An Analytical Approach to Polyominoes and a solution to the Goldbach conjecture

By

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ABSTRACT. Always, when viewing papers whose writers show polyominoes graphically, this question crossed my mind, are there any equations which may be given to avoid the need for drawings? Polyominoes are sometimes called by the number of faces (like triomeno or tetraomino). In this paper, I try to formulate polyomino shapes and establish a correspondence between them and polynominals. About the final part where I refer to the Goldbach conjecture, I must to say that my aim is to give a geometric representation of the proof of this conjecture so that if a special chain of subsets such as, $I_0 \subset I_1 \subset \cdots \subset I_n$ exists in a set Ω, where both ends of the chain include trivial subsets, and if the conjecture be true for at least one arbitrary member of this chain, then it will be true for all the other members of the chain.

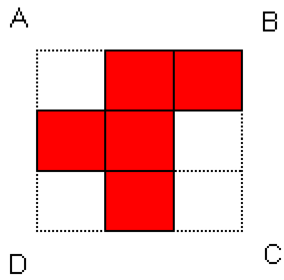
1. Introduction

A polyomino is a set of unit squares connected to each other edge-to-edge or an edge - connected union of cells in the planar square lattice. Polyominoes were studied for the first time by Solomon W. Golomb (1953) and the name (polyominoes) was invented by him. His publication on them, attracted others, and the ideas relating to them began to appear in the Scientific American journal. Several questions about them remained unsolved, leaving challenging problems for combination geometry. In this paper, we make use of their characters to solve Goldbach's conjecture. Since Mathematical Induction is required for the proof, the numerical structure must be compatible with the structure of the set of polyominoes in such a way that the existing definitions in the numerical structure represent properties within the range of the polyominoes. In addition, an algebraic structure must be defined on the polyominoes.

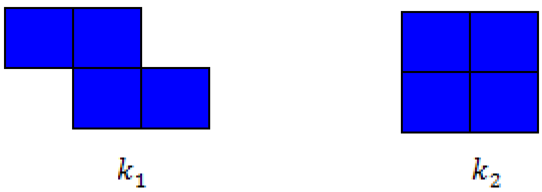
2. Analytical approach

For writing the equation of a polyomino, some definitions are necessary:

Definition 2.1. If P be a polyomino then we call the rectangular created by the extension of any four outer edges of P cells (in any direction) as a convex hull of P, namely, ABCD is the convex hull of the polyomino which shows with the red squares.

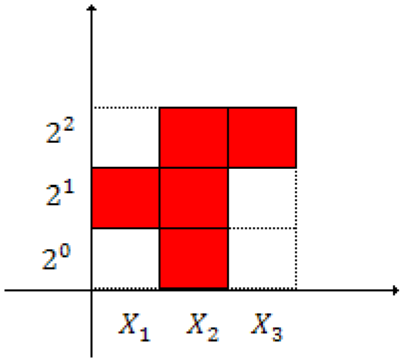


Definition 2.2. For any polyomino like Ω , the squares number of Ω is called the cardinal of Ω and will be shown by $|\Omega|$. Also if there exists a pattern structure (formation) of Ω like Ω_1 where in this new formation, the convex hull of Ω_1 coincides with the Ω_1 itself, and squares of the new arrangement has a rectangular form, then we call Ω a convex polyomino. Like k_1 that can be arranged as k_2:



Any polyomino that isn't convex is called a concavo polyomino.

Now for writing the equation of the above polyomino (ABCD), we put its convex hull in the coordinate axis plane in the manner that the D sits on the origin, DC on the X axis and DA on the Y axis. We can see that in its convex hull, the squares that belong to the polyomino are full and the others which belong to the convex hull are empty. We take any full square equal to 1 and any empty square equal to 0 and then we can get each column of the convex hull as a representation of a number in the base of 2 .Then the first row of the convex hull can show the 2^0, the second row 2^1 and the third 2^2 etc. Thus for ABCD, the first column has only one full square at 2^1 position so this column will represent 2, the second column has full squares at 2^0, 2^1 and 2^2 position. Then this column is a representation of $2^0+2^1+2^2=6$ and also, the third column has only one full square at 2^2 .So this reveals 4. Furthermore we can show the first column by X_1 , the second column by X_2 and the third column by X_3 etc. Now if we call this polyomino by Ω , then we can introduce its equation by $\Omega = 2X_1 + 7X_2 + 4X_3$ .The number of variables shows the column of squares and writing each variable coefficient in the binary expansion will tell us which square is full and which one is empty. We can shortly demonstrate this process by the following picture:



Similarly, we can write any polyomino equation by this method. Now, we try putting an algebraic system on polyominoes.

Supposing that A and B are polyominoes, we define the A+B as a new polyomino like C that satisfies the sum condition in Z_2 . If two polyominoes do not have overlapping sections, then we show the assembly of them as a one polyomino which will be a representation of their sum but if they overlap, then the full and empty squares sum like 1 and 0 in Z_2 . In an overlapping situation, each square must completely wear another square and the partial wearing of a square will not be allowed. Namely, we try to sum two polyomino like $P_1 = 15X_1$ and $P_2 = 13X_1$. As their equation says, we will have a one-column polyomino because P_1 and P_2 have only one column on X_1 .Thus we must sum their coefficients according to their overlapping squares, which show their places by a power of 2 as follow:

$$15 + 13 = (2^3 + 2^2 + 2^1 + 2^0) + (2^3 + 2^2 + 2^0) =$$

$$(2^3 + 2^3) + (2^2 + 2^2) + (2^0 + 2^0) + 2^1 = 0 + 0 + 0 + 2 = 2$$

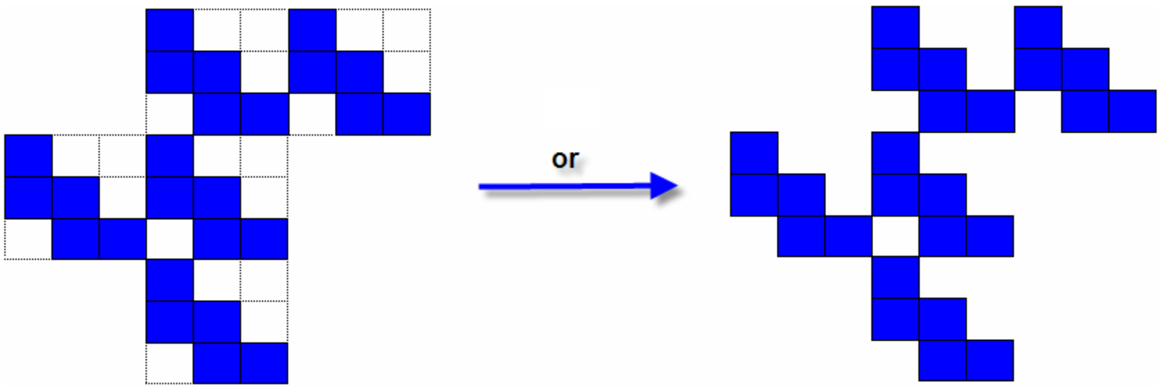
Then $P_1 + P_2 = 15X_1 + 13X_1 = 2X_1$

(According to this new sum method, we write a binary expansion of two numbers first of all, then we take 0, the sum of the same powers that exist in each number expansion and then write the reminders and add them.)

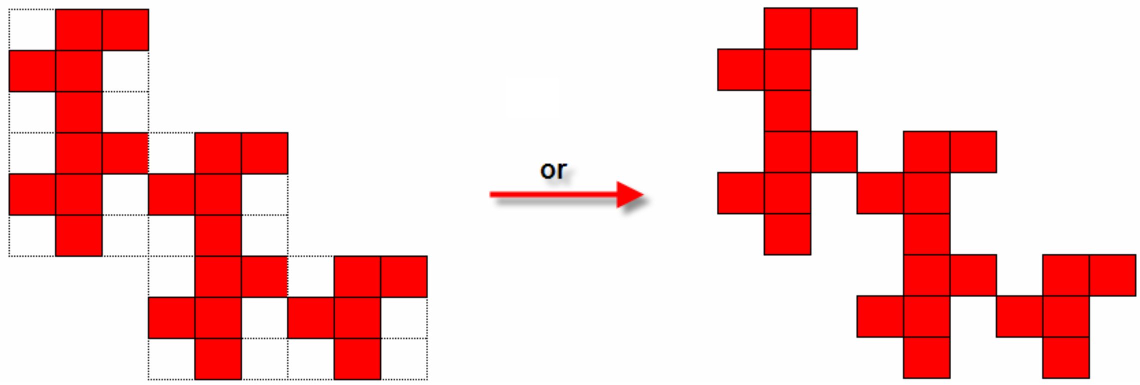
The multiplication of $\Omega = 2X_1 + 7X_2 + 4X_3$ and $\mu = 6X_1 + 3X_2 + X_3$ will be defined by $\Omega\mu = \Omega(\mu)$ which means that instead of each square of Ω ,we will put the convex hull of μ .

We show the $\Omega\mu$ and $\mu\Omega$ formally as follow:

For $\Omega\mu$:



And for $\mu\Omega$:

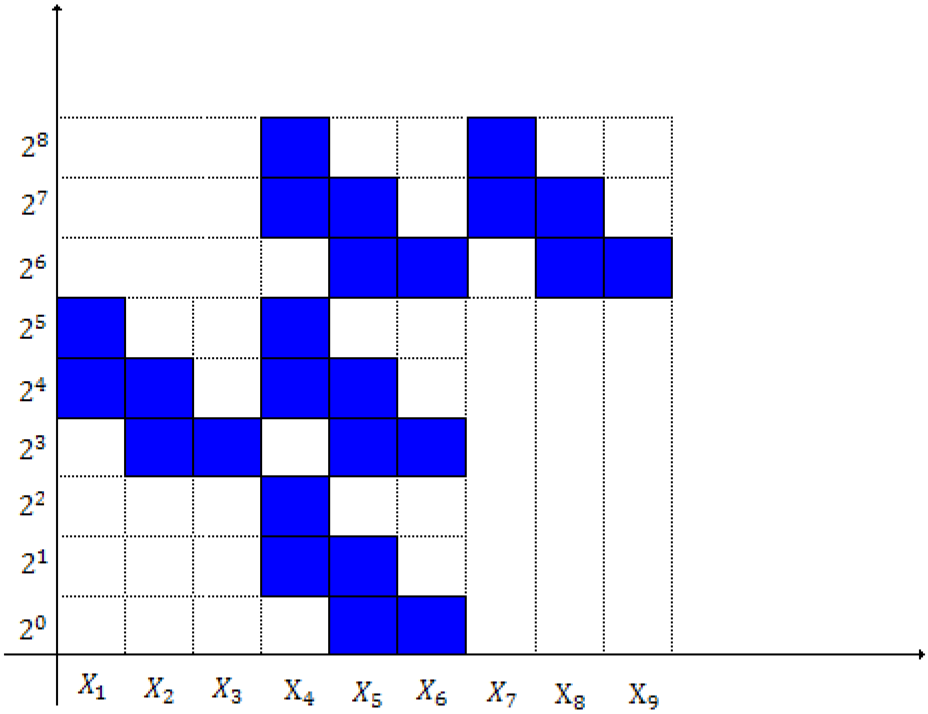


We have $\Omega\mu \neq \mu\Omega$. Another important point that exists in any multiplication like $\Omega\mu$, is that the μ (or its convex hull) is placed in the womb of Ω. Also we could say that $\Omega\mu$ is tiled with μ (or its convex hull) as well as tiled with squares. Therefore if we have $P = p_1 p_2 p_3 \dots p_k$ we can say that each p_i is inside of p_{i-1} (telescoping property) and also a P tiled with each p_i (or its convex hull). The explanations above contain important points which will be used in the final part for the proof of the conjecture and the same method will be used. By various juxtapositions of squares, the convex hull's shape could be a square or rectangle.

If $\phi = \Omega\mu$, then:

$\phi = 48X_1 + 24X_2 + 8X_3 + 438X_4 + 219X_5 + 73X_6 + 822X_7 + 192X_8 + 64X_9$

The following picture shows the shape of $\phi = \Omega\mu$:



We say that Ω is divisible to μ if there is a polyomino like ψ so that $\Omega = \mu\psi$,then we can write $\Omega/_{\mu} = \psi$.

Note: without loss of generality, the index and power of each variable can be replaced with each other and following this method, the equation $\Omega = 2X_1 + 7X_2 + 4X_3$ can reform to a new equation like $\Omega = 2X^1 + 7X^2 + 4X^3$. Thus we denoted a polyomino by a polynomial and also, we can extend the equation to $\Omega(x) = 2x^1 + 7x^2 + 4x^3$. This fact describes a correspondence between polyominoes and polynomials.

Now, it is time to perform the application of what has been designed.

3. Application of Polyominoes

Goldbach conjecture: Any even number can be written as the sum of two prime numbers.

To simplifying our expressions, we will put “2n is a GB” instead of “we can write 2n as the sum of two prime numbers” and also "CH" instead of "convex hull" and finally a polyomino with n squares will be shown by P_n .

Proof: Now, we investigation the conjecture in the set of polyominoes which as a result, the Goldbach conjecture will change to the new proposition. On the other hand, we must prove that "any P_{2n} can be written as the sum of two concavo polyomino " or"P_{2n} is a GB". Repeating this procedure and modifying the latter proposition by preserving the main properties and concepts, we will have the following proposition:

"If for some CH the P_{2n} be a GB, then for all of the squares, P_{2n} is a GB"

The sufficient tools to prove the latest proposition are now available.

Suppose that P_{2n} is a GB and we try to prove that $P_{2(n+1)}$ is a GB. If n+1 is a prime, then nothing else remains to be proven. So we assume that n+1 is a composite. That is: $2(n+1) = 2q_1q_2 \dots q_k$ where each q_i (i=1, 2,…, k) is a factor of n+1. Then in the polyominoes environment, we have:

$$P_{2(n+1)} = P_{2q_1q_2\dots q_k} = P_2P_{q_1}P_{q_2}\dots P_{q_k} = P_2(P_{q_1}(P_{q_2}(\dots(P_{q_{k-1}}(CH(P_{q_k}))) \dots) = P_2P_{q_1}P_{q_2}\dots P_{q_{k-1}}\left(CH(P_{q_k})\right) = P_{2q_1q_2\dots q_{k-1}}\left(CH(P_{q_k})\right)$$

On the other hand, we use of $P_{q_k}'s$ convex hull and according to the assumption (because $2q_1q_2 \dots q_{k-1} < 2n$) the $P_{2q_1q_2\dots q_{k-1}}$ it can be written as the sum of two concavo polyominoes whose squares have been replaced by the convex hull of P_{q_k}(we can consider P_{q_k} 's squares formation in a manner that its CH is a bigger square. Then we will have a more tangible pattern). Thus, all of the squares being in $P_{2(n+1)}$ can be written as the sum of two concavo polyomino or the Goldbach Conjecture is true. By a similar process, we can prove that any even number can be written as the minus of two prime numbers.

Conclusion.

After Golomb's work, these shapes have not been seriously considered, but the simplicity and nearness of them to numeration is always attractive. The set of polyominoes have a patterned structure which includes properties of the Natural numbers and an analytic method could help to develop it as was used for Euclidean geometry. These creatures are linkable to fuzzy sets, finite geometry and algebra. So these characteristics will help them enter technological applications. The above statements are a model that could make the Goldbach conjecture touchable.

REFRENCES

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