presence of entangled states - could have advantages over usual electronic digital computers.

The main message, see pct. 2) in Remark 1 in section 4, is that for two quantum systems X and Y, one has

$$X \text{ } \textit{quanutm composed with } Y \ =$$

$$= \ (\ X \ classical \ composed \ with \ Y \ ) \ classical \ composed \ with$$

$$(\ quantumness \ of \ X \ and \ Y \ )$$

or more compactly written

$$(1.4) \qquad X \bigcirc_q Y \ = \ (\ X \bigcirc_c Y \ ) \bigcirc_c (\ quantumness \ of \ X \ and \ Y \ )$$

where $\bigcirc_c$ denotes the classical composition of systems, that is, the Cartesian product of their state spaces, while $\bigcirc_q$ denotes the quantum composition of systems, thus the tensor product of their state spaces, see [3].

It follows that the quantum composition $X \bigcirc_q Y$ is *reduced* to two classical compositions, namely, first

$$X \bigcirc_c Y$$

the result of which is then further composed classically with

$$quantumness \ of \ X \ and \ Y$$

Needless to say, by the above reduction the quantum aspects are not eliminated completely, since they remain in the

$$quantumness \ of \ X \ and \ Y$$

However, the interest in such reduction is in the consequent relegation of whatever the quantum aspects may ever be as such to a specific well circumscribed place, namely, as a mere component in two successive classical compositions.

2. Constructing Tensor Products

Let us start with a more general, and thus simpler setup in order to better highlight what is going on. Let E and F be two abelian groups.

Then their tensor product $E \bigotimes F$ is constructed in the following *five steps*, [1-3].

Step 1 :

Let G be the *free monoid* generated by the elements of the usual Cartesian product $E \times F$. In other words, the elements of G are all the finite sequences

$$(2.1) \qquad (a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$$

where $n \geq 1$ and $a_1, a_2, a_3, \ldots, a_n \in E, b_1, b_2, b_3, \ldots, b_n \in F$. We also include the *empty sequence*, which thus corresponds to $n = 0$.

Step 2 :

We recall that the *monoidal composition* of these sequences is done simply by their concatenation. Furthermore, in order to simplify the notation, the comas between pairs of brackets in (2.1) will be omitted. It will be convenient to denote the resulting *monoid* by

$$(2.2) \qquad (G, \diamond)$$

Clearly, $(G, \diamond)$ is a *noncommutative* monoid, whenever at least one of the groups E or F has more than one single element. Further, we have the *injective* mapping

$$(2.3) \qquad E \times F \ni (a, b) \longmapsto (a, b) \in G$$

which in fact is the *embedding*

$$(2.4) \qquad E \times F \subseteq G$$

and this embedding is *strict*, whenever $m + n < mn$, where E has at least m elements, while F has at least n elements.

Step 3 :

We define an *equivalence relation* $\approx$ on G as follows. Given two elements

$$g = (a_1, b_1)(a_2, b_2)(a_3, b_3) \dots (a_n, b_n),$$

$$h = (c_1, d_1)(c_2, d_2)(c_3, d_3) \dots (c_m, d_m) \in G$$

they are equivalent, if and only if any of the following conditions holds :

(2.5) $\quad g = h$

or one of the elements g or h can be obtained from the other by a finite number of applications of any of the following operations :

(2.6) a permutation of pairs (a, b) in g

(2.7) a permutation of pairs (c, d) in h

(2.8) replacement of a pair $((a' + a''), b)$ in g with the pair of pairs $(a', b)(a'', b)$, or vice-versa

(2.9) replacement of a pair $(a, (b' + b''))$ in g with the pair of pairs $(a, b')(a, b'')$, or vice-versa

(2.10) replacement of a pair $((c' + c''), d)$ in h with the pair of pairs $(c', d)(c'', d)$, or vice-versa

(2.11) replacement of a pair $(c, (d' + d''))$ in h with the pair of pairs $(c, d')(c, d'')$, or vice-versa

where $+$ is the group operation in the respective abelian groups E and F.

It follows easily that $\approx$ is an equivalence relation which is *compatible* with the monoid $(G, \diamond)$.

Step 4 :

Finally, one defines the tensor product as the *quotient space*

$$(2.12) \qquad E \bigotimes F \ = \ G/ \approx$$

and in view of (2.3), (2.4), obtains the *injective* mapping

$$(2.13) \qquad E \times F \ni (a, b) \ \longmapsto \ a \otimes b \in E \bigotimes F$$

where $a \otimes b$ denotes the *coset*, or in other words, the *equivalence class* of $(a, b) \in G$, see (2.4), with respect to the equivalence relation $\approx$ on G.

Step 5 :

Since the equivalence $\approx$ is compatible with the monoid structure of $(G, \diamond)$, and in view of (2.8), (2-9), it follows that the tensor product $E \bigotimes F$ obtains an *abelian group* structure.

It is useful to note the fact that in the above steps 1 and 2, there is absolutely no need for any structure on the sets E and F, and thus they can be arbitrary nonvoid sets.

Furthermore, in step 3 above, the only place the structure on the sets E and F appears is in (2.8), (2.9). And the way this structure is involved allows for *wide ranging* generalizations, far beyond any algebra, [2-6].

3. Universal Property of Tensor Products

For convenience, we recall here certain main features of the usual tensor product of vector spaces, and relate them to certain properties of Cartesian products.

Let $\mathbb{K}$ be a field and E, F, G vector spaces over $\mathbb{K}$.

3.1. Cartesian Product of Vector Spaces

Then $E \times F$ is the vector space over $\mathbb{K}$ where the operations are given by

$$\lambda(x,y) + \mu(u,v) \;=\; (\lambda x + \mu u, \lambda y + \mu v)$$

for any $x, y \in E,\ u, v \in F,\ \lambda, \mu \in \mathbb{K}$.

3.2. Linear Mappings

Let $\mathcal{L}(E, F)$ be the set of all mappings

$$f : E \;\longrightarrow\; F$$

such that

$$f(\lambda x + \mu u) \;=\; \lambda f(x) + \mu f(u)$$

for $u, v \in E,\ \lambda, \mu \in \mathbb{K}$.

3.3. Bilinear Mappings

Let $\mathcal{L}(E, F; G)$ be the set of all mappings

$$g : E \times F \;\longrightarrow\; G$$

such that for $x \in E$ fixed, the mapping $F \ni y \longmapsto g(x,y) \in G$ is linear in y, and similarly, for $y \in F$ fixed, the mapping $E \ni x \longmapsto g(x,y) \in G$ is linear in $x \in E$.

It is easy to see that

$$\mathcal{L}(E, F; G) \;=\; \mathcal{L}(E, \mathcal{L}(F, G))$$

3.4. Tensor Products

The aim of the tensor product $E \bigotimes F$ is to establish a close connection between the *bilinear* mappings in $\mathcal{L}(E, F; G)$ and the *linear* mappings in $\mathcal{L}(E \bigotimes F, G)$.

Namely, the *tensor product* $E \bigotimes F$ is :

(3.4.1) a vector space over $\mathbb{K}$, together with

(3.4.2) a bilinear mapping $t : E \times F \longrightarrow E \bigotimes F$, such that we have the following :

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$$\forall \quad V \text{ vector space over } \mathbb{K}, \quad g \in \mathcal{L}(E, F; V) \text{ bilinear mapping } :$$

$$\exists\,! \quad h \in \mathcal{L}(E \bigotimes F, V) \text{ linear mapping } :$$

$$h \circ t \;=\; g$$

or in other words :

(3.4.3) the diagram commutes

$$\begin{tikzcd} E \times F \arrow[rr, "t"] \arrow[dr, "g"'] & & E \bigotimes F \arrow[dl, "\exists\,!\; h"] \\ & V & \end{tikzcd}$$

and

(3.4.4) the tensor product $E \bigotimes F$ is *unique* up to vector space isomorphism.

Therefore we have the *injective* mapping

$$\mathcal{L}(E, F; V) \ni g \longmapsto h \in \mathcal{L}(E \bigotimes F, V) \quad \text{with} \quad h \circ t = g$$

The converse mapping

$$\mathcal{L}(E \bigotimes F, V) \ni h \longmapsto g = h \circ t \in \mathcal{L}(E, F; V)$$

obviously exists. Thus we have the *bijective* mapping

$$(3.4.5) \qquad \mathcal{L}(E \bigotimes F, V) \ni h \longmapsto g = h \circ t \in \mathcal{L}(E, F; V)$$

3.5 An Application of the Universal Property of Tensor Products

We shall particularize (3.4.3) as follows. Let $V = E \times F$ and $g : E \times F \longrightarrow E \times F$ any bilinear mapping. Then we obtain a unique linear mapping

$$(3.5.1) \qquad h_g : E \bigotimes F \longrightarrow E \times F$$

such that

$$(3.5.2) \qquad g = h_g \circ t$$

and clearly

$$(3.5.3) \qquad g \text{ surjective} \implies h_g \text{ surjective}$$

moreover, in general

$$(3.5.4) \qquad g(E \times F) \subseteq h_g(E \bigotimes F)$$

Now, if we consider the image of h_g, namely

$$(3.5.5) \qquad (E \times F)_g = h_g(E \bigotimes F) \subseteq E \times F$$

which in view of the linearity of h_g is a *vector subspace* in $E \times F$, then obviously we have the *vector space isomorphism*

(3.5.6) $\quad (E \times F)_g \;\cong\; (E \bigotimes F)/ker\, h_g$

In this way, we obtain the *vector space isomorphism*

(3.5.7) $\quad E \bigotimes F \;\cong\; (E \times F)_g \times ker\, h_g \;\subseteq\; (E \times F) \times ker\, h_g$

in view of

Lemma

Let A be a vector space over the field $\mathbb{K}$, while B is a vector subspace in A. Then we have the vector space isomorphism

(3.5.8) $\quad A \;\cong\; B \times (A/B)$

Proof.

Let C be a vector subspace in A, such that $A \;=\; B \oplus C$. Then we have the vector space isomorphism $C \;\cong\; A/B$. However, we also have the vector space isomorphism $A \;\cong\; B \times C$.

$\square$

In view of (3.5.7), we introduce

Definition 1.

Given two vector spaces E and F on a field $\mathbb{K}$, and a bilinear mapping $g : E \times F \longrightarrow E \times F$.

Then the vector subspace $ker\, h_g$ in $E \bigotimes F$ is called the *g-quantumness* of the tensor product $E \bigotimes F$.

$\square$

Now in view of (3.5.3), (3.5.5) and (3.5.7), we obtain

(3.5.9) $\quad g \;\text{ surjective } \;\Longrightarrow\; E \bigotimes F \;\cong\; (E \times F) \times ker\, h_g$

As for $ker\, h_g$, (3.4.3) gives for $\sum_{1 \leq i \leq n} x_i \otimes y_i \in E \bigotimes F$ the relation

$$h_g(\sum_{1 \leq i \leq n} x_i \otimes y_i) \;=\; (0,0) \in E \times F$$

if and only if $\sum_{1 \leq i \leq n} h_g(x_i \otimes y_i) = (0,0) \in E \times F$.

Thus we obtain

$$(3.5.10) \qquad ker\, h_g \;=\; \{ \sum_{1 \leq i \leq n} x_i \otimes y_i \in E \bigotimes F \mid \sum_{1 \leq i \leq n} g(x_i, y_i) = (0,0) \in E \times F \}$$

We further note that every bilinear mapping $g : E \times F \longrightarrow E \times F$ is of the form

$$(3.5.11) \qquad E \times F \ni (x,y) \longmapsto g(x,y) = (g_E(x,y), g_F(x,y)) \in E \times F$$

where

$$(3.5.12) \qquad g_E : E \times F \longrightarrow E, \qquad g_F : E \times F \longrightarrow F$$

are bilinear mappings. Consequently, (3.5.10) takes the form

$$(3.5.13) \qquad ker\, h_g \;=\; \{ \sum_{1 \leq i \leq n} x_i \otimes y_i \in E \bigotimes F \mid \sum_{1 \leq i \leq n} g_E(x_i, y_i) = 0 \in E, \; \sum_{1 \leq i \leq n} g_F(x_i, y_i) = 0 \in F \}$$

Remark 1.

1) The decomposition (3.5.7), and its particular instance in (3.5.9), of the tensor product $E \bigotimes F$ of two arbitrary vector spaces has, in terms of (1.1) - (1.3), the interest of bringing a certain clarity about the difference between the simple Cartesian product $E \times F$, and on the other hand, the more involved tensor product $E \bigotimes F$. And that clarification is made in terms of a further simple Cartesian product, as well as of a special vector subspace $ker\, h_g$ in the tensor product

$E \bigotimes F$, a vector subspace called g-quantumness of E and F.

Here g is an arbitrary bilinear operator on $E \times F$. Therefore, there is in fact a whole *family* of decompositions (3.5.7), (3.5.9), corresponding to the various mentioned bilinear mappings g.

The respective quantum mechanical interpretation of the family of decompositions (3.5.7), (3.5.9) was mentioned in section 1.

2) In case the dimensions of E and F are finite, then (3.5.5) obviously gives

(3.5.14) $\quad dim\,(E \times F)_g \;\leq\; dim\,E + dim\,F$

hence in view of (3.5.7), one has

(3.5.15) $\quad dim\,ker\,h_g \;\geq\; dim\,E\,dim\,F - dim\,E - dim\,F$

while in the particular case of (3.5.9), one obtains

(3.5.16) $\quad dim\,ker\,h_g \;=\; dim\,E\,dim\,F - dim\,E - dim\,F$

3) All the above results extend trivially to any finite number of quantum compositions.

3.6. Universality Property of Cartesian Products

Let A, B be two nonvoid sets. Their cartesian product is :

(3.6.1) a set $A \times B$, together with

(3.6.2) two projection mappings $p_A : A \times A \;\longrightarrow\; A$,
$p_B : A \times B \;\longrightarrow\; B$, such that we have the following :

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$$\forall \quad Z \text{ nonvoid set}, \quad f : Z \longrightarrow A, \quad g : Z \longrightarrow B \ :$$

$$\exists\,! \ \ h : Z \longrightarrow A \times B \ :$$

$$f \ = \ p_A \circ h, \quad g \ = \ p_B \circ h$$

or in other words :

(3.6.3) the diagram commutes

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\exists\,!\ h"] & \\
A & & B \\
 & A \times B \arrow[ul, "p_A"] \arrow[ur, "p_B"'] &
\end{tikzcd}$$

3.7. Cartesian and Tensor Products seen together

$$
\begin{tikzcd}
& \underline{\underline{E}} & & \underline{\underline{E \bigotimes F}} \arrow[dd, "\exists ! \ l"] \\
\forall\, G \arrow[ur, "\forall\, f"] \arrow[rr, "\exists ! \ h"] \arrow[dr, "\forall\, g"'] & & \underline{\underline{E \times F}} \arrow[ul, "\underline{\underline{pr_E}}"'] \arrow[dl, "\underline{\underline{pr_F}}"] \arrow[ur, "\underline{\underline{t}}"] \arrow[dr, "\forall\, k"'] & \\
& \underline{\underline{F}} & & \forall\, V
\end{tikzcd}
$$

4. The Quantumness of Composite Quantum Systems

Based on the above, and specifically (3.5.7) and its particular instance (3.5.9), let us see in some detail what is the difference between the Cartesian product $E \times F$ and the tensor product $E \bigotimes F$ in (1.1) - (1.3), since it is precisely this difference which stands for the assumed *quantumness* that distinguishes between the classical composition of the respective systems leading to the Cartesian product $E \times F$, and on the other hand, the quantum composition which gives the tensor product $E \bigotimes F$.
For convenience, we shall assume here that both E and F are vector spaces on a given field $\mathbb{K}$, a case which obviously contains in particular the situation of quantum interest, when E and F are complex Hilbert spaces.

Clearly, $E \bigotimes F$ is in general considerably larger than $E \times F$. An important fact here, however, is that we have (3.5.7), and in particular, (3.5.9). Consequently, we obtain

Theorem 1.

Given two vector spaces E and F on a field $\mathbb{K}$, and a bilinear mapping

$g : E \times F \longrightarrow E \times F$.
Then for every $\sum_{1 \leq i \leq n} x_i \otimes y_i \in E \bigotimes F$, there exists a *unique* representation

$$(4.1) \qquad \sum{}_{1 \leq i \leq n} x_i \otimes y_i = x \otimes y + \sum{}_{1 \leq j \leq m} u_j \otimes v_j$$

where $x_i, x, u_j \in E, y_i, y, v_j \in F$, and

$$(4.2) \qquad \sum{}_{1 \leq j \leq m} g(u_j, v_j) \ = \ (0, 0) \in E \times F$$

which condition, in view of (3.5.11), (3.5.12), is equivalent with

$$(4.3) \qquad \sum{}_{1 \leq j \leq m} g_E(u_j, v_j) \ = \ 0 \in E, \qquad \sum{}_{1 \leq j \leq m} g_F(u_j, v_j) \ = \ 0 \in F$$

Consequently, $x \otimes y \in (E \times F)_g$ is the classical term in (4.1), while $\sum{}_{1 \leq j \leq m} u_j \otimes v_j \in ker\, h_g$ corresponds to the g-quantumness term.

Further, we have the following implicit conditions on the classical term $x \otimes y \in (E \times F)_g$, namely

$$(4.4) \qquad \sum{}_{1 \leq i \leq n} g(x_i, y_i) \ = \ g(x, y) \in E \times F$$

or equivalently

$$(4.5) \qquad \begin{aligned} \sum{}_{1 \leq i \leq n} g_E(x_i, y_i) \ &= \ g_E(x, y) \in E, \\ \sum{}_{1 \leq i \leq n} g_F(x_i, y_i) \ &= \ g_F(x, y) \in F \end{aligned}$$

Corollary 1.

Given two vector spaces E and F on a field $\mathbb{K}$, and a bilinear mapping $g : E \times F \longrightarrow E \times F$.
Then the quantum composition $E \bigotimes F$ is the Cartesian composition of the vector subspace $(E \times F)_g$ of the Cartesian composition $E \times F$, with the g-quantumness space $ker\, h_g$.

5. Examples

1) Clearly, for $u \in E,\ v \in F$, we have

(5.1) $\quad u \otimes v \in ker\, h_g \iff g_E(u,v) = 0 \in E, \quad g_F(u,v) = 0 \in F$

2) For $u_1, u_2 \in E,\ v_1, v_2 \in F$, we have

(5.2) $\quad u_1 \otimes v_1 + u_2 \otimes v_2 \in ker\, h_g \iff$

$$\iff \begin{pmatrix} g_E(u_1,v_1) + g_E(u_2,v_2) = 0 \in E, \\ \\ g_F(u_1,v_1) + g_F(u_2,v_2) = 0 \in F \end{pmatrix}$$

3) Let now $E = F = \mathbb{R}^2$, with $\mathbb{K} = \mathbb{R}$. Further, let any bilinear mapping $g : E \times F \longrightarrow E \times F$.

We consider the non-normalized Bell state-like element

(5.3) $\quad z = |\,a > \otimes |\,a > \ + |\,b > \otimes |\,b > \in E \bigotimes F$

where $|\,a >, |\,b > \in \mathbb{R}^2$ form an orthonormal basis in $\mathbb{R}^2$, and consider its unique representation (4.1), namely

(5.4) $\quad z = x \otimes y + \sum_{1 \leq j \leq m} u_j \otimes v_j$

In order to find the terms in the right hand above, we recall (4.4) and obtain

(5.5) $\quad g(x,y) = g(|\,a >, |\,a >) + g(|\,b >, |\,b >)$

while (4.2) gives

(5.6) $\quad \sum_{1 \leq j \leq m} g(u_j, v_j) = (0,0) \in E \times F$

Let us now choose a specific bilinear mapping $g : E \times F \longrightarrow E \times F$. For instance, given $(x,y) = ((x_1,x_2),(y_1,y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$, we can take

$$(5.7) \quad g(x,y) = ((x_1\,\overline{y_1}, x_2\,\overline{y_2}), (x_1\,\overline{y_2}, x_2\,\overline{y_1})) \in \mathbb{R}^2 \times \mathbb{R}^2$$

Then, assuming that $|\,a> = (a_1, a_2),\ |\,b> = (b_1, b_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, we obtain from (5.5) the relations

$$(5.8) \quad x_1\,\overline{y_1} = |a_1|^2 + |b_1|^2, \quad x_2\,\overline{y_2} = |a_2|^2 + |b_2|^2,$$

$$x_1\,\overline{y_2} = x_2\,\overline{y_1} = a_1\,\overline{a_2} + b_1\,\overline{b_2}$$

As for the relation (5.6), it becomes

$$(5.9) \quad \sum{}_{1 \leq j \leq m}\, u_{j,1}\,\overline{v_{j,1}} = \sum{}_{1 \leq j \leq m}\, u_{j,2}\,\overline{v_{j,2}} =$$

$$= \sum{}_{1 \leq j \leq m}\, u_{j,1}\,\overline{v_{j,2}} = \sum{}_{1 \leq j \leq m}\, u_{j,2}\,\overline{v_{j,1}} = 0$$

if we assume that $u_j = (u_{j,1}, u_{j,2}), v_j = (v_{j,1}, v_{j,2}) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Now, in the decomposition (5.4), the unknowns are x, y, u_j, v_j, each of which is a pair of complex numbers. Thus (5.4) contains $4m + 4$ unknown complex numbers, namely $x_1, x_2, y_1, y_2, u_{j,1}, u_{j,2}, v_{j,1}, v_{j,2}$.

On the other hand, (5.8), (5.9) are 8 equations in those $4m + 4$ complex numbers. Thus it may appear that, in general, one should have $4m + 4 \leq 8$, which means that $m \leq 1$.

However, the g-quantumness term $\sum_{1 \leq j \leq m} u_j \otimes v_j$ in (5.4), although unique in its value in $ker\, h_g$, need not have a unique form as such, due to a well known property of sums of tensor products. And then, it may happen that $m > 1$.

This issue is addressed in principle next.

6. Rank of Quantumness

In view of the above, we are led to

Definition 2.

Given two vector spaces E and F on a field $\mathbb{K}$, and a bilinear mapping $g : E \times F \longrightarrow E \times F$.

Then for $z \in ker\, h_g$, we define

(6.1) $\quad rank_g\, z$

to be smallest $m \in \mathbb{N}$ for which we have

(6.2) $\quad z = \sum_{1 \leq j \leq m} u_j \otimes v_j$

with

(6.3) $\quad \sum_{1 \leq j \leq m} g(u_j, v_j) = (0, 0) \in E \times F$

$\square$

In view of the unique representation (4.1), one can extend Definition 2, as follows

Definition 3.

Given $z \in (E \bigotimes F)$, we define

(6.4) $\quad rank_g\, z = rank_g\, w$

where one has the unique representation (4.1)

(6.5) $\quad z = x \otimes y + w$

with $w \in ker\, h_g$.

References

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