The Diophantine Equations $a^2 \pm mb^2 = c^n$, $a^3 \pm mb^3 = d^2$ and $y_1^4 \pm my_2^4 = R^2$

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Abstract

The Diophantine equations $a^2 \pm mb^2 = c^n$, and $a^3 \pm mb^3 = d^2$ have infinitely many nonzero integer solutions, Using the methods of infinite descent and infinite ascent we prove $y_1^4 \pm my_2^4 = R^2$.

The Diophantine equation

$$a^2 + b^2 = c^3, \tag{1}$$

has infinitely many nonzero integer solutions. But it is difficult to prove this [1,2]. In this paper we prove some theorems.

Theorem 1. The Diophantine equation

$$a^2 + mb^2 = c^n \tag{2}$$

has infinitely many nonzero integer solutions.

We define supercomplex number [3]

$$z = \begin{pmatrix} x & -my \\ y & x \end{pmatrix} = x + yJ, \tag{3}$$

where

$$J = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}, \quad J^2 = -m.$$

Then from equation (3)

$$z^n = (x + yJ)^n = a + bJ. \tag{4}$$

Let n be an odd number

$$a=\sum_{k=0}^{(n-1)/2}\binom{n}{2k}(-m)^k x^{n-2k}y^{2k}, b=\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}(-m)^k x^{n-2k-1}y^{2k+1},$$

Let n be an even number

$$a=\sum_{k=0}^{n/2}\binom{n}{2k}(-m)^k x^{n-2k}y^{2k}, b=\sum_{k=0}^{n/2-1}\binom{n}{2k+1}(-m)^k x^{n-2k-1}y^{2k+1}.$$

Then from (4) the circulant matrix

$$\begin{pmatrix} x & -my \\ y & x \end{pmatrix}^n = \begin{pmatrix} a & -mb \\ b & a \end{pmatrix}, \tag{5}$$

Then from (5) circulant determinant

$$\begin{vmatrix} x & -my \\ y & x \end{vmatrix}^n = \begin{vmatrix} a & -mb \\ b & a \end{vmatrix}, \tag{6}$$

Then from equation (6)

$$c^n = a^2 + mb^2, \tag{7}$$

where

$$c = x^2 + my^2 .$$

We prove that (2) has infinitely many nonzero integer solutions.

Theorem 2. The Diophantine equation

$$a^2 - mb^2 = c^n \tag{8}$$

has infinitely nonzero integer solutions.

Define supercomplex number [3]

$$z = \begin{pmatrix} x & my \\ y & x \end{pmatrix} = x + yJ , \tag{9}$$

where

$$J = \begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix}, \quad J^2 = m .$$

Then from equation (9)

$$z^n = (x + yJ)^n = a + bJ . \tag{10}$$

Let n be an odd number

$$a=\sum_{k=0}^{(n-1)/2}\binom{n}{2k}m^k x^{n-2k}y^{2k}, b=\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}m^k x^{n-2k-1}y^{2k+1}.$$

Let n be an even number

$$a=\sum_{k=0}^{n/2}\binom{n}{2k}m^k x^{n-2k}y^{2k}, b=\sum_{k=0}^{n/2-1}\binom{n}{2k+1}m^k x^{n-2k-1}y^{2k+1}.$$

Then from (10) circulant matrix

$$\begin{pmatrix} x & my \\ y & x \end{pmatrix}^n = \begin{pmatrix} a & mb \\ b & a \end{pmatrix}. \qquad (11)$$

Then from (11) circulant determinant

$$\begin{vmatrix} x & my \\ y & x \end{vmatrix}^n = \begin{vmatrix} a & mb \\ b & a \end{vmatrix}. \qquad (12)$$

Then from equation (12)

$$c^n = a^2 - mb^2, \qquad (13)$$

where

$$c = x^2 - my^2.$$

We prove that (8) has infinitely many nonzero integer solutions.

Theorem 3. The Diophantine equation

$$a^3 + mb^3 + m^2c^3 - 3mabc = d^n \qquad (14)$$

has infinitely many nonzero integer solutions

Define supercomplex number [3]

$$w = \begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix} = x + yJ + zJ^2, \qquad (15)$$

where

$$J = \begin{pmatrix} 0 & 0 & m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = m.$$

Then from (15)

$$w^n = (x + yJ + zJ^2)^n = a + bJ + cJ^2 \qquad (16)$$

Then from equation (16) circulant matrix

$$\begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix}^n = \begin{pmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{pmatrix} \qquad (17)$$

Then from equation (17) circulant determinant

$$\begin{vmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{vmatrix}^n = \begin{vmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{vmatrix} \qquad (18)$$

Then from equation (18)

$$d^n = a^3 + mb^3 + m^2c^3 - 3mabc \qquad (19)$$

where

$$d = x^3 + my^3 + m^2 z^3 - 3mxyz \qquad (20)$$

We prove that (14) has infinitely many nonzero integer solutions.

Suppose $n = 2$ and $c = 0$. Then from (19)

$$a^3 + mb^3 = d^2 \qquad (21)$$

when $n = 2$ from (16)

$$a = x^2 + 2myz \neq 0, \quad b = 2xy + mz^2 \neq 0, \quad c = y^2 + 2xz = 0 \qquad (22)$$

Then from (22) $y^2 = -2xz$.

Let

$$z = -2, \ x = P^2, \quad y = 2P, \qquad (23)$$

where $P > 1$ is an odd number.

Substituting (23) into (20) and (22)

$$d = P^6 + 20mP^3 - 8m^2, \quad a = P^4 - 8mP, \quad b = 4P^3 + 4m \qquad (24)$$

Using equation (24) we prove that (21) has infinitely many nonzero integer solutions.

Theorem 4. The Diophantine equation

$$a^3 - mb^3 + m^2 c^3 + 3mabc = d^n \qquad (25)$$

has infinitely many nonzero integer solutions.

Define supercomplex number [3]

$$w = \begin{pmatrix} x & -mz & -my \\ y & x & -mz \\ z & y & x \end{pmatrix} = x + yJ + zJ^2, \qquad (26)$$

where

$$J = \begin{pmatrix} 0 & 0 & -m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & -m & 0 \\ 0 & 0 & -m \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = -m,$$

Then from (26)

$$w^n = (x + yJ + zJ^2)^n = a + bJ + cJ^2 \qquad (27)$$

Then from (27) circulant matrix

$$\begin{pmatrix} x & -mz & -my \\ y & x & -mz \\ z & y & x \end{pmatrix}^n = \begin{pmatrix} a & -mc & -mb \\ b & a & -mc \\ c & b & a \end{pmatrix}, \qquad (28)$$

Then from equation (28) circulant determinant

$$\begin{vmatrix} x & -mz & -my \\ y & x & -mz \\ z & y & x \end{vmatrix}^n = \begin{vmatrix} a & -mc & -mb \\ b & a & -mc \\ c & b & a \end{vmatrix}. \qquad (29)$$

Then from (29)

$$d^n = a^3 - mb^3 + m^2c^3 + 3mabc \qquad (30)$$

where

$$d = x^3 - my^3 + m^2z^3 + 3mxyz. \qquad (31)$$

We prove that (25) has infinitely many nonzero integer solutions.
Suppose $n = 2$ and $c = 0$. Then from (30)

$$a^3 - mb^3 = d^2 \qquad (32)$$

When $n = 2$ from (27)

$$a = x^2 - 2myz \neq 0, \quad b = -mz^2 + 2xy \neq 0, \quad c = y^2 + 2xz = 0 \qquad (33)$$

Then from (33) $y^2 = -2xz$

Let $z = -2, \quad x = P^2, \quad y = 2P$, (34)

where $P > 1$ is an odd numer.
Substitutin (34) into (31) and (33)

$$d = P^6 - 20mP^3 - 8m^2, \quad a = P^4 + 8mP, \quad b = 4P^3 - 4m \qquad (35)$$

Using (35) we prove that (32) has infinitely many nonzero integer solutions.
Theorem 5. Define supercomplex number

$$w = \begin{pmatrix} x_1 & -mx_4 & -mx_3 & -mx_2 \\ x_2 & x_1 & -mx_4 & -mx_3 \\ x_3 & x_2 & x_1 & -mx_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} = x_1 + x_2J + x_3J^2 + x_4J^3, \qquad (36)$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & -m \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & -m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J^4 = -m$$

Then from (36)

$$w^n = (x_1 + x_2J + x_3J^2 + x_4J^3)^n = y_1 + y_2J + y_3J^2 + y_4J^3 \qquad (37)$$

Then from (37)

$$R^n = |y_i|, \qquad (38)$$

where

$$R = x_1^4 + m(x_2^4 + 2x_1^2x_3^2 - 4x_1x_2^2x_3 + 4x_2x_1^2x_4) + m^2(x_3^4 + 2x_2^2x_4^2 + 4x_1x_3x_4^2 - 4x_2x_4x_3^2) + m^3x_4^4,$$

$$|y_i| = y_1^4 + m(y_2^4 + 2y_1^2y_3^2 - 4y_1y_2^2y_3 + 4y_2y_1^2y_4) + m^2(y_3^4 + 2y_2^2y_4^2 + 4y_1y_3y_4^2 - 4y_2y_4y_3^2) + m^3y_4^4.$$

（39）

We prove that (38) has infinitely many nonzero integer solutions,

Suppose $n = 2,\ y_1 \neq 0,\ y_2 \neq 0,\ y_3 = 0$ and $y_4 = 0$, from (38) and (39)

$$R^2 = y_1^4 + my_2^4 \quad (40)$$

When $n = 2$ from (37)

$$y_1 = x_1^2 - mx_3^2 - 2mx_2x_4 \neq 0, \quad (41)$$

$$y_2 = 2(x_1x_2 - mx_3x_4) \neq 0, \quad (42)$$

$$y_3 = x_2^2 - mx_4^2 + 2x_1x_3 = 0, \quad (43)$$

$$y_4 = 2(x_1x_4 + x_2x_3) = 0, \quad (44)$$

Then from (44)

$$x_3 = -\frac{x_1x_4}{x_2}. \quad (45)$$

Substituting (45) into (43)

$$x_4 = \frac{-x_1^2 \pm \sqrt{x_1^4 + mx_2^4}}{mx_2}. \quad (46)$$

Then from (46)

$$R_1^2 = x_1^4 + mx_2^4. \quad (47)$$

If (47) has no nonzero integer solutions, $R_1 < R$, using the method of infinite descent we prove (40) has no nonzero integer solutions. If (47) has one nonzero integer solution, $R_1 < R$, using the method of infinite ascent we prove (40) has infinitely many nonzero integer solutions.

Suppose $m = 1$ from (47)

$$R_1^2 = x_1^4 + x_2^4 \quad (48)$$

has no integer solutions.

Suppose $m = 2$ from (47)

$$R_1^2 = x_1^4 + 2x_2^4 \quad (49)$$

has no nonzero integer solutions.

Suppose $m=8$ [4] from (47)

$$R_1^2=x_1^4+8\times x_2^4 \tag{50}$$

We have a solution $3^2=1^4+8(1^4)$. Let $x_1=1, x_2=1$. Then from (45) and (46) $x_3=1/2$, $x_4=-1/2$. Then from (41) and (42) $y_1=7, y_2=6$, $7^4+8\times 6^4=113^2$

Let $x_1=7, x_2=6, x_3=-14/9, x_4=4/3$. Then from (41) and (42) $y_1=-7967/81$, $y_2=9492/81$. $7967^4+8\times 9492^4=262621633^2$.

Suppose $m=73$ [5]. From (47)

$$R_1^2=x_1^2+73\times x_2^4. \tag{51}$$

We have $6^4+73(1^4)=37^2$, $1223^4+73\times 444^4=2252593^2$.

Suppose $m=89$ [5]. From (47),

$$R_1^2=x_1^4+89\times x_2^4. \tag{52}$$

We have $2^4+89\times 3^4=85^2$, $7193^4+89\times 1020^4=52662001^2$

If $m=R_1^2-1$ and $m=R_1^2-x_1^4$, then (40) has infinitely many nonzero integer solutions.

Theorem 6. The Diophantine equation

$$y_1^4-my_2^4=R^2, \tag{53}$$

where

$$y_1=x_1^2+mx_3^2+2mx_2x_4\neq 0, \tag{54}$$

$$y_2=2(x_1x_2+mx_3x_4)\neq 0, \tag{55}$$

$$y_3=x_2^2+mx_4^2+2x_1x_3=0, \tag{56}$$

$$y_4=2(x_1x_4+x_2x_3)=0. \tag{57}$$

Then from (57)

$$x_3=-\frac{x_1x_4}{x_2}. \tag{58}$$

Substituting (58) into (56)

$$x_4 = \frac{x_1^2 \pm \sqrt{x_1^4 - mx_2^4}}{mx_2}. \quad (59)$$

Then from (59)

$$x_1^4 - mx_2^4 = R_1^2. \quad (60)$$

If (60) has no nonzero integer solutions. $R_1 < R$, using the method of infinite descent we prove (53) has no nonzero integer solutions. If (60) has one nonzero integer solution, $R_1 < R$, using the method of infinite ascent we prove (53) has infinitely many nonzero integer solutions.

Suppose $m = 1$ from (60)

$$x_1^4 - x_2^4 = R_1^2. \quad (61)$$

has no nonzero integer solutions.

Suppose $m = 2$, from (60)

$$x_1^4 - 2x_2^4 = R_1^2, \quad (62)$$

has no nonzero integer solutions.

Suppose $m = 7$ from (60)

$$x_1^4 - 7x_2^4 = R_1^2. \quad (63)$$

We have one solution

$$2^4 - 7(1^4) = 3^2.$$

Let $x_1 = 2, x_2 = 1$. Then from (58) and (59) $x_3 = -2$, $x_4 = 1$. Then from (54) and (55) $y_1 = 2\times 23$, $y_2 = -2\times 12$. $23^4 - 7\times 12^4 = 367^2$. We prove (63) has infinitely many nonzero integer solutions.

If $m = x_1^4 - R_1^2$, then (53) has infinitely many nonzero integer solutions. Our method [3] is used in studies of the Diophantine equations

$$y_1^n \pm my_2^n = R^e, n = 2,3,4,...; e = 2,3,4,...; m = 1,2,3,... \quad (61)$$

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