Serious anomalies in the reported geometry of Einstein's gravitational field

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Careful reading of the reported geometry of Einstein's gravitational field reveals that the physicists have committed fatal errors in the elementary differential geometry of a pseudo-Riemannian metric manifold. These elementary errors in mathematics invalidate much of the reported physics of Einstein's gravitational field. The consequences for astrophysical theory are significant.

I. Introduction

In the usual interpretation of Hilbert's [1, 2, 3] version of Schwarzschild's solution, the quantity r therein has *never* been properly identified. The physicists have variously and vaguely called it " the radius" of a sphere [4, 5], the "radius of a 2-sphere" [6], the "coordinate radius"[7], the "radial coordinate" [8, 9], the "radial space coordinate" [10], the "areal radius" [7, 11], the "reduced circumference" [12], and even "*a gauge choice: it defines the coordinate* r" [13]. In the particular case of $r = 2GM/c^2$ it is invariably referred to by the physicists as the "Schwarzschild radius" or the "gravitational radius". However, the irrefutable geometrical fact is that r, in the spatial section of Hilbert's version of the Schwarzschild/Droste line-element, is the radius of Gaussian curvature [14, 15, 16, 17], and as such it *does not* in fact determine the geodesic radial distance from the centre of spherical symmetry located at an arbitrary point in the related pseudo-Riemannian metric manifold. It does not in fact determine any distance at all in the spherically symmetric metric manifold. It is the radius of Gaussian curvature merely by virtue of its formal geometric relationship to the Gaussian curvature. It must also be emphasized that a geometry is completely determined by the *form* of its line-element, a fact that the physicists, with few exceptions [18], have not realised.

It immediately follows from the invalidity of Ric $=0$ that Einstein's conceptions of the conservation and localisation of gravitational energy are erroneous and in conflict with the usual conservation of energy and momentum, and that the current search for Einstein's gravitational waves is ill-conceived. Also, the concepts of black holes and their interactions are ill-conceived and the two-body problem has been neither correctly formulated nor solved by means of the General Theory of Relativity.

II. Gaussian curvature

Recall that Hilbert's version of the "Schwarzschild" solution is (using $c = G = 1$),

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (1)$$

wherein r can, by assumption (i.e. without any proof), in some way or another, go down to zero, and m is allegedly the mass causing the gravitational field. Schwarzschild's [19] actual solution, for comparision, is

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (2)$$

$$R = R(r) = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}, \quad 0 \le r < \infty,$$

$$\alpha = const.$$

Note that (2) is singular only when $r = 0$ (in which case the metric does not actually apply), and that the constant α is indeterminable (Schwarzschild did not assign any value to the constant α for this reason).

For a 2-D spherically symmetric geometric surface [20] determined by

$$ds^2 = R_c^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

$$R_c = R_c(r),$$

the Riemannian curvature (which depends upon both position and direction) reduces to the Gaussian curvature K (which depends only upon position), given by [14, 21, 22, 23, 24],

$$K = \frac{R_{1212}}{g},$$

where $R_{ijkm} = g_{in} R^{n}_{.jkm}$ is the Riemann tensor of the first kind and $g = g_{11} g_{22} = g_{\theta\theta} g_{\varphi\varphi}$ (because the metric tensor is diagonal). Recall that

$$R^{1}_{.212} = \frac{\partial \Gamma^{1}_{22}}{\partial x^{1}} - \frac{\partial \Gamma^{1}_{21}}{\partial x^{2}} + \Gamma^{k}_{22} \Gamma^{1}{}_{k1} - \Gamma^{k}{}_{21} \Gamma^{1}_{k2},$$

$$\Gamma^{\alpha}_{\alpha\beta} = \Gamma^{\alpha}_{\beta\alpha} = \frac{\partial}{\partial x^{\beta}} \left(\frac{1}{2} \ln |g_{\alpha\alpha}| \right),$$

$$\Gamma^{\alpha}_{\beta\beta} = -\frac{1}{2 g_{\alpha\alpha}} \frac{\partial g_{\beta\beta}}{\partial x^{\alpha}}, \quad (\alpha \neq \beta),$$

and all other $\Gamma^{\alpha}_{\beta\gamma}$ vanish. In the above, $k, \alpha, \beta = 1, 2$, $x^{1} = \theta$ and $x^{2} = \phi$, of course. Straightforward calculation gives for expression (3),

$$K = \frac{1}{R_c^2},$$

so that R_c is the inverse square root of the Gaussian curvature, i. e. the radius of Gaussian curvature, and so r in Hilbert's "Schwarzschild's solution" is the radius of Gaussian curvature. The geodesic (i.e. proper) radius, R_p, of the spatial section of Schwarzschild's solution (2), up to a constant of integration, is given by

$$R_p = \int \frac{dR(r)}{\sqrt{1 - \frac{\alpha}{R(r)}}}, \tag{4}$$

and for Hilbert's "Schwarzschild's solution" (1), by

$$R_p = \int \frac{dr}{\sqrt{1 - \frac{2m}{r}}}.$$

Thus the proper radius and the radius of Gaussian curvature *are not the same.* The radius of Gaussian curvature does not determine the geodesic radial distance from the arbitrary point at the centre of spherical symmetry of the metric manifold. It is a "radius" only in the sense of it being the inverse square root of the Gaussian curvature. A detailed development of the foregoing, from first principles, is given in [14] and [15].

Note that in (2), if $\alpha = 0$ Minkowski space is recovered:

$$ds^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right),$$

$$0 \leq r < \infty.$$

In this case the radius of Gaussian curvature is r and the proper radius is

$$R_p = \int_0^r dr = r,$$

so that the radius of Gaussian curvature and the proper radius are identical. It is for this reason that in the spacetime of Minkowski the radius of Gaussian curvature can be substituted for the proper radius (i.e. the geodesic radius). However, in the case of a (pseudo-) Riemannian manifold, such as (1) and (2) above, only the great circumference and surface area can be determined via the radius of Gaussian curvature. Distances from the arbitrary point at the centre of spherical symmetry to a geodesic spherical surface in a Riemannian metric manifold can only be determined via the proper radius, except for particular points (if any) in the manifold where the radius of Gaussian curvature and the geodesic radius happen to be identical, and volumes by a triple integral involving a function of the radius of Gaussian curvature. In the case of Schwarzschild's solution (2) (and hence also for (1)), the radius of Gaussian curvature, $R_c = R(r)$, and the proper radius, R_p, are identical only at $R_c \approx 1.467\alpha$. When the radius of Gaussian curvature, R_c, is greater than $\approx 1.467\alpha$, $R_p > R_c$, and when the radius of Gaussian curvature is less than $\approx 1.467\alpha$, $R_p < R_c$.

The upper and lower bounds on the Gaussian curvature (and hence on the radius of Gaussian curvature) are not arbitrary, but are determined by the proper radius in accordance with the intrinsic geometric structure of the line-element (which completely determines the geometry), manifest in the integral (4). Thus, one cannot merely assume, as the physicists have done, that the radius of Gaussian curvature for (1) and (2) can vary from zero to infinity. Indeed, in the case of (2) (and hence also of (1)), as R_p varies from zero to infinity, the Gaussian curvature varies from $1/\alpha^2$ to zero and so the radius of Gaussian curvature correspondingly varies from α to infinity, as easily determined by evaluation of the constant of integration associated with the indefinite integral (4). Moreover, in the same way, it is easily shown that expressions (1) and (2) can be generalised [17] to all real values, but one, of the variable r, so that both (1) and (2) are particular cases of the general radius of Gaussian curvature, given by

$$R_c = R_c(r) = \left(\left| r - r_0 \right|^n + \alpha^n \right)^{\frac{1}{n}}, \tag{5}$$

$$r \in \Re, \quad n \in \Re^{+}, \quad r \neq r_0,$$

wherein r_0 and n are entirely arbitrary constants. Choosing $n = 3$, $r_0 = 0$ and $r > r_0$ yields Schwarzschild's solution (2). Choosing $n = 1$, $r_0 = \alpha$ and $r > r_0$ yields line-element (1) as determined by Johannes Droste [25] in May 1916, independently of Schwarzschild. Choosing $n = 1$, $r_0 = \alpha$ and $r < r_0$ gives $R_c = 2\alpha - r$, with line-element

$$ds^2 = \left(1 - \frac{\alpha}{2\alpha - r}\right) dt^2 - \left(1 - \frac{\alpha}{2\alpha - r}\right)^{-1} dr^2 - (2\alpha - r)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Using relations (5) directly, all real values of $r \neq r_0$ are permitted. In any case, however, the related line-element is singular only at the arbitrary parametric point $r = r_0$ on the real line (or half real line, as the case may be), which is the only parametric point on the real line (or half real line, as the case may be) at which the line-element fails (at $R_p(r_0) = 0 \ \forall \ r_0 \ \forall \ n$). Indeed, substituting (5) for $R(r)$ in (4), and evaluating the constant of integration gives

$$R_p = \sqrt{R_c\,(R_c - \alpha)} + \alpha \ln\left(\frac{\sqrt{R_c} + \sqrt{R_c - \alpha}}{\sqrt{\alpha}}\right),$$

where $R_c = R_c(r)$ is given by (5).

Note that in the Standard Model interpretation of (1), only g_{00} and g_{11} are modified by the presence of the constant m. However, according to (2) and (5) *all* the components of the metric tensor are modified by the constant α, and since (1) is a particular case of (5), all the components of the metric tensor of (1) are modified by the constant α as well.

The Kruskal-Szekeres coordinates do not take into account the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section of the Schwarzschild manifold. These coordinates thereby violate the geometric form of the line-element, producing a completely separate pseudo-Riemannain manifold that does not form part of the solution space of the Schwarzschild manifold [36], and are consequently invalid. The concept of the Black Hole is therefore invalid.

III. The non-existence of point-mass singularities

According to Special Relativity, infinite densities are forbidden because their existence implies that a material object can acquire the speed of light c in vacuo (or equivalently, the existence of infinite energies), thereby violating the very basis of Special Relativity. Since General Relativity cannot violate Special Relativity, General Relativity must thereby also forbid infinite densities. Point-mass singularities are alleged to be infinitely dense objects. Therefore, point-mass singularities are forbidden by the Theory of Relativity.

Let a cuboid rest-mass m_0 have sides of length L_0. Let m_0 have a relative speed $v < c$ in the direction of one of three mutually orthogonal Cartesian axes attached to an observer of rest-mass M_0. According to the observer M_0, the moving mass m is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{6}$$

and the volume V thereof is

$$V = L_0^3 \sqrt{1 - \frac{v^2}{c^2}}. \tag{7}$$

Thus, the density D is

$$D = \frac{m}{V} = \frac{m_0}{L_0^3\left(1 - \frac{v^2}{c^2}\right)}, \tag{8}$$

and so $v \to c \ \Rightarrow \ D \to \infty$. Since by (6) no material object can acquire the speed c (this would require an infinite energy), infinite densities are forbidden by Special Relativity, and so point-mass singularities are forbidden. Since General Relativity cannot violate Special Relativity, it too must thereby forbid infinite densities and hence forbid point-mass singularities [1, 15, 17, 19]. Point-charges too are therefore forbidden by the Theory of Relativity since there can be no charge without mass.

IV. Ric = 0 is inadmissible

According to Einstein [26], his 'Principle of Equivalence' (equivalence of gravitational and inertial mass) requires that Special Relativity manifest in any freely falling inertial frame located in a sufficiently small region of the gravitational field. Now Special Relativity permits the presence of arbitrarily large (but not infinite) masses in spacetime, which are subject to the mass dilation relation (expression (6) above; and hence also to expressions (7) and (8) as well), and the definition of a relativistic inertial frame requires the *a priori* presence of two masses; the mass of the observer and the mass of the observed (to define relative motion of material bodies). In addition, at any instant the masses defining the freely falling inertial frame (and hence any other masses present therein) can have a speed up to but not including the speed of light in vacuo, by the action of the gravitational field. However, $R_{\mu\nu} = 0$ precludes, by definition, the presence of any masses and energies in the gravitational field because the energy-momentum tensor $T_{\mu\nu} = 0$ by hypothesis. Therefore, Special Relativity cannot manifest in any "freely falling" inertial frame in the spacetime of $R_{\mu\nu} = 0$. Indeed, a "freely falling" inertial frame cannot even be present since its very definition requires the presence of two masses which are, at any instant, subject to mass dilation under the action of the gravitational field. Similarly the equivalence of gravitational and inertial mass cannot manifest in the absence of matter in the gravitational field. Thus,

$R_{\mu\nu} = 0$ violates Einstein's 'Principle of Equivalence' and is therefore inadmissible – it does not describe Einstein's gravitational field. Matter can only be introduced into Einstein's gravitational field via the energy-momentum tensor since it alone is what specifies that which physically causes the curvature of spacetime (i.e. the gravitational field). Clearly, the standard *a posteriori* and *ad hoc* introduction of matter as the physical cause of spacetime curvature, into the so-called "Schwarzschild solution" for $R_{\mu\nu} = 0$, violates the requirements of Einstein's theory because the energy-momentum tensor is set to zero in that case.

V. Gravitational energy cannot be localised

Since $R_{\mu\nu} = 0$ does not describe Einstein's gravitational field, the energy-momentum tensor can never be zero (i.e. if $T_{\mu\nu} = 0$ there is no gravitational field), so Einstein's field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}$$

can be written as [21, 27, 28]

$$\frac{1}{\kappa} G_{\mu\nu} + T_{\mu\nu} = 0, \tag{9}$$

wherein the $G_{\mu\nu}/\kappa$ are the components of a gravitational energy tensor. Thus, $G_{\mu\nu}/\kappa$ and $T_{\mu\nu}$ *vanish identically*; the total energy is always zero; there is no localisation of gravitational energy (i.e. there are no Einstein gravitational waves). The current international search for Einstein's gravitational waves is destined to detect nothing.

It is of interest to note that Einstein's pseudo-tensor is frequently utilised as a basis for the localisation of gravitational energy [9, 18, 21, 26, 29, 30]. From the foregoing it is evident that this cannot be correct. This is reaffirmed by the fact that Einstein's pseudo-tensor is mathematically (and hence also physically) meaningless, because it implies the existence of an invariant that has no mathematical existence [28]. Indeed, Einstein's pseudo-tensor, $\sqrt{-g}\, t^{\mu}_{\nu}$, is defined as [9, 18, 21, 26, 28, 29, 30],

$$\sqrt{-g}\, t^{\mu}_{\nu} = \frac{1}{2}\left(\delta^{\mu}_{\nu} L - \frac{\partial L}{\partial g^{\sigma\rho}_{,\mu}} g^{\sigma\rho}_{,\nu}\right)$$

wherein L is given by

$$L = -g^{\alpha\beta}\left(\Gamma^{\gamma}_{\alpha\kappa}\Gamma^{\kappa}_{\beta\gamma} - \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\kappa}_{\gamma\kappa}\right).$$

Contracting the pseudo-tensor and applying Euler's theorem yields,

$$\sqrt{-g}\, t^{\mu}_{\mu} = L,$$

which is a 1st-order intrinsic differential invariant that depends only upon the components of the metric tensor and their 1st derivatives. However, the mathematicians Ricci and Levi-Civita [31] proven in 1900 that such invariants *do not exist*. Consequently, everything built upon Einstein's pseudo-tensor is invalid. Eddington's [30] other objections to the pseudo-tensor are therefore quite well-founded.

Similarly, Einstein's field equations cannot be linearised because linearisation implies the existence of a tensor that, except for the trivial case of being zero, *does not otherwise exist*, as proven by Hermann Weyl in 1944 [32].

Since it has already been proven elsewhere [33] that the so-called "cosmological constant" must be precisely zero, expression (9) can contain no other terms.

Einstein's General Theory of Relativity is therefore in conflict with the usual conservation of energy and momentum. The usual conservation of energy and momentum is well established experimentally, so if the usual conservation of energy and momentum is valid, then General Relativity is invalid, taking with it the alleged expansion of the Universe and the Big Bang cosmology.

VI. The two-body problem

Einstein's field equations are non-linear, so the 'Principle of Superposition' cannot apply. Therefore, before one can talk of relativistic binary systems it must first be proven that the two-body system is theoretically well-defined by General Relativity. This can be done in only two ways:

(a) Derivation of an exact solution to Einstein's field equations for the two-body configuration of matter; or

(b) Proof of an existence theorem.

There are no known solutions to Einstein's field equations for the interaction of two (or more) masses, so option (a) has never been fulfilled. No existence theorem has ever been proven, by which Einstein's field equations even admit of latent solutions for such configurations of matter, and so option (b) has never been fulfilled. The black hole is allegedly obtained from a line-element satisfying $\mathrm{Ric} = 0$. Ignoring for the moment that $\mathrm{Ric} = 0$ violates Einstein's 'Principle of Equivalence', and, for the sake of argument, assuming that black holes are predicted by General Relativity, since $\mathrm{Ric} = 0$ is a statement that there is no matter in the Universe, one cannot simply insert a second black hole into the spacetime of $\mathrm{Ric} = 0$ of a given black hole so that the resulting two black holes (each obtained separately from $\mathrm{Ric} = 0$) mutually interact in a mutual spacetime that *by definition contains no matter*! One cannot simply assert by an analogy with Newton's theory that two black holes can be components of binary systems, collide or merge [34, 35], because the 'Principle of Superposition' does not apply in Einstein's theory. Moreover, General Relativity has to date been unable to account for the simple experimental fact that two fixed bodies will attract one another when released.

Thus, the concepts of black holes, black hole binaries, collisions and mergers are all invalid.

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