Sharp concentration of the rainbow connection of random graphs

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Abstract

An edge-colored graph G is rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Similarly, a vertex-colored graph G is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. We prove that both $rc(G)$ and $rvc(G)$ have sharp concentration in classical random graph model $G(n,p)$.

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1. Introduction

We follow the terminology and notation of [4] in this letter. A natural and interesting connectivity measure of a graph was recently introduced in [6] and has attracted many attention of researchers. An edge-colored graph G is called rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. Hence, if a graph is rainbow edge-connected, then it must also be connected. Also notice that any connected graph has a trivial edge coloring that makes it rainbow edge-connected. The rainbow connection of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow edge-connected.

If G has n vertices then $rc(G) \leq n-1$, since one can color the edges of a given spanning tree of G with distinct colors, and color the remaining edges with one of the already used colors. Obviously, $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree. An easy observation gives $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of G. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been addressed in the work [5, 10, 11], which indicate that $rc(G)$ is upper bounded by the reciprocal of $\delta(G)$ up to a multiplicative constant (which we will discuss later). Some related concepts such as rainbow path [9], rainbow tree [8] and rainbow k-connectivity [7] have also been investigated recently.

The authors in [10] introduce a vertex coloring edition. A vertex-colored graph G is called rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. Denote the rainbow vertex-connection of a connected graph G by $rvc(G)$, which is defined as the smallest number of colors that are needed in order to make G rainbow vertex-connected. It is clear that $rvcG \leq n-2$, and $rvcG = 0$ if and only if G is complete. Similarly, we have $rvcG \geq diam(G) - 1$.

Note that $rc(G)$ and $rvc(G)$ are both monotonic property in the sense that if we add an edge to G we cannot increase its rainbow edge/vertex-connection. Therefore, it is desirable to study the random graph setting [3]. Motivating this idea, in this paper we consider the rainbow edge/vertex-connection in Erdős-Rényi random graph model $G(n,p)$ with n vertices and edge probability $p \in [0,1]$. Based on some known bounds of diameter and degree of $G(n,p)$, we establish the following concentration results:

Theorem 1. *Suppose that* $\omega = \omega(n) \to -\infty$ *and* $c = c(n) \to 0$. *Let* $d = d(n) \geq 2$ *be a natural number and* $0 < p = p(n) < 1$. *If*

$$np = \ln n + \frac{20n \ln\ln n}{d+1} - \omega, \tag{1}$$

$$p^d n^{d-1} = \ln\left(\frac{n^2}{c}\right) \tag{2}$$

and

$$\frac{pn}{(\ln n)^3} \to \infty \tag{3}$$

hold, then $rc(G(n,p)) = d$ *almost surely as* $n \to \infty$.

Theorem 2. *Suppose that* $\omega = \omega(n) \to -\infty$ *and* $c = c(n) \to 0$. *Let* $d = d(n) \geq 2$ *be a* *natural number and* $0 < p = p(n) < 1$. *If*

$$np = \ln n + \frac{11n \ln\ln n}{d} - \omega, \tag{4}$$

$$p^d n^{d-1} = \ln\left(\frac{n^2}{c}\right) \tag{5}$$

and

$$\frac{pn}{(\ln n)^3} \to \infty \tag{6}$$

hold, then $rvc(G(n,p)) = d-1$ *almost surely as* $n \to \infty$.

2. Proof of Theorem 1 and 2

In this section, we will first prove Theorem 1 and then Theorem 2 can be derived similarly.

Let $\delta(G)$ be the minimum degree of a graph G. The following lemma gives upper bounds of rainbow edge/vertex-connection.

Lemma 1.([10]) *A connected graph* G *with* n *vertices has* $rc(G) < 20n/\delta(G)$ *and* $rvc(G) < 11n/\delta(G)$.

Proof of Theorem 1. By Lemma 1 and the comments in the Section 1, we have

$$diam(G(n,p)) \leq rc(G(n,p)) < 20n/\delta(G(n,p)) \tag{7}$$

if $G(n,p)$ is connected.

To get the concentration result, we need to estimate the diameter and minimum degree of random graph $G(n,p)$. It follows from the assumptions (2) and (3) that $diam(G(n,p)) = d$ almost surely (see [2] or [3] pp.259). By the assumption (1), we get $\delta(G(n,p)) = 20n/(d+1)$ (see [1] or [3] pp.65). Now we almost conclude our proof by (7).

There are nevertheless two things remain to check: (i) The assumptions (1)-(3) are reasonable, that is, there really exist such p and d. (ii) $G(n,p)$ is almost surely connected.

Define $c = c(n) \to 0$ by the equation

$$\ln\ln\left(\frac{n^2}{c}\right) = (\ln n)\cdot \ln\ln n \tag{8}$$

and let $\omega(n) \to -\infty$ sufficiently slowly. By the assumption (1), we define a function of d

$$f(d) := (np)^d = \left(\ln n + \frac{20n \ln\ln n}{d+1} - \omega\right)^d. \tag{9}$$

Take $d = \ln n$, and we obtain

$$\begin{aligned}
\ln f(d) &= (\ln n) \cdot \ln\Big(\ln n + \frac{20n \ln\ln n}{1+\ln n} - \omega\Big) \\
&\geq (\ln n) \cdot \ln\Big(\frac{n \ln\ln n}{\ln n}\Big) \\
&\geq \ln n + (\ln n) \cdot \ln\ln n \\
&= \ln\Big(n \cdot \ln\Big(\frac{n^2}{c}\Big)\Big) \qquad (10)
\end{aligned}$$

where the last equality holds by the definition (8).

Take $d = \ln\ln n$, and we have

$$\begin{aligned}
\ln f(d) &= (\ln\ln n) \cdot \ln(\ln n + 20n - \omega) \\
&\leq (\ln\ln n) \cdot \ln(21n) \\
&\leq \ln n + (\ln n) \cdot \ln\ln n \\
&= \ln\Big(n \cdot \ln\Big(\frac{n^2}{c}\Big)\Big) \qquad (11)
\end{aligned}$$

where the last equality holds by the definition (8).

From (10), (11) and the fact that $f(d)$ is continuous, we derive that there exists some $d \in [\ln\ln n, \ln n]$ such that $\ln f(d) = \ln(n \ln(n^2/c))$ holds. Consequently, the assumption (2) holds. For such d, by (9), we have

$$np = \Omega\Big(\frac{n \ln\ln n}{\ln n}\Big), \qquad (12)$$

which clearly satisfies the assumption (3), and $G(n,p)$ is connected almost surely (c.f. [3] pp.164).

Hence, both (i) and (ii) have been checked and the proof is finally completed. □

Proof of Theorem 2. It can be proved similarly by noting the fact

$$diam(G(n,p)) - 1 \leq rvc(G(n,p)) < 11n/\delta(G(n,p)). \qquad (13)$$

We leave the details to the interested readers.□

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