Sequences on Graphs with Symmetries

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Abstract: An interesting symmetry on multiplication of numbers found by Prof.Smarandache recently. By considering integers or elements in groups on graphs, we extend this symmetry on graphs and find geometrical symmetries. For extending further, Smarandache's or combinatorial systems are also discussed on general mathematical systems in this paper, particularly, the CC conjecture presented by myself six years ago, which enables one to construct symmetrical systems in mathematical sciences.

Key Words: Smarandache sequence, labeling, Smarandache beauty, graph, group, Smarandache system, combinatorial system, CC conjecture.

AMS(2010): 05C21, 05E18

§1. Sequences

Let $\mathbb{Z}^+$ be the set of non-negative integers and Γ a group. We consider sequences $\{i(n)|n \in \mathbb{Z}^+\}$ and $\{g_n \in \Gamma | n \in \mathbb{Z}^+\}$ in this paper. There are many interesting sequences appeared in literature. For example, the sequences presented by Prof.Smarandache in references [1]-[3] following:

(1) **Consecutive sequence**

$1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, \cdots;$

(2) **Digital sequence**

$1, 11, 111, 1111, 11111, 11111, 1111111, 11111111, \cdots$

(3) **Circular sequence**

[1]Reported at *The 7th Conference on Number Theory and Smarandache's Notion, March 25-27, 2011, Xian, P.R.China*

$1,\ 12, 21,\ 123, 231, 312,\ 1234, 2341, 3412, 4123, \cdots;$

(4) **Symmetric sequence**

$1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321, \cdots;$

(5) **Divisor product sequence**

$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19, \cdots;$

(6) **Cube-free sieve**

$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, \cdots.$

He also found three nice symmetries for these integer sequences recently.

First Symmetry

$$\begin{aligned}
1 \times 8 + 1 &= 9 \\
12 \times 8 + 2 &= 98 \\
123 \times 8 + 3 &= 987 \\
1234 \times 8 + 4 &= 9876 \\
12345 \times 8 + 5 &= 98765 \\
123456 \times 8 + 6 &= 987654 \\
1234567 \times 8 + 7 &= 9876543 \\
12345678 \times 8 + 8 &= 98765432 \\
123456789 \times 8 + 9 &= 987654321
\end{aligned}$$

Second Symmetry

$$\begin{aligned}
1 \times 9 + 2 &= 11 \\
12 \times 9 + 3 &= 111 \\
123 \times 9 + 4 &= 1111 \\
1234 \times 9 + 5 &= 11111 \\
12345 \times 9 + 6 &= 111111 \\
123456 \times 9 + 7 &= 1111111 \\
1234567 \times 9 + 8 &= 11111111 \\
12345678 \times 9 + 9 &= 111111111 \\
123456789 \times 9 + 10 &= 1111111111
\end{aligned}$$

Third Symmetry

$$\begin{aligned}
1 \times 1 &= 1\\
11 \times 11 &= 121\\
111 \times 111 &= 12321\\
1111 \times 1111 &= 1234321\\
11111 \times 11111 &= 12345431\\
111111 \times 111111 &= 12345654321\\
1111111 \times 1111111 &= 1234567654321\\
11111111 \times 11111111 &= 13456787654321\\
111111111 \times 111111111 &= 12345678987654321
\end{aligned}$$

Notice that Smarandache sequences on integers are not symmetric in global, but a group always presents symmetric figure in geometry. These previous symmetries show that one can get a global symmetry by means of that non-symmetry.

[1] F.Smarandache, *Only Problems, Not Solutions!* Xiquan Publishing House, Chicago, 1990.

[2] F.Smarandache, *Sequences of Numbers Involved in Unsolved Problems*, Hexis, Phoenix, Arizona, 2006.

[3] D.Deleanu, *A Dictionary of Smarandache Mathematics*, Buxton University Press, London & New York, 2006.

§2. Graphs with Labelings

2.1 Graph. A *graph* G is an ordered 3-tuple $(V(G), E(G); I(G))$, where $V(G), E(G)$ are finite sets, $V(G) \neq \emptyset$ and $I(G) : E(G) \to V(G) \times V(G)$.

$V(G)$–vertex set, $E(G)$–edge set, $|V(G)|$–order, $|E(G)|$–size of a graph G.

A graph $H = (V_1, E_1; I_1)$ is a *subgraph* of a graph $G = (V, E; I)$ if $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \to V_1 \times V_1$, denoted by $H \subset G$.

2.2 Example. A graph G is shown in Fig.2.1, where, $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5,\ e_6, e_7, e_8, e_9, e_{10}\}$ and $I(e_i) = (v_i, v_i), 1 \leq i \leq 4; I(e_5) = (v_1, v_2) = (v_2, v_1), I(e_8) = (v_3, v_4) = (v_4, v_3), I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2)$, $I(e_8) = I(e_9) = (v_4, v_1) = (v_1, v_4)$.

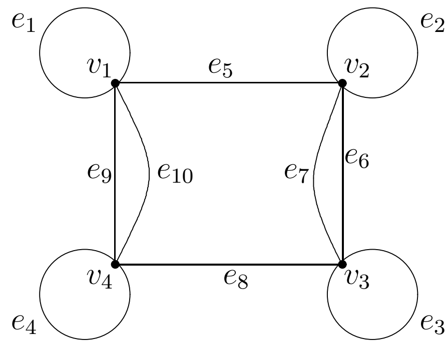


Fig. 2.1

An automorphism of a graph G is a $1-1$ mapping $\theta : V(G) \rightarrow V(G)$ such that

$$\theta(u,v) = (\theta(u), \theta(v)) \in E(G)$$

holds for $\forall(u,v) \in E(G)$. All such automorphisms form a group under composition operation, denoted by Aut$G$. A graph G is *vertex-transitive* if Aut$G$ is transitive on $V(G)$.

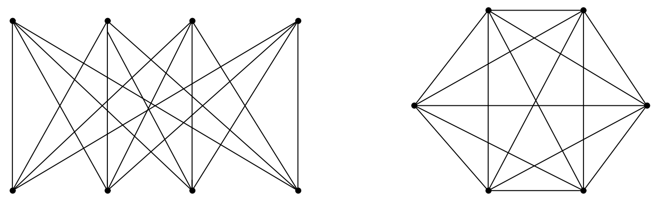
2.3 Graph Family. A *graph family* $\mathscr{F}_P$ is the set of graphs whose each element possesses a graph property P.

Walk. A walk of a graph G is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $e_i = (u_i, u_{i+1})$ for $1 \leq i \leq n$.

Path and Circuit. A walk such that all the vertices are distinct and a circuit or a cycle is such a walk $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $u_1 = u_n$ and distinct vertices. A graph $G = (V, E; I)$ is *connected* if there is a path connecting any two vertices in this graph.

Tree. A tree is a connected graph without cycles.

n-Partite Graph. A graph G is n-partite for an integer $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \cdots, V_n$ such that every edge joints a vertex of $V_i$to a vertex of V_j, $j \neq i$, $1 \leq i, j \leq n$. A *complete n-partite graph* G is such an n-partite graph with edges $uv \in E(G)$ for $\forall u \in V_i$ and $v \in V_j$ for $1 \leq i, j \leq n$, denoted by $K(p_1, p_2, \cdots, p_n)$ if $|V_i| = p_i$ for integers $1 \leq i \leq n$. Particularly, if $|V_i| = 1$ for integers $1 \leq i \leq n$, such a complete n-partite graph is called *complete* *graph* and denoted by K_n.

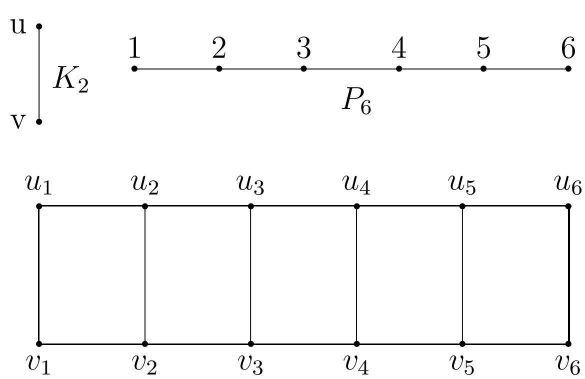


The complete bipartite graph $K(4,4)$ The complete graph K_6

Fig.2.2

Cartesian Product. A Cartesian product $G_1 \times G_2$ of graphs G_1 with G_2 is defined by $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

The graph $K_2 \times P_6$ is shown in Fig.2.3 following.



$K_2 \times P_6$

Fig.2.3

Union. The *union* $G \cup H$ of graphs G and H is a graph $(V(G \cup H), E(G \cup H), I(G \cup H))$ with

$$V(G \cup H) = V(G) \cup V(H), \quad E(G \cup H) = E(G) \cup E(H) \quad \text{and} \quad I(G \cup H) = I(G) \cup I(H).$$

2.4 Labeling. Let G be a graph and $N \subset \mathbb{Z}^+$. A *labeling* of G is a mapping $l_G: V(G) \cup E(G) \to N$ with each labeling on an edge (u, v) is induced by a ruler $r(l_G(u), l_G(v))$ with additional conditions.

Classical Labeling Ruler. The following rulers are usually found in literature.

Ruler R1. $r(l_G(u), l_G(v)) = |l_G(u) - l_G(v)|$.

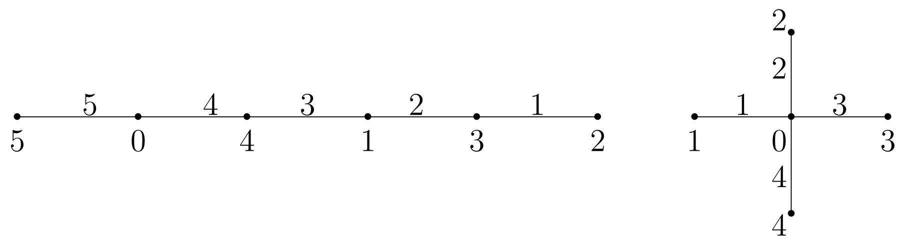


Fig.2.4

Such a labeling l_G is called to be a graceful labeling of G if $l_G(V(G)) \subset \{0, 1, 2, \cdots, |V(G)|\}$ and $l_G(E(G)) = \{1, 2, \cdots, |E(G)|\}$. For example, the graceful labelings of P_6 and $S_{1.4}$ are shown in Fig.2.4.

Graceful Tree Conjecture (A.Rose, 1966) *Any tree is graceful.*

There are hundreds papers on this conjecture. But it is opened until today.

Ruler R2. $r(l_G(u), l_G(v)) = l_G(u) + l_G(v)$.

Such a labeling l_G on a graph G with q edges is called to be harmonious on G if $l_G(V(G)) \subset \mathbb{Z}(\mathrm{mod} q)$ such that the resulting edge labels $l_G(E(G)) = \{1, 2, \cdots, |E(G)|\}$ by the induced labeling $l_G(u, v) = l_G(u) + l_G(v) \pmod{q}$ for $\forall (u, v) \in E(G)$. For example, ta harmonious labeling of P_6 are shown in Fig.2.5 following.

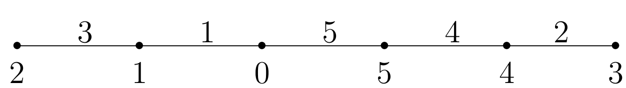


Fig.2.5

Update results on classical labeling on graphs can be found in a survey paper [4] of Gallian.

[4] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic J.Combinatorics*, # DS6, 16(2009), 1-219.

Smarandachely Labeling. There are many new labelings on graphs appeared in *International J.Math.Combin.* in recent years. Such as those shown in the following.

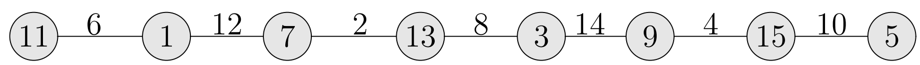


Fig.2.6

Smarandachely k-Constrained Labeling. A *Smarandachely k-constrained labeling* of a graph $G(V, E)$ is a bijective mapping $f : V \cup E \to \{1, 2, .., |V| + |E|\}$ with the additional conditions that $|f(u) - f(v)| \geq k$ whenever $uv \in E$, $|f(u) - f(uv)| \geq k$ and $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a Smarandachely k-constrained total graph, abbreviated as $k - CTG$. An example for $k = 5$ on P_7 is shown in Fig.2.6.

The minimum positive integer n such that the graph $G \cup \overline{K}_n$ is a $k - CTG$ is called *k-constrained number* of the graph G and denoted by $t_k(G)$, the corresponding labeling is called a minimum k-constrained total labeling of G.

Update Results for $t_k(G)$ are as follows:

(1) $t_2(P_n)=\begin{cases} 2 \;\; if \;\; n = 2, \\ 1 \;\; if \;\; n = 3, \\ 0 \;\; else. \end{cases}$

(2) $t_2(C_n) = 0$ if $n \geq 4$ and $t_2(C_3) = 2$.

(3) $t_2(K_n) = 0$ if $n \geq 4$.

(4) $t_2(K(m, n))=\begin{cases} 2 \;\; if \;\; n = 1 \;\; and \;\; m = 1, \\ 1 \;\; if \;\; n = 1 \;\; and \;\; m \geq 2, \\ 0 \;\; else. \end{cases}$

(5) $t_k(P_n)=\begin{cases} 0 \;\; if \;\; k \leq k_0, \\ 2(k - k_0) - 1 \;\; if \;\; k > k_0 \;\; and \;\; 2n \equiv 0 (mod\ 3), \\ 2(k - k_0) \;\; if \; k > k_0 \;\; and \;\; 2n \equiv 1 \; or \; 2 (mod\ 3). \end{cases}$

(6) $t_k(C_n) = \begin{cases} 0 \;\; if \;\; k \leq k_0, \\ 2(k - k_0) \;\; if \;\; k > k_0 \;\; and \;\; 2n \equiv 0 \; (mod\ 3), \\ 3(k - k_0) \;\; if \; k > k_0 \;\; and \;\; 2n \equiv 1 \; or \; 2 (mod\ 3), \end{cases}$

where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. More results on $t_k(G)$ cam be found in references.

[5] ShreedharK, B. Sooryanarayana and Raghunath P., Smarandachely k-Constrained

labeling of Graphs, *International J.Math. Combin.* Vol.1 (2009), 50-60.

[6] P. Devadas Rao, B. Sooryanarayana and M. Jayalakshmi, Smarandachely k-Constrained Number of Paths and Cycles, *International J.Math. Combin.* Vol.3 (2009), 48-60.

Smarandachely Super m-Mean Labeling. Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced *Smarandachely edge m-labeling* f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph. Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

A Smarandache super 2-mean labeling on P_6^2 is shown in Fig.2.7.

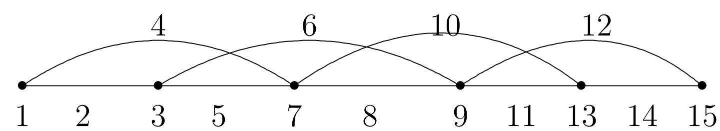


Fig.2.7

Now we have know graphs P_n, C_n, K_n,K(2,n), $(n \geq 4)$, $K(1, n)$ for $1 \leq n \leq 4$, $C_m \times P_n$ for $n \geq 1, m = 3, 5$ have Smarandachely super 2-mean labeling. More results on Smarandachely super m-mean labeling of graphs can be found in references following.

[7] R. Vasuki and A. Nagarajan, Some results on super mean graphs, *International J.Math. Combin.* Vol.3 (2009), 82-96.

[8] R. Vasuki and A. Nagarajan, Some results on super mean graphs, *International J.Math. Combin.* Vol.3 (2009), 82-96.

[9] Selvam Avadayappan and R. Vasuki, New families of mean graphs, *International J.Math. Combin.* Vol.2 (2010), 68-80.

[10] A. Nagarajan, A.Nellai Murugan and S.Navaneetha Krishnan, On near mean graphs, *International J.Math. Combin.* Vol.4 (2010), 94-99.

§3. Smarandache Sequences on Symmetric Graphs

Let $l_G^S : V(G) \to \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$ be a vertex labeling of a graph G with edge labeling $l_G^S(u,v)$ induced by $l_G^S(u)l_G^S(v)$ for $(u,v) \in E(G)$ such that $l_G^S(E(G)) = \{1, 121, 12321, 1234321, 123454321, 12345654321, 1234567654321, 123456787654321, 12345678987654321\}$, i.e., $l_G^S(V(G) \cup E(G))$ contains all numbers appeared in the Smarandachely third symmetry. Denote all graphs with l_G^S labeling by $\mathscr{L}^S$. We know the following result.

Theorem 3.1 *Let $G \in \mathscr{L}^S$. Then $G = \bigcup\limits_{i=1}^{n} H_i$ for an integer $n \geq 9$, where each H_i is a connected graph. Furthermore, if G is vertex-transitive graph, then $G = nH$ for an integer $n \geq 9$, where H is a vertex-transitive graph.*

Proof Let $C(i)$ be the connected component with a label i for a vertex u, where $i \in \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$. Then all vertices v in $C(i)$ must be with label $l_G^S(v) = i$. Otherwise, if there is a vertex v with $l_G^S(v) = j \in \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\} \setminus \{i\}$, let $P(u,v)$ be a path connecting vertices u and v. Then there must be an edge (x,y) on $P(u,v)$ such that $l_G^S(x) = i$, $l_G^S(y) = j$. By definition, $i \times j \notin l_G^S(E(G))$, a contradiction. So there are at least 9 components in G.

Now if G is vertex-transitive, we are easily know that each connected component $C(i)$ must be vertex-transitive and all components are isomorphic. $\square$

The smallest graph in $\mathscr{L}_v^S$ is the graph $9K_2$, shown in Fig.3.1 following.

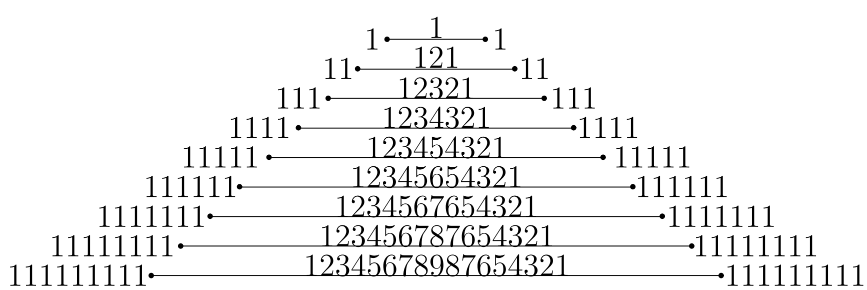


Fig.3.1

It should be noted that each graph in $\mathscr{L}_v^S$ is not connected. For finding a connected one, we construct a graph $\widehat{Q}_k$ following on the digital sequence

$$1, 11, 111, 1111, 11111, \cdots, \underbrace{11\cdots 1}_{k}.$$

by

$$V(\widetilde{Q}_k) = \{1, 11, \cdots, \underbrace{11\cdots 1}_{k}\} \bigcup \{1', 11', \cdots, \underbrace{11\cdots 1'}_{k}\},$$

$$E(\widetilde{Q}_k) = \{(1, \underbrace{11\cdots 1}_{k}), (x, x'), (x, y) | x, y \in V(\widetilde{Q}) \text{ differ in precisely one } 1\}.$$

Now label $x \in V(\widetilde{Q})$ by $l_G(x) = l_G(x') = x$ and $(u, v) \in E(\widetilde{Q})$ by $l_G(u)l_G(v)$. Then we have the following result for the graph $\widetilde{Q}_k$.

Theorem 3.2 *For any integer $m \geq 3$, the graph $\widetilde{Q}_m$ is a connected vertex-transitive graph of order $2m$ with edge labels*

$$l_G(E(\widetilde{Q})) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence.

Proof Clearly, $\widetilde{Q}_m$ is connected. We prove it is a vertex-transitive graph. For simplicity, denote $\underbrace{11\cdots 1}_{i}$, $\underbrace{11\cdots 1'}_{i}$ by $\overline{i}$ and $\overline{i}'$, respectively. Then $V(\widetilde{Q}_m) = \{\overline{1}, \overline{2}, \cdots, \overline{m}\}$. We define an operation $+$ on $V(\widetilde{Q}_k)$ by

$$\overline{k} + \overline{l} = \underbrace{11\cdots 1}_{k+l(\mathrm{mod}k)} \quad \text{and} \quad \overline{k}' + \overline{l}' = \overline{k+l}', \quad \overline{k}'' = \overline{k}$$

for integers $1 \leq k, l \leq m$. Then an element $\overline{i}$ naturally induces a mapping

$$i^* : \ \overline{x} \to \overline{x+i}, \quad \text{for} \ \ \overline{x} \in V(\widetilde{Q}_m).$$

It should be noted that i^* is an automorphism of $\widetilde{Q}_m$ because tuples $\overline{x}$ and $\overline{y}$ differ in precisely one 1 if and only if $\overline{x+i}$ and $\overline{y+i}$ differ in precisely one 1 by definition. On the other hand, the mapping $\tau : \overline{x} \to \overline{x}'$ for $\forall \overline{x} \in$ is clearly an automorphism of $\widetilde{Q}$. Whence,

$$\mathscr{G} = \langle\ \tau, i^* \mid 1 \leq i \leq m \rangle \preceq \mathrm{Aut}\widetilde{\mathrm{Q}},$$

which acts transitively on $V(\widetilde{Q})$ because $(\overline{y-x})^*(\overline{x}) = \overline{y}$ for $\overline{x}, \overline{y} \in V(\widetilde{Q})$ and $\tau : \overline{x} \to \overline{x}'$.

Calculation shows easily that

$$l_G(E(\widetilde{Q})) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence. This completes the proof. $\square$

By the definition of graph $\widetilde{Q}_m$, w consequently get the following result by Theorem 3.2.

Corollary 3.3 *For any integer $m \geq 3$, $\widetilde{Q}_m \simeq C_m \times P_2$.*

The smallest graph containing the third symmetry is $\widetilde{Q}_9$ shown in Fig.3.2 following,

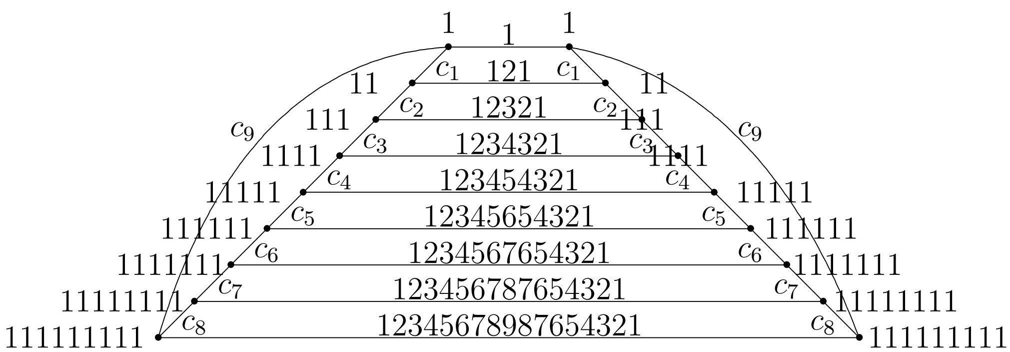


Fig.3.2

where $c_1 = 11$, $c_2 = 1221$, $c_3 = 123321$, $c_4 = 12344321$, $c_5 = 12344321$, $c_5 = 1234554321$, $c_6 = 123456654321$, $c_7 = 12345677654321$, $c_8 = 1234567887654321$, $c_9 = 123456789987654321$.

§4. Groups on Symmetric Graphs

In fact, the Smarandache digital or symmetric sequences are subsequences of $\mathbb{Z}$, a special infinite Abelian group. We consider generalized labelings on vertex-transitive graphs following.

Problem 4.1 *Let $(\Gamma; \circ)$ be a group generated by $x_1, x_2, \cdots, x_n$. Thus $\Gamma = \langle x_1, x_2, \cdots, x_n | W_1, \cdots \rangle$. Find connected vertex-transitive graphs G with generalized labeling $l_G : V(G) \to \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ and induced edge labeling $l_G(u, v) = l_G(u) \circ l_G(v)$*

for $(u,v)\in E(G)$ *such that*

$$l_G(E(G))=\{1_\Gamma, x_1^2, x_1\circ x_2, x_2^2, x_2\circ x_3,\cdots,x_{n-1}\circ x_n, x_n^2\}.$$

Similarly, we know the following result.

Theorem 4.2 *Let* $(\Gamma;\circ)$ *be a group generated by* $x_1,x_2,\cdots,x_n$ *for an integer* $n\geq 1$. *Then there are vertex-transitive graphs* G *with a labeling* $l_G: V(G)\rightarrow \{1_\Gamma, x_1,x_2,\cdots,x_n\}$ *such that the induced edge labeling by* $l_G(u,v)=l_G(u)\circ l_G(v)$ *with*

$$l_G(E(G))=\{1_\Gamma, x_1^2, x_1\circ x_2, x_2^2, x_2\circ x_3,\cdots,x_{n-1}\circ x_n, x_n^2\}.$$

Proof For any integer $m\geq 1$, define a graph $\widehat{Q}_{m,n,k}$ by

$$V(\widehat{Q}_{m,n,k})=\left(\bigcup_{i=0}^{m-1}U^{(i)}[x]\right)\bigcup\left(\bigcup_{i=0}^{m-1}W^{(i)}[y]\right)\bigcup\cdots\bigcup\left(\bigcup_{i=0}^{m-1}U^{(i)}[z]\right)$$

where $|\{U^{(i)}[x],v^{(i)}[y],\cdots,W^{(i)}[z]\}|=k$, $U^{(i)}[x]=\{x_0^{(i)},x_1^{(i)},x_2^{(i)},\cdots,x_n^{(i)}\}$, $V^{(i)}[y]=\{(y_0)^{(i)},y_1^{(i)},y_2^{(i)},\cdots,y_n^{(i)}\}$, $\cdots$, $W^{(i)}[z]=\{(z_0)^{(i)},z_1^{(i)},z_2^{(i)},\cdots,z_n^{(i)}\}$ for integers $0\leq i\leq m-1$ and

$$E(\widehat{Q}_{m,n})=E_1\bigcup E_2\bigcup E_3,$$

where $E_1=\{(x_l^{(i)},y_l^{(i)}),\cdots,(z_l^{(i)},x_l^{(i)})|0\leq l\leq n-1,\ 0\leq i\leq m-1\}$, $E_2=\{(x_l^{(i)},x_{l+1}^{(i)}),(y_l^{(i)},y_{l+1}^{(i)}),\cdots,(z_l^{(i)},z_{l+1}^{(i)})|0\leq l\leq n-1,\ 0\leq i\leq m-1,$ where $l+1\equiv(\mathrm{mod}n)\}$ and $E_3=\{(x_l^{(i)},x_l^{(i+1)}),(y_l^{(i)},y_l^{(i+1)}),\cdots,(z_l^{(i)},z_l^{(i+1)})|0\leq l\leq n-1,\ 0\leq i\leq m-1,$ where $i+1\equiv(\mathrm{mod}m)\}$. Then is clear that $\widehat{Q}_{m,n,k}$ is connected.

We prove this graph is vertex-transitive. In fact, by defining three mappings

$$\theta: x_l^{(i)}\rightarrow x_{l+1}^{(i)},\ y_l^{(i)}\rightarrow y_{l+1}^{(i)},\cdots,z_l^{(i)}\rightarrow z_{l+1}^{(i)},$$
$$\tau: x_l^{(i)}\rightarrow y_l^{(i)},\cdots,z_l^{(i)}\rightarrow x_l^{(i)},$$
$$\sigma: x_l^{(i)}\rightarrow x_l^{(i+1)},\ y_l^{(i)}\rightarrow y_l^{(i+1)},\cdots,z_l^{(i)}\rightarrow z_l^{(i+1)},$$

where $1\leq l\leq n,\ 1\leq i\leq m,\ i+1(\mathrm{mod}m),\ l+1(\mathrm{mod}n)$. Then it is easily to check that $\theta,\ \tau$ and σ are automorphisms of the graph $\widehat{Q}_{m,n,k}$ and the subgroup $\langle\theta,\tau,\sigma\rangle$ acts transitively on $V(\widehat{Q}_{m,n,k})$.

Now we define a labeling $l_{\widehat{Q}}$ on vertices of $\widehat{Q}_{m,n,k}$ by

$$l_{\widehat{Q}}(x_0^{(i)})=l_{\widehat{Q}}(y_0^{(i)})=\cdots=l_{\widehat{Q}}(z_0^{(i)})=1_\Gamma,$$

$$l_{\widehat{Q}}(x_l^{(i)}) = l_{\widehat{Q}}(y_l^{(i)}) = \cdots = l_{\widehat{Q}}(z_l^{(i)}) = x_l, \quad 1 \leq i \leq m, \ 1 \leq l \leq n.$$

Then we know that $l_G(E(G)) = \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ and

$$l_G(E(G)) = \{1_\Gamma, x_1^2, x_1 \circ x_2, x_2^2, x_2 \circ x_3, \cdots, x_{n-1} \circ x_n, x_n^2\}. \qquad \square$$

The following result is a conclusion by the proof of Theorem 4.3.

Corollary 4.3 *For integers $m, n \geq 1$, $\widehat{Q}_{m,n,k} \simeq C_m \times C_n \times C_k$.*

For example, the graph $\widehat{Q}_{5,3,2}$ is shown in Fig.4.1 following.

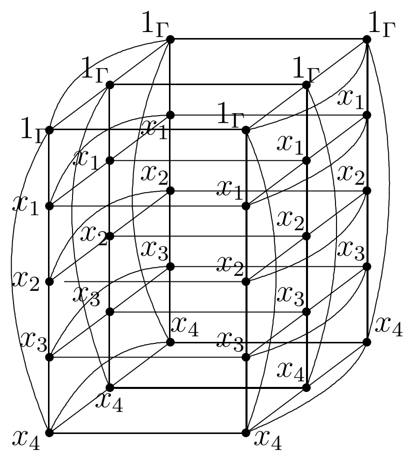


Fig.4.1

Particularly, let Γ be a subgroup of $(\mathbb{Z}_{111111111}, \times)$ generated by

$$\{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$$

and $m = 1$. We get the Symmetric sequence on the symmetric graph shown in Fig.3.2 again. Denote by $N_G[x]$ all vertices in a graph G labeled by $x \in \Gamma$. Then we also know the following result.

Corollary 4.4 $|N_{\widehat{Q}_{m,n,k}}[x]| = mk$ *for* $\forall x \in \{1_\Gamma, x_1, \cdots, x_n\}$ *and integers* $m, n, k \geq 1$.

§5. Speculation

It should be noted that the essence we have done in Sections 3 and 4 is a combinatorial notion, i.e., combining mathematical systems with that of graphs. Recently, Sridevi et al consider the Fibonacci sequence on graphs. Let G be a graph and

$\{F_0, F_1, F_2, \cdots, F_q, \cdots\}$ be the Fibonacci sequence, where F_q is the q^{th} Fibonacci number. An injective labeling $l_G : V(G) \to \{F_0, F_1, F_2, \cdots, F_q\}$ is called to be *super Fibonacci graceful* if the induced edge labeling by $l_G(u, v) = |l_G(u) - l_G(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$ with initial values $F_0 = F_1 = 1$. They proved a few graphs, such as those of $C_n \oplus P_m$, $C_n \oplus K_{1,m}$ have super Fibonacci labelings in [18]. For example, a super Fibonacci labeling of $C_6 \oplus P_6$ is shown in Fig.5.1.

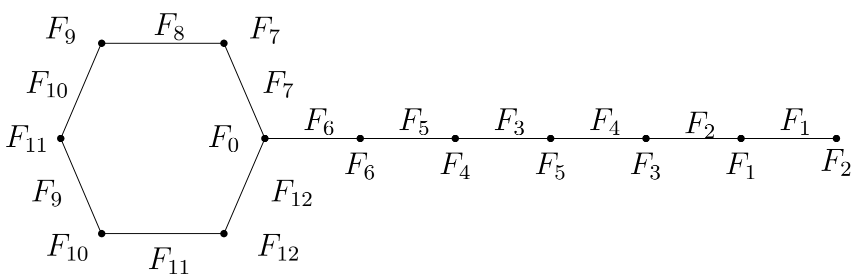


Fig.5.1

All of these are not just one mathematical system. In fact, they are applications of Smarandache multi-space and CC conjecture for developing modern mathematics, which appeals one to find combinatorial structures for classical mathematical systems, i.e., the following problem.

Problem 5.1 *Construct classical mathematical systems combinatorially and characterize them. For example, classical algebraic systems, such as those of groups, rings and fields by combinatorial principle.*

Generally, we have the following Smarandache multi-spaces following.

Definition 5.2([11],[13]) *For an integer $m \geq 2$, let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be m mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Definition 5.3([17]) *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ for an integer m, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\}, \quad E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

We have known a few Smarandache multi-spaces in classical mathematics. For examples, these rings and fields are group multi-space, and topological groups, topological rings and topological fields are typical multi-space are both groups, rings, or fields and topological spaces. Usually, if $m \geq 3$, a Smarandache multi-space must be underlying a combinatorial structure G. Whence, it becomes a combinatorial space in that case. I have presented the CC conjecture for developing modern mathematical science in 2005 [12], then formally reported it at *The 2th Conference on Graph Theory and Combinatorics of China* (2006, Tianjing, China)([14]-[17]).

CC Conjecture(Mao, 2005) *Any mathematical system* $(\Sigma; \mathcal{R})$ *is a combinatorial system* $\mathscr{C}_G(l_{ij}, 1 \leq i, j \leq m)$.

This conjecture is not just an open problem, but more likes a deeply thought, which opens a entirely way for advancing the modern mathematical sciences. In fact, it indeed means a *combinatorial notion* on mathematical objects following for researchers.

(1) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(2) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(3) One can make one combination of different branches in mathematics and find new results after then.

(4) One can understand our WORLD by this combinatorial notion, establish combinatorial models for it and then find its behavior, for example,

what is true colors of the Universe, for instance its dimension?

and $\cdots$. For its application to geometry and physics, the reader is refereed to books [13]-[14] and [17] of mine.

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