The Powers of π are Irrational

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Abstract

Transcendence of a number implies the irrationality of powers of a number, but in the case of π there are no separate proofs that powers of π are irrational. We investigate this curiosity. Transcendence proofs for e involve what we call Hermite's technique; for π's transcendence Lindemann's adaptation of Hermite's technique is used. Hermite's technique is presented and its usage is demonstrated with irrationality proofs of e and powers of e. Applying Lindemann's adaptation to a complex polynomial, π is shown to be irrational. This use of a complex polynomial generalizes and powers of π are shown to be irrational. The complex polynomials used involve roots of i and yield regular polygons in the complex plane. One can use graphs of these polygons to visualize various mechanisms used to proof π^2, π^3, and π^4 are irrational. The transcendence of π and e are easy generalizations from these irrational cases.

1 Introduction

There are curiosities in the treatment of the transcendence and irrationality of π and e. One such curiosity is the absence of proofs that powers of π are irrational. Certainly π^2 is proven to be irrational frequently [4, 21, 37], but not higher powers.[1] Proofs of the irrationality of powers of e [12, 28, 32][2] are given in text books, making it all the more odd that the irrationality of powers of π are not even mentioned.

[1]Haňcl does give a proof of the irrationality of π^4 [11].

[2]An attempt is made to give references to both primary and easily available secondary sources.

Another curiosity is that the proofs of the irrationality of π (and π^2), e, and the powers of e are not presented as special cases of transcendence. In both cases, e and π, it is obvious that the techniques used to prove transcendence of these numbers must work to prove their irrationality and the irrationality of their powers: if the polynomial $nx^k - m$ is an integral polynomial and $n/m = \pi^k$, (i.e. π^k is assumed rational) $k \geq 1$, then π is its root, a contradiction of transcendence.

Ideally the proofs of the irrationality of e and π would serve as base cases for transcendence proofs. We might also hope that proofs of the irrationality of e would suggest a means for proving that π is irrational. Historically, the transcendence proofs of these two constants followed this pattern. We demonstrate that this ideal sequence is possible, giving, apparently for the first time, an independent (from transcendence) proof that powers of π are irrational. Transcendence proofs for both numbers follow relatively easily from proofs that their powers are irrational. We demonstrate this.

2 The case of e

In this article four results are given: the irrationality of e, e^j, π, and π^j, where j here and throughout this article is a natural number. We wish to use the latest (easiest) transcendence proofs to establish the irrationality of e and π and their powers. In the case of e the transcendence proof of Hurwitz is the one we use [16, 18].

2.1 e is irrational

One can use a property of e with an assumption of its irrationality in the following way.[3] If we use a linear function to embed the assumption that e is rational, we have $f(x) = c_1 x + c_0$ is such that $f(e) = 0$ or $e = -c_0/c_1$, where c_j is an integer here and throughout this article. We have then, using composition and the mean value theorem (MVT),

$$\frac{f(e^1) - f(e^0)}{1 - 0} = [f(e^\zeta)]' = c_1 e^\zeta$$

or

$$-c_1 - c_0 = [f(e^\zeta)]' = c_1 e^\zeta,$$

[3]The proof of e's irrationality that occurs in textbooks, such as [3, 31], is attributed to Fourier [9].

with $\zeta \in (0,1)$. This has the pattern of a left side consisting of the sum of integers and the right side a power. If the sum on the left side can be made to have a factor of $n!$, all else remaining the same on the right side, a contradiction follows: an integer equals a power over a factorial. This is the general pattern of what we call Hermite's technique.

Composition of two functions gives the idea; the details follow from considering how a polynomial can yield integers divisible by $n!$. If r, an integer, is a root of multiplicity n of an integral polynomial, r will also be a root of the polynomial's 0 through $n-1$ derivatives [5]. After the nth derivative all surviving coefficients will have factors of $n!$. When these derivatives are evaluated at such a root they will yield integers divisible by $n!$. The sum of all derivatives of such a polynomial evaluated at such a root will also be divisible by $n!$.

Suppose then that $H(x)$ is the sum of the derivatives of a polynomial with roots 0 and 1 each of multiplicity n: that is let $h(x) = x^n(1-x)^n$ and $H(x)$ be the sum of its derivatives. We have, using the mean value theorem on the product $e^{-x}H(x)$,

$$\frac{e^{-1}H(1) - H(0)}{1-0} = -e^{-\zeta}(H(\zeta) - H'(\zeta)) = -e^{-\zeta}h(\zeta), \tag{1}$$

for some $\zeta \in (0,1)$. Our assumption that $e = -c_0/c_1$ can be combined with this auxiliary work to give

$$H(1) + \frac{c_0}{c_1}H(0) = -e^{-\zeta}\zeta^n(1-\zeta)^n \tag{2}$$

or

$$c_1H(1) + c_0H(0) = -c_1e^{-\zeta}\zeta^n(1-\zeta)^n, \tag{3}$$

an integer divisible by $n!$ equals a product of powers. As the product of powers never can equal zero, we have a contradiction: we know $\zeta^n/n! \to 0$ as $n \to \infty$ for any real or complex ζ.

2.2 Hermite's technique with Leibniz's formula

Hermite's technique uses the sum of the derivatives of a polynomial. Leibniz's formula,

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x), \tag{4}$$

gives the nth derivative of product of two functions. Hermite's technique, as used in transcendence theorems of e and π, generates non-zero integers divisible by $(p-1)!$ (p a prime) by stipulating that one root of a polynomial have multiplicity $p-1$ and all other roots have multiplicity p. One can view this situation as the product of two polynomials: $f(x) = x^{p-1}$ and $g(x) = c^M[\prod(x-r_s)]^p$. Such a product has a root $x = 0$ of multiplicity $p-1$ and r_s roots of multiplicity p. We take c as some constant to be specified raised to some natural number power M, also to be specified: the c^M factor is used to insure integer coefficients in $[\prod(z - z_s)]^p$. Using repeated applications of Leibniz's formula, we have

$$H(x) = \sum_{n=0}^{sp+p-1} \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}(x). \tag{5}$$

The term with $f^{(p-1)}(x)g^{(0)}(x)$ in this sum is critical to establishing the non-zero integer property. We call it the pivot of the Leibniz table for $H(x)$. All the terms in (5) can be depicted in a Leibniz table [21]; this allows the divisibility patterns to become apparent and the importance of the pivot to be seen.

In Table 1 all the derivatives of $f(x)$ are given along the top and all the derivatives of $g(x)$ are given in the left most column. The interior cells are the summands of (5) where the binomial coefficients of (5), immaterial to the divisibility pattern we are interested in, are omitted. One can determine $H(0)$ be noting that the top row will all be zero at $x = 0$, except for the last entry: $(p-1)!$. The left column will have p factors in all but the first entry. The product of p with $(p-1)!$ yields a $p!$ factor in all terms, except the pivot term. Looking at the pivot, then, we must stipulate that p is greater than $\max\{c, \prod r_s\}$ in order to force $(p-1)!|H(0)$, but $p \nmid H(0)$. Similar reasoning gives that $p!| \sum H(r_s)$.

The combination of this divisibility patterns gives $p \nmid (H(0) + \sum H(r_s))$ and this means $H(0) + \sum H(r_s)$ is potentially a non-zero integer: the coefficients have the right pattern for this property. If the roots, r_s, are those of an integer polynomial, then, using Newton's identities [35, p. 38], the sum of their powers is an integer and the non-zero integer is generated.

	z^{p-1}	$(p-1)z^{p-2}$	$\dots$	$(p-1)!$
$c^M[\prod(z-z_s)]^p$				*
$p\dots$				
$\vdots$				
$p!\dots$				
$\vdots$				

Table 1: Leibniz table with an asterisk indicating its pivot.

2.3 e^j is irrational

Let $h(x) = x^{p-1}(j-x)^p$ and $H(x)$ be the sum of the derivatives of $h(x)$. Assume, to obtain a contradiction, $e^j = -c_1/c_0$. The pivot of $h(x)$ is $(p-1)!(j-x)^p$; at $x = 0$, this is $(p-1)!j$, so we stipulate that $p > j$ to ensure that $p \nmid H(0)$. Using the MVT, we have

$$\frac{e^{-j}H(j) - e^0 H(0)}{j-0} = -e^{\zeta}h(\zeta),$$

for some $\zeta \in (0, j)$. Using our assumption, we obtain

$$c_1 H(j) + c_0 H(0) = -jc_1 e^{j-\zeta}h(\zeta). \tag{6}$$

We must stipulate that $p > |c_0|$. Upon division of both sides of (6) by $(p-1)!$, the absolute value of the left side shrinks to less than one, a contradiction.

The proofs of the irrationality of e and powers of e just given appear to be new, although relatively easy applications of the transcendence of e proof by Hurwitz. Existing irrationality proofs for rational powers of e [12, 28, 32], an easy generalization from e^j is irrational, are needlessly difficult and use Hermite's original transcendence proof [13] of e. We, thus, have provided an update for irrationality proofs for rational powers of e via the more recent evolution[4] of Hermite's original transcendence proof: that of Hurwitz.

[4] A telltale sign of an early transcendence and irrationality proof is the unmotivated definition of a complicated polynomial.

3 The case of π

Upon Hermite's success with proving the transcendence of e [2, 13, 14], Lindemann took up the case of proving the transcendence of π. He succeeded in 1882 [2, 25]. The existing proofs of π's irrationality circa 1882 are complicated and unrelated to Lindemann's transcendence proof [2, 15, 23, 24, 26]. For a review of these (and other) irrationality proofs see [4, 36].

After Lindemann's transcendence proof, then, the natural question (a puzzle) is posed: can (or why can't) an irrationality proof for π be constructed based on Lindemann and, by extension, Hermite's transcendence proofs? Niven, perhaps seeing this puzzling lacuna, succeeded in proving π is irrational using Hermite's original technique in 1947 [29]. His polynomials, however, do not generalize to prove all powers of π are irrational, nor do they connect with a proof of the transcendence of π directly as a special case. We show here that this ideal, relative to economics of effort, can be realized. The e case points the way to π, as it did historically. Complex polynomials, forbidden in transcendence proofs, are the key.

3.1 A template for a proof

If the MVT worked for complex variables, a proof of π's irrationality could be easily achieved with the themes just used for proving e is irrational. Briefly, with suitably defined $h(z)$ and $H(z)$, we would have

$$\frac{e^{-\pi i}H(\pi i) - e^0 H(0)}{\pi i - 0} = -e^{-\zeta}h(\zeta). \tag{7}$$

With some algebra, this becomes

$$e^{\pi i}H(0) = H(\pi i) + \pi i e^{\pi i - \zeta}h(\zeta). \tag{8}$$

Adding $H(0)$ to both sides, using Euler's identity, and dividing both sides by $(p-1)!$ gives the contradiction: we obtain a non-zero integer, perhaps a non-zero Gaussian integer, with absolute value less than one. Gaussian integers, numbers of the form $x + iy$ with x and y integers, are a subject of abstract algebra [1, 8] and number theory [12].

Unfortunately the MVT does not work with complex variables: the function e^{ix} on the general interval $[a, b]$ is a counterexample [30, p. 39]. Complex integration, however, does give a means of achieving an irrationality proof for π.

3.2 Complex calculus

We wish to show how transcendence techniques for π can prove its irrationality. As with the transcendence of e, transcendence of π proofs have evolved from Lindemann's original proof. We use Niven's proof [27] and Niven, curiously enough, cites Hurwitz's transcendence of e proof [18] as his source for the use of $e^{-z}H(z)$.[5] He introduces several innovations. One of which is complex integration. The numerator of the left side of (7) does remind one of an evaluation of a definite integral.

3.3 π is irrational

Assume $\pi = m/n$. Using Hermite's technique, let $h(z) = z^{p-1}(nz - mi)^p$ and $H(z)$ be defined as the sum of its derivatives. The product rule for derivatives gives

$$\frac{d}{dx}e^{-z}H(z) = -e^{-z}(H(z) - H'(z)) = -e^{-z}h(z). \tag{9}$$

Forming the complex integral with both sides of (9), we have

$$\int_0^{\pi i} \frac{d}{dz}[e^{-z}H(z)]dz = -\int_0^{\pi i} e^{-z}h(z)dz \tag{10}$$

and this implies, using the fundamental theorem of calculus [33, Theorem 3, p. 97],

$$e^{-\pi i}H(\pi i) - H(0) = -\int_0^{\pi i} e^{-z}h(z)dz. \tag{11}$$

Multiplying (11) by $e^{\pi i}$ and using some algebra gives

$$e^{\pi i}H(0) = H(\pi i) + e^{\pi i}\int_0^{\pi i} e^{-z}h(z)dz \tag{12}$$

and, on adding $H(0)$ to both sides of (12), we have, using Euler's formula,

$$0 = H(0)(e^{\pi i} + 1) = H(0) + H(\pi i) + e^{\pi i}\int_0^{\pi i} e^{-z}h(z)dz. \tag{13}$$

[5]Niven's proof is used, although it is not cited, in [7, 34], the distinguishing characteristic being the use of complex integration. In contrast, the older [12, 17] use derivatives only versions, such as the one by Gordan [10]. One can modify our proof of the irrationality of π into a derivatives only version and the previous use of the mean value theorem for e proofs into real integral versions.

We need $(H(0) + H(\pi i))/(p-1)!$ to be non-zero. The pivot of the Leibniz table for $h(z)$ is $(p-1)!(nz - mi)^p$; at $z = 0$ this is $(p-1)!(-mi)^p$; we must stipulate that $p > m$ to ensure this expression is not divisible by p; with this stipulation and for all odd prime p, $(p-1)!(-mi)^p$ is a purely imaginary Gaussian integer not divisible by p.

To complete the proof it is easy to show (see Theorem 1 in section 6) that the integral of (13) becomes less than one upon division by $(p-1)!$ for sufficiently large p. This being accomplished, (13) reduces to

$$0 = R + \epsilon, \tag{14}$$

where R is a non-zero Gaussian integer and $|\epsilon| < 1$: a contradiction.

4 The powers of π are irrational

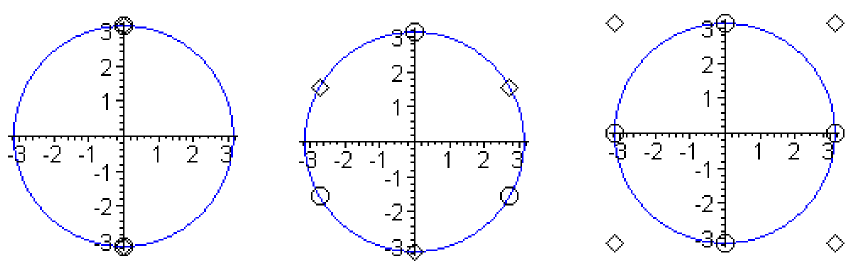

Figure 1: The roots r_k and R_k for the $j = 2$, 3, and 4 cases.

Niven's proof of π's irrationality was generalized to the π^2 case by Iwamoto in 1949 [20] but not to higher powers. Original proofs of π's irrationality seem to allow for exactly one generalization – the squared case. Other recent examples of this are [6, 19, 21, 22, 37]. Will the approach given here suffer the same fate?

4.1 π^2 is irrational

Consider $z^2 - (\pi i)^2 = (z - \pi i)(z + \pi i)$. We can get the same pattern as (13) using

$$\begin{aligned} 0 &= H(0)(1 + e^{\pi i})(1 + e^{-\pi i}) && (15)\\ &= H(0)(2 + e^{\pi i} + e^{-\pi i}) && (16)\\ &= 2H(0) + H(\pi i) + H(-\pi i) + 2\epsilon, && (17) \end{aligned}$$

where ϵ gives a sum of two integral expressions that go to 0 upon division by a factorial. Assuming $\pi^2 = m/n$, we define

$$h(z) = n^{2p+p-1} z^{p-1}[(z-mi/n)(z+mi/n)]^p = n^{2p+p-1} z^{p-1}[z^2+m/n]^p. \tag{18}$$

The power of n insures that all the coefficients of $h(z)$ and hence of $H(z)$ are integers. The sum $H(\pi i) + H(-\pi i)$ will be an integer divisible by $(p-1)!$, as the sum of odd powers of πi and $-\pi i$ cancel each other out and even powers fall under the rationality assumption. The pivot for $H(0)$ is $n^{2p+p-1}(m/n)^p = n^{2p-1}m^p$, so stipulating that $p > \max\{mn\}$, completes this application of Hermite's technique. As before we obtain a contradiction: π^2 is irrational.

The pattern for proving powers of π are irrational and π is transcendental follow from this π^2 case!

4.2 π^3 and π^4 are irrational

Using the roots, r_k, of $f(z) = z^j - (\pi i)^j$ we can repeat this pattern for general j powers. The roots of $f(z)$ are the jth roots of the complex number

$$\pi(i^{\text{j mod 4}})^{1/j}. \tag{19}$$

There are four cases corresponding to the value of i^j:

$$\begin{aligned} &\pi(\cos(\pi/2 + 2k\pi) + i\sin(\pi/2 + 2k\pi))^{1/j} \text{ for } i^j = i && (20)\\ &\pi(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))^{1/j} \text{ for } i^j = -1 && (21)\\ &\pi(\cos(3\pi/2 + 2k\pi) + i\sin(3\pi/2 + 2k\pi))^{1/j} \text{ for } i^j = -i && (22)\\ &\pi(\cos(2k\pi) + i\sin(2k\pi))^{1/j} \text{ for } i^j = 1. && (23) \end{aligned}$$

Let r_k, $1 \le k \le j$, designate the roots for each of these cases. One such root will always be πi, insuring the zero value on the left of

$$0 = H(0)\left(\prod(1 + e^{r_k})\right) = AH(0) + \sum H(R_k) + \epsilon. \tag{24}$$

j	r_k
1	πi
2	$\pm\pi i$
3	$\pi i, \pi\left(\frac{\pm\sqrt{3}}{2} - \frac{i}{2}\right)$
4	$\pm\pi i, \pm\pi$

Table 2: Roots of $f(z)$ for first four powers.

j	R_k non-zero	No. of zero roots	$\prod(z - R_k)$
1	πi	0	$z - \pi i$
2	$\pm\pi i$	1	$z^2 + \pi^2$
3	$\pm\pi i, \pi\left(\pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}\right)$	1	$z^6 + \pi^6$
4	$\pm\pi i, \pm\pi, \pi(\pm 1 \pm i)$	3	$(z^4 - \pi^4)^2(z^4 + 4\pi^4)$

Table 3: A few details for first four powers of π.

Table 2 gives these roots, r_k, for $j = 1$, 2, 3, and 4. In Table 3 we have the non-zero sums of these roots, R_k, taken one through j at a time (per the algebra of $\prod(1 + e^{r_k})$), the number of such sums that total zero (reflected in the constant A in (24)), and the polynomial that results from the non-zero roots. The ϵ in (24) reflects terms that shrink to zero upon division by $(p-1)!$. The two sets of roots, r_k (circles) and R_k (circles and diamonds), are depicted for cases $j = 2$, 3, and 4 in Figure 1.[6]

As before we assume, to get a contradiction, $\pi^j = m/n$. Examining the polynomials generated by the R_k roots shows that the coefficients generated involve powers of π that are under the rationality assumption for each case. Hence, multiplying the polynomials in Table 3 by a sufficiently large power of n, n^{jp+p-1}, as we did with the π^2 case, insures all coefficients in $h(z)$ are integers or Gaussian integers. The sums of powers of the roots R_k will be, per Newton's identities, Gaussian integers. A Leibniz table indicates a lower bound on the prime p that must be used. These powers of π are irrational. Curiously, the general case is easier to prove than these specific cases.

[6]These graphs show roots of i, not unity. Like roots of unity, powers of i revolve around a circle. Roots of unity are a topic of abstract algebra [8] and complex analysis [33].

4.3 The general case

For the general jth power of π, the polynomial $f(z) = z^j - (\pi i)^j$ has only one coefficient and, under a rationality assumption, this is a Gaussian rational. The sums of these roots of this polynomial taken one through j at a time will be expressible with this rational.[7] This follows from the fundamental theorem of symmetric functions [35, chapter 8]. We can form a polynomial

$$h(z) = n^{jp+p-1} z^{p-1} [\prod (z - R_k)]^p \tag{25}$$

using these roots. It will have Gaussian integer coefficients. The sum of powers of such roots will be also expressible using this same single coefficient. This follows as such sums are symmetric functions of this single elementary symmetric function, the constant coefficient $(\pi i)^j$. Newton's identities confirm the power of n needed and indicate a possible factor of j with this coefficient. We see this with the $j = 4$ polynomial given in Table 3. Using this information with a Leibniz table we stipulate that $p > \max\{j, A, m, n\}$ where A is defined by (24). General powers of π are irrational.

5 The transcendence of e and π

It's a curious twist in the saga of the transcendence of e and π that with the above proofs of the irrationality of the powers of these numbers, transcendence proper for π is now easier to prove than that for e. We demonstrate this in this section.

The pattern for transcendence proofs is to combine an inference from the assumption that the number in question solves a integer polynomial with a property of the number and arrive at a contradiction. Given $f(z)$ is an integer polynomial, regardless of its roots, we can make the following inferences. The first two were proven earlier.

Theorem 1 *If $f(x)$ is an integer polynomial of degree n with roots r_k there exists an integer, N, and integer polynomials $H(x)$ such that*

[7] Consider $(x - r_1)(x - r_2)(x - (r_1 + r_2))$. A transposition of the roots r_1 and r_2 gives the same polynomial via commutativity of multiplication and addition. Permutations are compositions of transpositions. So permutations of such polynomials built from sums of roots give the same polynomial; the R_k roots yield coefficients that are symmetric functions of the r_k roots.

for all prime numbers p, with $p > N$

$$H(0) + \sum_{k=1}^{n} H(r_k) \tag{26}$$

is divisible by $(p-1)!$ and not divisible by p.

Corollary 1 *Under the assumptions of Theorem 1,*

$$H(0) + \sum_{k=1}^{n} H(r_k) \tag{27}$$

is a non-zero integer.

The following result is implicitly given in proofs of the transcendence of e and π.

Theorem 2 *For any real or complex polynomial $f(z) = (z-r_1)\dots(z-r_n) = z^n + c_1 z^{n-1} + \dots + c_n$, if $r_a = \max\{|r_k|\}$, then*

$$|f(r_k)| \leq (2r_a)^n, \tag{28}$$

for every root r_k of $f(z)$.

Proof. The elementary symmetric functions are generated by expanding the root form of the polynomial. So c_j is the sum of the product of roots r_k taken j at a time. We have then

$$|f(z)| \leq |f(|z|)| \leq |z|^n + |c_1||z|^{n-1} + \dots + |c_n|. \tag{29}$$

Let C_k indicate the sum of the absolute values of the roots taken j at a time. This implies, using once again the triangle inequality, that

$$|z|^n + |c_1||z|^{n-1} + \dots + |c_n| \leq |z|^n + C_1|z|^{n-1} + \dots + C_n. \tag{30}$$

The right hand side of this inequality is the same as

$$(|z| + |r_1|)\dots(|z| + |r_n|), \tag{31}$$

so we have

$$f(|z|) \leq (|z| + |r_1|)\dots(|z| + |r_n|). \tag{32}$$

This inequality is true for any z, so it is true for all r_k. Thus

$$f(|r_k|) \leq (|r_k| + |r_1|)\dots(|r_k| + |r_n|) \tag{33}$$

and given the definition of r_a, we have

$$f(|r_k|) \leq (|r_k| + |r_1|)\dots(|r_k| + |r_n|) \leq (2r_a)^n. \tag{34}$$

This with (29) gives (28).

Corollary 2 *Under the assumptions of Theorem 2,*

$$\lim_{m\to\infty} \frac{[f(r_k)]^m}{m!} = 0 \tag{35}$$

and

$$\lim_{m\to\infty} \sum_{k=1}^{n} \frac{[f(r_k)]^m}{m!} = 0. \tag{36}$$

Corollary 3 *Under the assumptions of Theorem 2,*

$$\left| \int_0^{r_k} f(z)dz \right| \le r_a(2r_a)^n \tag{37}$$

and

$$\sum \left| \int_0^{r_k} f(z)dz \right| \le r_a \sum (2r_a)^n. \tag{38}$$

Theorem 3 *e is transcendental.*

Proof. Suppose e solves $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$. We showed previously that all the powers of e are irrational using polynomials with $x = 0$ and $x = j$ roots. We combine these polynomials and define

$$h(x) = x^{p-1} \left[\prod_{k=1}^{n} (k - x) \right]^p, \tag{39}$$

and, as usual, define $H(x)$ as the sum of the derivatives of $h(x)$.

We have, with repeated uses of the fundamental theorem of calculus (or the MVT),

$$c_n H(n) - e^n c_n H(0) = \epsilon_n \tag{40}$$

$$c_{n-1} H(n-1) - e^{n-1} c_{n-1} H(0) = \epsilon_{n-1} \tag{41}$$

$$\vdots \tag{42}$$

$$c_1 H(1) - e c_1 H(0) = \epsilon_1. \tag{43}$$

Adding all of the equations above gives

$$\sum_{k=0}^{n} c_k H(k) = \sum_{k=1}^{n} \epsilon_k. \tag{44}$$

We use $c_0 = -(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x)$ in this equation. The left hand side is an integer divisible by $(p-1)!$, but not by p, and the right hand side, using Corollary 2, goes to zero upon division by $(p-1)!$. This yields a non-zero integer less than one, a contradiction.

Theorem 4 *π is transcendental.*

Proof. Suppose πi solves $c_j x^j + c_{j-1}x^{j-1} + \cdots + c_0 = 0$. Exactly as in the proof that the powers of π are irrational, we form roots R_k from the r_k roots of this polynomial. Let

$$h(z) = n^{jp+p-1} z^{p-1}[\prod(z - R_k)]^p \tag{45}$$

where R_k are the roots r_k summed one through j at a time. Let $H(z)$ be the sum of $h(z)$ derivatives. Using Euler's formula, we have

$$0 = H(0)\left(\prod(1 + e^{r_k})\right) = AH(0) + \sum H(R_k) + \epsilon, \tag{46}$$

where the ϵ term, using Corollary 3, goes to zero upon division by $(p-1)!$. As $AH(0) + \sum H(R_k)$, using the same reasoning as with the powers of π proof, is a non-zero integer, we have a contradiction.

6 Conclusion

Apart from the general connecting the dots theme of this article, there are additional benefits to this approach to the transcendence of π. Technology allows for ease of computation and experimentation, as well as the generation of visual elements to accompany proofs. The author confirmed many hand calculations for Tables 2 and 3 using Maple and Excel. It is also possible with Maple (other computer algebra systems) to generate polygons giving the R_k roots for the $j = 3$ and 4 cases: $j = 5$ is challenging. With such illustrations one can get a visual corroboration of roots and sums of roots; one can see the polynomial generated in the $j = 3$ case: $z^6 + \pi^6$. Modern pedagogy encourages students to explore and find solutions on their own; the use of the treatment given here in classrooms might thus enable students to find their own, so to speak, proofs about π: its powers.

Finally, it is hoped that this article will encourage others to reconsider classic mathematics and continue to explore it and simplify it for the service of a new generation of mathematicians: they need to build upon things difficult to us made a little easier. Such is, may I proffer, the nature of good cultural evolution.

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