FLORENTIN SMARANDACHE

PROBLEMS WITH AND WITHOUT … PROBLEMS!

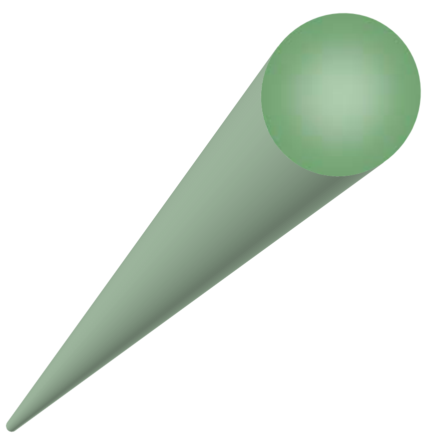
Translated from French

Kappa & Omega

2011

FLORENTIN SMARANDACHE

PROBLEMS WITH
AND
WITHOUT …
PROBLEMS!



Translated from French into English

Original title:
"Problèmes avec et sans… problèmes!", par Florentin Smarandache, Edition Française, Traduit du Roumain par l'auteur et Sophie Mugnier, 1983,
SOCIETE MAROCAINE D'IMPRESSION ET D'EQUIPEMENT S.A.R.L.
SOMIPRESS, Fès, Marocco,
Siege Sociale: 72 Boulevard Hassan II, Tel: 238-68 – FES (v.n.).

This book can be ordered in a paper bound reprint from:

Books on Demand

ProQuest Information & Learning

(University of Microfilm International)

300 N. Zeeb Road

P.O. Box 1346, Ann Arbor

MI 48106-1346, USA

Tel.: 1-800-521-0600 (Customer Service)

http://wwwlib.umi.com/bod/basic

6744 W. Northview Ave.
Glendale, AZ 85303, USA

Peer-reviewers:

Prof. Ion Pătrașcu, Frații Buzești College, Craiova, Romania.

Prof. Gabriel Tica, Bailesti College, Bailesti, Jud. Dolj, Romania.
Dr. Fu Yuhua, 13-603, Liufangbeili, Liufang Street, Chaoyang district
Beijing, 100028, P. R. China.

ISBN-13: 9781599731421

Printed in the United States of America

PRE//FACE

This book is addressed to College honor students, researchers, and professors.

It contains 136 original problems published by the author in various scientific journals around the world.

The problems could be used to preparing for courses, exams, and Olympiads in mathematics.

Many of these have a generalized form.

For each problem we provide a detailed solution.

I was a *professeur coopérant* between 1982-1984, teaching mathematics in French language at *Lycée Sidi EL Hassan Lyoussi* in Sefrou, Province de Fès, Morocco.

I used many of these problems for selecting and training, together with other Moroccan professors, in Rabat city, of the Moroccan student team for the **International Olympiad of Mathematics** in Paris, France, 1983.

The Author

Contents

AMUSING PROBLEMS

1.1.

Four teams of soccer: A, B, C and D participated in a tournament. The final ranking has not been established, the stars will indicate the figures undecided (classification, teams, the number of games played. Victories, null games, defeats, points marked, points missed):

1.	A	3	***	5 - 2	6
2.	B	*	**2	3 – 3	*
3.	C	3	***	4 - *	*
4.	D	3	***	1 – 4	*

The teams have been separated by criteria known to the soccer rules and the ranking the same.

a) Complete the grid.

b) Find the result of all disputed games.

Solution

a) The team A played 3 games, so A played against B. The same for C and D. Or B B has to play 3 games. A has 6 points in 3 games, then A won all the games, therefore A has 3 victories, zero defeats. B has 2 defeats, therefore B is on the second place. It results that the third game of B is a victory, because if it would be a null game, then C and D would have 6x2-(6+1) = 5 points, which will mean that at least one of them would have more points than B. Therefore B has 2 points.

C and D have together 6x2-(6+2) = 4 points. Then C and D have both 2 points, because otherwise it will result that at least one of them, C or D, will have more than B.

Therefore C has a victory, zero null games, and two defeats. The same for D. (C and D can not get the 2 points from two null games, because A and B did not have any null matches), C got (5+3+4+1)-(2+3+4) = 4 goals. The complete ranking is:

1.	A	3	3 0 0	5 – 2	6
2.	B	3	1 0 2	3 – 3	2
3.	C	3	1 0 2	4 – 4	2
4.	D	3	1 0 2	1 – 4	2

We know that in a ranking, the sum of marked goals by all the teams is equal to the sum of the goals received by all the teams.

b) We determine the scenario of the games.

A has 3 victories, then A – B = 1, A – C = 1, A – D = 1

B and C have the same number of points, the same difference between the goals marked and the goals received, the same number of victories, but B has a place superior than C, it results that B – C =1, from where B – D = 2, then C – D = 1.

D marked only one goal and received 2 goals; the difference is 5 – 2 = 3. Then A took the results: 1 – 0, 1 – 0, 3 – 2 or 1 – 0, 2 -1, 2 – 1. Because A – D = 1 and D marked its own goal against B, it results that: A – D = 1 – 0, from which it results that C – D = 3 – 0.

The situation is:

1.	A	2	2 0 0	4 – 2	4
2.	B	2	1 0 1	3 – 2	2
3.	C	2	0 0 2	1 – 4	0

with the scenario anterior.

A can have the results: 1 – 0, 3- 2, 2 – 1, 2 – 1. We observe that we cannot have the result A – B = 3 – 2, because B marked only 2 goals. Then, it results the alternative 2 – 1, 2 – 1, from which A – B = 2 -1, A – C = 2 -1 and we obtain B – C = 2 – 0. The exact results are:

A –B = 2 – 1, A – C = 2 – 1, A – D = 1 – 0, B – C = 2 – 0, B – D = 0 -1, C – D = 3 – 0.

1.2.

At the end of a soccer tournament of the teams A, B, C, D, the classment is the following:

1.	A	3	2 0 1	2 – 1	4
2.	B	3	2 0 1	2 – 1	4
3.	C	3	1 1 1	4 – 4	3
4.	D	3	0 1 2	3 – 5	1

The criteria for the teams rating was:

a) the number of accumulated points
b) the difference between the marked goals and the received goals
c) the number of victories
d) the direct victories against a team.

Find all the games results, knowing that for a victory a team gains 2 points, for a tight game it gets 1 point, and for a defeat 0 (zero) points.
The first column represents the team order, second column the team, third column the number of played games, fourth column the number of victories, fifth column the number of tight games, sixth column thenumber of defeats, seventh column the numbers of marked goals, eighth column the number of received goals, and last column the number of points.

Solution

Firstly we determine the exact estimation of the played games.

The teams A and B have the same number of points, the same difference between the marked goals and the goals received, the same number of victories, but A is on the first place while B is on the second place. From here A – B = 1 (that is, A gained a game). B has two victories and one defeat, then B – C = 1 and B – D = 1. Then C – D = X (where X means an equal game). A has a defeat, then A – C = 2. The exact estimations are:

A – B = 1, A – C = 2, A- D = 1, B – C = 1, B – D = 1, C – D = X

Now we determine the results.

Because A has 2 victories and marked only 2 goals, then its victories have been obtained as 1 – 0, 1 – 0. We have then A – B = 1 – 0 and A – D = 1 – 0 , from where A – C = 0 – 1.

Similarly for B we have A – D = 1 – 0, from which A – C = 0, and therefore C – D = 3 – 3. The exact results are:

A – B = 1 – 0, A – C = 0 – 1, A – D = 1 – 0, B – C = 1 – 0, B – D = 1 – 0, C – D = 3 – 3.

1.3.

In the elaboration of a soccer ranking which follows, were made four errors, the order of the teams remaining the same.

1.	A	3	2 1 0	1 – 0	5
2.	B	2	1 0 1	5 – 4	2
3	C	3	1 0 2	6 – 6	2
4.	D	3	0 2 1	2 – 5	2

a) What are the errors?

b) Correcting these errors, find the results of all the games played.

Solution

a) Because A, C, D played 3 games its results that B also played 3 games (it is not possible to have 2 games played by each team, because it would be more than 4 errors in the ranking). The third game played by B cannot be a victory, because it would have in total $5 + 4 + 2 + 2 = 13 \neq 12$ points. (it is not possible to make other modifications on the points of A, C, or D, because we would get more than 4 errors).

In the same way, the third game of B cannot be a defeat. Therefore B has a null game (the third error). A has 2 victories and one single marked goal. The number of marked goals is $1 + 5 + 6 + 2 = 14 \neq 15 = 0 + 4 + 6 + 5$ which is the number of received goals by all the teams. From which it results that A has marked 2 goals (the fourth error). (It is not possible to make modifications on the received goals for A or for others for the same reason.

b) The correct ranking is

1.	A	3	2 1 0	2 – 0	5
2.	B	3	1 1 1	5 – 4	3
3	C	3	1 0 2	6 – 6	2
4.	D	3	0 2 1	2 – 5	2

1) It is necessary to establish the exact forecasts.

D has 2 null games and A and B each have a null game.

Then A – D = X, B – D = X. The team A has also 2 victories. Then A – B = 1, A – C = 1. From B – D = X and A – B = 1 it results that B – C = 1, because B has one victory. In the same way C – D = 1.

The exact forecasts are:

A – B = 1, A – C = 1, A – D = X, B – C = 1, B – D = X, C – D = 1.

2) Now, it is sufficient to establish the exact results.

A has 2 victories, and 2 marked goals. Then A – B = 1 – 0, A – C = 1 – 0.

Because A did not receive any goals, we have A – D = 0 – 0.

Excluding the team A from the ranking (with all its results), we obtain the following sub-ranking:

2.	B	2	1 1 0	5 – 3	3
3	C	2	1 0 1	6 – 5	2

4.	D	2	0 1 1	2 – 5	1

with the known forecasts: B – C = 1, B – D = X, C – D = 1
B has one victory and no defeats, and the difference of goals is 5 – 3 = 2.
From which: B – C = 2 -0 or 3 -1, or 4 -2, or 5 – 3.
C has one victory and the difference of goals 6 – 5 = 1; but because B has won over C by 2 goals, C wins therefore by 3 goals against D. From which C – D = 3 – 0 or 4 -1.

If we have C – D = 3 – 0, then we have B – C = 5 – 3, and B – D = 0 – 0. But this means that D has zero marked goals. This is a contradiction. Therefore C – D = 4 – 1. And we have : B – C = 4 – 2, B – D = 1 – 1.
These last results verify the ranking
The results are:
A – B = 1 – 0, A – C = 1 – 0, A – D = 0 – 0, B – C = 4 – 2, B – D = 1 – 1, C – D = 4 – 1.

The problem has been uniquely resolved.

The problem is completely proved.

1.4.

For the preliminaries of the world soccer championship are disputed, back and forth, the games of a group of 5 teams from which will be qualifies the first 2 teams.

Determine the minimum number of points for which a team can qualify, and also the results that will enable the qualification.

Generalize the problem for the case of a group of n teams, from which must qualify the first m ($1 \le m \le n$).

Solution

We'll resolve the exercise, directly, for the general case, the initial case will result as a particularization.

In the group of n teams will be played $2\left[(n-1)+(n-2)+...+(2+1)\right]=n(n-1)$ games. The total number of points is $2n(n-1)$. If $m=1$, the minimum number of points will be $2n(n-1):2=2(n-1)$ points (all teams have the same number of points, but that that will have the largest difference between the goals marked and the goals received will qualify. If one team has less than $2(n-1)$ points, then there will be another team which will have more that $2(n-1)$, because the total number of points is equal to $2n(n-1)$. If $m=n$, obvious, the minimum number of points is zero.

In the case that $1<m<n$. The qualified team with the minimum number of points will be the m th. When it is the minimum of points it means that the $m-1$ first teams would have obtained the maximum possible points. Then the h th team, $1 \le h \le m-1$, would have $4(n-i)$ points. The $m-1$ teams would have $4(n-1)+4(n-2)+...+4(n-m+1)=2(m-1)(2n-m)$ points. From the total number of points we remove the points of the first $m-1$ teams and we find $2(n-m)(n-m+1)$, which represent the $(n-m+1)$ points of the remaining teams. Then $\frac{2(n-m)(n-m+1)}{n-m+1}=2(n-m)$, which is the minimum number of points for a team to qualify.

1.5.

At a forecasting game regarding 13 soccer games, a person plays utilizing m doubles and n triples, $0 \le m+n \le 13$, $n,m \in N$.

a) In the case that he'll obtain a variant with 13 exact results, implicitly how many variants of 12 and of 11 exact results he'll obtain?

b) Also, if he gets 12 exact results, implicitly how many variants of 11 and of 10 exact results will he obtain?

Solution

There are 13 games and for each there are three possibilities: 1, X or 2 (that is, regarding the first team, victory, null match or defeat). There are 12 possible variants (more than 1000000). Having m the doubles and n the triples, it results that we have $13-m-n$ solitaries, which means the games for which we give only one answer. There are $2^m \cdot 2^n$ variants in total.

a) We obtain $m+2n$ variant with 12 exact results.

If $m \ge 2$ and $n \ge 2$, we have $C_m^2 + 4C_n^2 + 2mn$ variants with 11 exact results; if $m \ge 2$ and $n < 2$ we have $C_m^2 + 2mn$; if $m < 2$, $n < 2$ we have $2mn$; if $m < 2$, $n \ge 2$ we have $4C_n^2 + 2mn$

b) The case: when the solitaries are false. Then it results:

$$\begin{cases} m+2n \text{ variants with 11 exact results} \\ C_m^2 + 4C_n^2 + 2mn \text{ variants with 10 exact results, or } m \ge 2,\ n \ge 2 \\ 4C_n^2 + 2mn, \text{ if } m < 2,\ n \ge 2, \text{ variants have 10 exact results} \\ C_m^2 + 2mn, \text{ if } m \ge 2 \text{ and } n < 2 \text{ variants with 10 exact results} \\ 2mn, \text{ if } m < 2,\ n < 2 \text{ variants with 10 exact results.} \end{cases}$$

In the case that the double is false, we have

$(m-1)+2n$ variants with 10 exact results

$C_m^2 - 1 + 4C_n^2 + 2(m-1)n$ variants with 10 exact results, if $m \ge 3$, $n \ge 2$

$4C_n^2 + 2(m-1)n$ variants with 10 exact results if $m < 3$, $n \ge 2$

$C_m^2 - 1 + 2(m-1)n$ variants with 10 exact results if $m \ge 3$, $n < 2$

$2(m-1)n$ variants with 10 exact results if $m < 3$, $n < 2$.

1.6.

At a tour of chess participated 10 players $A_1, A_2, ..., A_{10}$ - each chess players played with each of the other players one game. For each victory one point is gained, for each null game half of a point, and for each defeat zero points.

At the end of the tour, the ranking was

1.	A_1	9.5 points
2.	A_2	9 points
3.	A_3	6 points
4-5	A_4	5 points
4-5	A_5	5 points
6.	A_6	4 points
7-9.	A_7	2 points
7-9.	A_8	2 points
7-9.	A_9	2 points
10.	A_{10}	1 point

Show that in this ranking there are at least three errors.

<u>Solution</u>

The first error: A_1 cannot accumulate more than the maximum of 9 points, because only 9 games were played, therefore there are no 9.5 points.

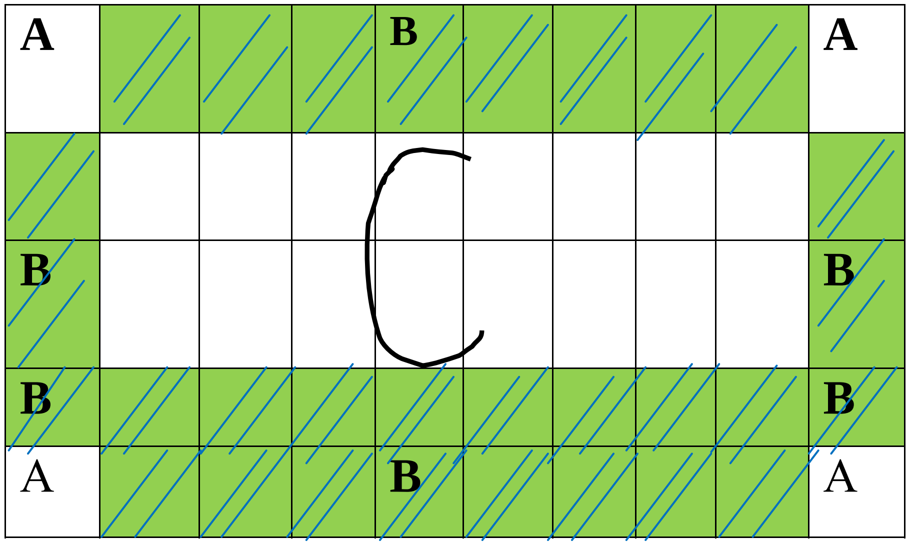
The second error: A_2 , situated on the second place and the ranking, cannot accumulate more than 8 points, not 9 points, because he cannot gain more than a maximum of 8 points (the 9^{th} game, played against A_1 was lost; against A_1 the player A_2 could not have a null match because it would result that A_2 should occupy the place 1 – 2, not 2).

The third error: in this tour there were played 9 + 8 + 7 +…+ 1 = 45 games, therefore the total number of points of the ranking must be 45, because

$9.5 + 9 + 6 + 2{,}5 + 4 + 3{,}2 + 1 = 45.5 \neq 45.$

1.7.

Given a grid of crosswords (of n lines, m columns and p black boxes), such that there are not two black cases that have a common side.



a) Prove that the number of the total words (horizontal and vertical) – we called word a box that contains only one letter.

b) Find the difference between the number of horizontal words and the number of vertical words.

Solution

a) We show that $N = n + m + CNB + 2CNC$, where
N = the number of the total words of the grille
CNB = the number of the black boxes in the B boxes
CNC = the number of the black boxes from the C boxes
We consider the grid divided in 3 zones.
1^0 the four corners of the grid (the A zone)
2^0 the border of the grid minus the four corners
3^0 the interior part of the grid (the C zone).
We assume that the grid at the beginning does not have any black boxes. Then, there are $n+m$ words.

- If we put a black box in the zone A, the number of the total words remains the same. (Then the number of the black boxes from the zone A does not present any importance)
- If we put a black box in the zone B, for example on the line 1 and column j, $1 < j < m$, the number of words being a unit, [because on the line 1 are formed now two words (before there was only one word), and on column j there is also only one word]. The situation is similar if we put a black box on the column 1 and line i,

$1 < i < n$, (we can reverse the grid: the horizontal to be the vertical and vice versa). Then, for each black box from the zone B we add a word to the total number of the words of the grid.

- If we place a black box in the zone C, for example on the line i, $1 < i < n$, and the column j, $1 < j < m$, then the number of the words formed by two unites: as on the line i, and on the column j are at this time two words, contrary to the previous situation where it was a single word in each. Therefore, for each black case from the zone C we add two words to the number of the words of the grid.

b) We divide the zone B in two parts:

- the zone BO = the horizontal part of B (the lines 1 and n)
- the zone BV = the vertical part of B (the columns 1 and m)

Then: $NO - NV = n - m + CNBO - CNBV$, where

NO = the number of the horizontal words
NV = the number of the vertical words
$CNBO$ = the number of the black cases of BO
$CNBV$ = the number of the black cases of BV

The proof of this proposition feats the precedent one, and we use the following result:

- If there is no black boxes on the zone A, the difference $NO - NV$ is equal to $n - m$
- If there is a black box on the zone A, the difference remains the same
- The same for the zone C
- If there is a black box on the zone BO, then the difference will be $n - m + 1$, and if the black box will be on the zone BV, then the difference will be $n - m - 1$.

ARITHMETIC

2.8.

Determine the last digit of the numbers of the sequence of Fermat:

$F_n = 2^{2^n} + 1$, with $n \in N$.

Solution

For $n = 0$ we have $F_0 = 3$, and for $n = 1$ we find $F_1 = 5$.

For $n \geq 2$ it results that $F_n = 2^{2^n} + 1 = 2^{4 \cdot 2^{n-2}} + 1 = 16^{2^{n-2}} + 1 = 16^K + 1$ which contains as last digit, 6+1=7, because the power of 16 ends in 6.

2.9.

Let p the product of the first n prime numbers.

Determine the set $F = \{\alpha \in N \mid \alpha ! = Mp\}$, Mp being the multiple of p.

Solution

Because $\alpha ! = Mp$, $1 \leq i \leq n$, we must have $\alpha \geq P_i$. Therefore $\alpha \geq \max_i \{P_i\} = P_n$.

$P_n ! = 1 \cdot P_1 \cdot P_2 \cdot 4 \cdot P_3 \cdots P_{n-1} \cdots P_n$, from where $P_n ! = Mp$.

P_n is the smallest number which has this property, because if there exists $\alpha' < P_n$ then $\alpha' ! \neq Mp$. If $\beta > P_n$ then, of course $\beta ! = Mp$. And $F = \{P_n, P_n + 1, P_n + 2, ...\}$.

2.10.

Find the smallest natural number such that its factorials are multiples of each of the numbers 1970, 1980, 1990, and 2000.

Solution

The greatest prime number which divides one of the numbers from the above is 199.

Let $\alpha \in N$ the number which we're seeking. Then $\alpha ! = M1990$, i.e. multiple of 1990, from where $\alpha = M199$..

Then $\alpha \geq 199$. We take $\alpha = 199$. $199! = M10$ because $10 < 199$. We also have $199! = M197$. Because $(10,197) = 1$, it results that $199! = M1970$..

$199! = M1990$.

$199! = M36$ and $199! = M55$, but $(36,55) = 1$; from here $199! = M36 \bullet 55 = M1980$.

$199! = M16$ and $199! = M125$ and $(166,125) = 1$; then $199! = M2000$.

We suppose by absurd that α is not the smallest. Then, it exists $\alpha' \leq 199$ such that $\alpha' ! = M199$; contradiction.

2.11.

Let A and B natural numbers. We consider $M_1 = A + B,\ M_2 = A - B,\ M_3 = A \bullet B$. We note X_m the numbers formed only by the last m digits of X.

a) Show that to find the last m digits of M_1 it is sufficient to find the last m digits of the sum $A_m + B_m$. The same question for M_2 and M_3.

b) Generalization.

c) What it can be said about the last m digits of A^B?

Solution

a) We can write $A = M_{10^m} + A_m$ and the same $B = M_{10^m} + B_m$. Then

$$M_1 = A + B = M_{10^m} + \left(A_m + B_m\right)$$

The same:

$$M_2 = A - B = M_{10^m} + \left(A_m - B_m\right)$$

$$M_3 = A \bullet B = M_{10^m} + \left(A_m \bullet B_m\right)$$

b) Generalization:

If $E\left(A_1, A_2, ..., A_n\right)$ is an arithmetic expression in which we have only the operations +, -, and if $A_1, A_2, ..., A_n$ are natural numbers, then

$$E_m\left(A_1, A_2, ..., A_n\right) = E\left(A_{1,m}, ..., A_{n,m}\right)$$

where $A_{i,m}$ represent the last m digits of A_i.

The proof results from a).

c) A^B is a repeated multiplication. Therefore $\left(A^B\right)_m = A_m^B$.

2.12.

Knowing what h hour and m minutes, $1 \le h \le 12$, $0 \le m < 60$, find after how long the needles on the clock's face would form an angle α, with $0 \le \alpha < 360°$.

Solution

Firstly we determine the angular speed for each needle on the clock's face.

The big needle executes 360^o in an hour; therefore $V_G = 6° / \text{min}$.

The small needle executes 360^o in 12 hours; therefore $V_S = 0.5° / \text{min}$.

We compute the angle between the needles on the clock's face at the h and m minutes. The big needle would execute $6m^{o}$. The small needle would execute $\left(60h + m\right) \cdot 0.5 = 30h + 0.5m$ degrees.

We note by x (the minutes) the unknown of the problem.

The angle formed by the needles is $|6m-30h-0.5m|=|5.5m-30h|$. (We consider the angles positive, because in the problem it is not specified the direction of the angle).

A) The case in which $|5.5m-30h|\le\alpha$.

a) $5.5m-30h\ge 0 \Leftrightarrow$ the big needle executes a distance (in degrees) greater than the distance executed by the small needle. We have:

$$6x-0.5x=\alpha-(5.5m-30h)\Rightarrow x=\frac{\alpha+30h-5.5m}{5.5}.$$

b) $5.5m-30h<0$, the opposite situation. We have

$$6x-0.5x=\alpha-(5.5m-30h)\Rightarrow x=\frac{\alpha+30h-5.5m}{5.5}.$$

B) The case in which $|5.5m-30h|>\alpha$.

a) $5.5m-30h\ge 0$. We have:

$$6x-0.5x=\alpha+360-(5.5m-30h)\Rightarrow x=\frac{\alpha+360h+30h-5.5m}{5.5}.$$

b) $5.5m-30h<0$. We have:

$$6x-0.5x=|5.5m-30h|-\alpha=5.5m-30h-\alpha\Rightarrow x=\frac{30h-5.5m-\alpha}{5.5}.$$

2.13.

Let $a_1,...,a_{2n+1}$ integers and $b_1,...,b_{2n+1}$ the same numbers but in a different order. Prove that the expression $E=(a_1\pm b_1)\cdots(a_{2n+1}\pm b_{2n+1})$ is an even number, where the signs + and – are arbitrary in each parenthesis. (The generalization of problem A.7, page 105, of D. Gerll and G. Gerard, “Les olympiades internationales de mathématiques”, Hachette, 1976).

Solution

We suppose that the expression E is odd. It resuls that each parenthesis is odd, therefore each parenthesis contains an even number and the other an odd number.

We have then $2n+1$ even numbers. But, if in a parenthesis we find an a_{i_0} even, then there exists another parenthesis where we find a $b_{j_0}=a_{i_0}$, then b_{j_0} is even. And the number of evens is an odd number, which, obviously is different of $2n+1$. This is a contradiction.

2.14.

Resolve the equation: $X-\Phi(X)=24$, knowing that $\Phi(X)$ represents the number of positive numbers, smaller than X and relatively prime in rapport to X.

Solution

Because $\Phi(X) \in N$, It results that $X = 24 + \Phi(X) \in N^*$ and $X \geq 24$. Let $X = P_1^{\alpha_1} \cdots P_s^{\alpha_s}$, $\alpha_i \in N^*$, P_i different prime numbers, $i = \overline{1,s}$.

$\Phi(X) = P_1^{\alpha_1 - 1} \cdots P_s^{\alpha_s - 1} \cdot (P_1 - 1) \cdots (P_s - 1)$, Φ being the Euler's function from the number theory.

$X - \Phi(X) = P_1^{\alpha_1 - 1} \cdots P_s^{\alpha_s - 1} \cdot \left[P_1 \cdots P_s - (P_1 - 1) \cdots (P_s - 1)\right] - 24 = 2^3 \cdot 3^1$;

then, evidently, X has the form: $X = 2^{\alpha_1} \cdot 3^{\alpha_2}$, where $\alpha_1, \alpha_2 \in N^*$. Then $s = 2$.

We obtain $X - \Phi(X) = P_1^{\alpha_1 - 1} \cdot P_2^{\alpha_2 - 1} \cdot \left[P_1 \cdot P_s - (P_1 - 1) \cdot (P_2 - 1)\right]$, which means that

$X - \Phi(X) = 2^{\alpha_1 - 1} \cdot 3^{\alpha_2 - 1} \cdot [6 - 1, 2] = 2^3 \cdot 3^1$, where $X - \Phi(X) = 2^{\alpha_1 - 1} \cdot 3^{\alpha_2 - 1} = 2^1 \cdot 3^1$, from where $\alpha_1 = \alpha_2 = 2$ and in conclusion $X = 2^2 \cdot 3^2 = 36$.

2.15.

Let $\Phi(n)$ be Euler's totient function. Prove that: $\Phi(n)$ is a prime number if and only if $n \in \{0, \pm 3, \pm 4, \pm 6\}$.

Solution

The sufficiency:

$\Phi(0) = \Phi(\pm 3) = \Phi(\pm 4) = \Phi(\pm 6) = 2$, which is a prime number.

The necessity:

$\Phi(\pm 1) = \Phi(\pm 2) = 1$ which is not a prime number. Then $n \neq \{\pm 1, \pm 2\}$.

Let $n = P_1^{\alpha_1} \cdots P_s^{\alpha_s}$ with $P_1, \ldots, P_s$ different prime numbers. $\alpha_i \in N^*$, $i \in \{1, 2, \ldots, s\}$.

$\Phi(n) = P_1^{\alpha_1 - 1}(P_1 - 1) \cdots P_s^{\alpha_s - 1}(P_s - 1) = M_2$ for $n \notin \{\pm 1, \pm 2\}$ because $P_i - 1 = M_2$ where $P_j^{\alpha_j - 1} = M_2$.

Because $\Phi(n)$ is prime number, it results that $\Phi(n) = 2$. Then $P_i - 1 = 1$ or $P_i - 1 = 2$, or 3. Then $P_i = 1 \Rightarrow \alpha_j = 2$, therefore $n = 4, 3, 6$. And $n \in \{\pm 3, \pm 4, \pm 6\}$.

But $\Phi(0) = 2$ which is prime, then $n \in \{0, \pm 3, \pm 4, \pm 6\}$.

2.16.

Let m be an integer such that $\Phi(m) = M4$ (multiple of 4), where Φ represents Euler's indicator . Prove that it exists an even number of primitive solutions modulo m (an integer a is called a primitive solution modulo m, if $a^{\Phi(m)} \equiv 1 (\bmod m)$ and $a^K \not\equiv 1 (\bmod m)$ for $1 \leq K < \Phi(m)$).

Solution

1) If there does not exists a primitive solution modulo m, then we have 0 solutions and 0 is an even number.

2) If there exist primitive solutions, let r one of them. We have:

$(r,m)=1$, $r^{\Phi(m)} \equiv 1(\bmod m)$ and $r^K \not\equiv 1(\bmod m)$

For $1 \le K < \Phi(m)$. We'll show that $m-r$ is also a primitive solution modulo m.

A) Firstly, $m-r \ne r(\bmod m)$, because contrarily it would result that $2r \equiv 0(\bmod m)$, where $2r = t \cdot m$, with $t \in Z$. Because $\Phi(m) = M4$, we have $m \in \{0, \pm 1, \pm 2\}$.

α) $m = 2h$, $h \in Z - \{0, \pm 1\}$. We have $m \mid 2r \Rightarrow 2h \mid 2r \Rightarrow h \mid r \Rightarrow (r,m) = h \ne \pm 1$, which is absurd.

β) $m = 2h+1$, $h \in Z - \{-1, 0\}$. We have $m \mid 2r \Rightarrow m \mid r \Rightarrow (r,m) = m \ne \pm 1$, which is absurd. Therefore $m - r \not\equiv r(\bmod m)$

B) $(m, m-r) = d \Rightarrow d \mid m$ and $d \mid m-r \Rightarrow d \mid r \Rightarrow d = (r,m) = 1$, then $(m-r, m) = 1$.

$(m-r)^{\Phi(m)} \equiv 1(\bmod m)$, in accordance with Euler's theorem.

We suppose, by absurd, that exists $\mu \in N^*$, $\mu < \Phi_\mu(m)$ with $(m-r)^\mu \equiv 1(\bmod m)$. It results $1 \equiv (m-r)^\mu \equiv (-r) \equiv (-1)^\mu r^\mu (\bmod m)$. From where μ is odd (if not it results that $r^\mu \equiv 1(\bmod m)$ and $1 \le \mu < \Phi(m)$, in other words r will not be a primitive solution). Then $\mu = 2p+1$, with $p \in N$ and $r^\mu \equiv -1(\bmod p)$, where $r^{2\mu} \equiv 1(\bmod p)$. But $\mu < \Phi(m)$, which implies that $2\mu < 2\Phi(m)$. Because r is a primitive solution we have $2\mu = \Phi(m)$, where $\Phi(m) = 2(2p+1) \ne M_4$. Contradiction. And $(m-r)^\mu \not\equiv 1(\bmod m)$ for $1 \le \mu < \Phi(m)$, therefore $m-r$ is also a primitive solution.

2.17.

Let m a natural number ≥ 3, and $a_1, \ldots, a_p$ all the positive numbers smaller than m and different of m. Then $a_1 + a_2 + \ldots + a_p = Mm$ (multiple of m).

Solution

We prove that $p = M2$. We observe that if $0 < a < m$ and $(a,m) = 1$, then we also have $0 < m-a < m$ and $(m-a, m) = 1$, because:

$0 < a < m \Rightarrow -m < a-m < m-m \Rightarrow 0 < m-a < m$, let $d = (a-m, m)$, it results $d \mid m$, from where $d \mid a$, therefore d divides $(a,m) = 1$, and $d = 1$. We have that for $\forall a \in \{a_1, \ldots, a_p\}$, $\exists\, m-a \in \{a_1, \ldots, a_p\}$ such that $m-a \ne a$; (in the contrary case, it would have resulted that $m = 2a$ and $(a,m) = a \ne 1$ from where $m = 2$, which is impossible). (1)

But $a + (m-a) = m = Mm$ and, because of (1) we obtain the conclusion of the problem.

2.18.

Given three integer numbers a,b,c, such that $a^2+c^2\neq 0$ and $b^2+c^2\neq 0$, prove that

$$\frac{(a,b,c)\cdot c}{(a,c)\cdot(b,c)}\in Z$$

(The notation $(x_1,...,x_n)$ represents the largest common divisor of the numbers $x_1,...,x_n$)

Solution

Let $d=(a,b,c)$. This implies that $a=a'd,\ b=b'd,\ c=c'd$ and $(a',b',c')=1$. Then

$(a,c)=(a'd,c'd)=d\cdot(a',c')=d\cdot d_{13}$

(we note $(a',c')=d_{23}$); from where $c'=d_{23}\cdot\beta,\ \beta\in Z$.

But $(d_{13},d_{23})=((a',c'),(b',c'))=(a',b',c')=1$

Because $d_{13}\cdot\alpha=d_{23}\cdot\beta$, $(d_{13},d_{23})=1$, and that all numbers are integers, it results that d_{23} divides α, that is $\alpha=d_{23}\alpha'$ with $\alpha'\in Z$. Then $c=d\cdot d_{13}\cdot d_{23}\cdot\alpha'$, and that

$$\frac{(a,b,c)\cdot c}{(a,c)\cdot(b,c)}=\frac{d\cdot d\cdot d_{13}\cdot d_{23}\cdot\alpha'}{d\cdot d_{13}\cdot d\cdot d_{23}}=\alpha'\in Z.$$

The conditions from the problem ensure the existence of the expression and that the denominator is different of zero.

2.19.

Given $a_i,b_i\in N$, $i=\overline{1,n}$. Prove that:

$$(a_1,...,a_n,b_1,...,b_n)\geq\prod_{i=1}^{n}(a_i,b_i)$$

where (α,β) represents the greatest common divisor of the numbers α and β.

Solution

We'll apply the recurrence reasoning.

It is evident for $i=1$. We have to show that for $i=2$ we have $(a_1a_2,b_1b_2)\geq(a_1,b_1)\cdot(a_2,b_2)$. Having $a_1=a_{11}d_{a_1b_1}$, $b_1=b_{11}d_{a_1b_1}$ with $(a_{11},b_{11})=1$ and having $a_2=a_{21}d_{a_2b_2}$, $b_2=b_{21}d_{a_2b_2}$ with $(a_{21},b_{21})=1$, then

$$(a_1a_2,b_1b_2)=d_{a_1b_1}\cdot d_{a_2b_2}\cdot(a_{11}a_{21},b_{11}b_{21})\geq d_{a_1b_1}\cdot d_{a_2b_2}=(a_1,b_1)\cdot(a_2,b_2).$$

We suppose that the inequality is true for the values of i which are smaller than n. It results that:

$$(a_1...a_na_{n+1},b_1...b_nb_{n+1})\geq(a_1...a_n,b_1...b_n)\cdot(a_{n+1},b_{n+1})\geq\left(\prod_{i=1}^{n}(a_i,b_i)\right)\cdot(a_{n+1},b_{n+1})=\prod_{i=1}^{n+1}(a_i,b_i).$$

2.20

If $(a_i, b_i) \in N^2$, $i \in \{1,2,...,n\}$ and $[\alpha, \beta]$ represent the smallest common multiple of the numbers α and β, then $[a_1...a_n a_{n+1}, b_1...b_n] \le \prod_{i=1}^{n} [a_i, b_i]$

Solution

We'll apply the recurrence reasoning.

It is evident for $i = 1$. We have to show that for $i = 2$ we have $[a_1 a_2, b_1 b_2] \le [a_1, b_1] \cdot [a_2, b_2]$. Having $a_1 = a_{11} d_{a_1 b_1}$, $b_1 = b_{11} d_{a_1 b_1}$ with $(a_{11}, b_{11}) = 1$ and having $a_2 = a_{21} d_{a_2 b_2}$, $b_2 = b_{21} d_{a_2 b_2}$ with $(a_{21}, b_{21}) = 1$, then

$$[a_1 a_2, b_1 b_2] = d_{a_1 b_1} \cdot d_{a_2 b_2} \cdot [a_{11} a_{21}, b_{11} b_{21}] \le d_{a_1 b_1} \cdot d_{a_2 b_2} \cdot a_{11} \cdot a_{21} \cdot b_{11} \cdot b_{21} = [a_1, b_1] \cdot [a_2, b_2].$$

We suppose that the property is true for the values of $i \le n$. It results that it is true also for $i = n+1$ because:

$[a_1...a_n a_{n+1}, b_1...b_n b_{n+1}] \le [a_1...a_n, b_1...b_n] \cdot [a_{n+1}, b_{n+1}] \le \left(\prod_{i=1}^{n} [a_i, b_i] \right) \cdot [a_{n+1}, b_{n+1}]$. Then the problem is proved.

2.21.

Let m be a natural number, and $1 \le n \le 5$. Prove that if $9^m = \overline{a_1 \cdots a_m}$, then $9 \cdots 9^m = \overline{\underbrace{9 \cdots 9}_{n-1} a_1 \underbrace{0 \cdots 0}_{n-1} a_2 \underbrace{9 \cdots 9}_{n-1} \cdots a_n}$, where there are *n* digits of 9 in the left-hand side of this equation, with $n \in N^*$.

Solution

When $m = 1$, we have $9^1 = 9$, $\underbrace{9 \cdots 9^1}_{n} = \underbrace{9 \cdots 9}_{n-1} 9$.

When $m = 2$, we have:

$$9^2 = 81,\ \underbrace{9 \cdots 9^2}_{n} = \Big(1\underbrace{00 \cdots 0}_{n} - 1\Big)^2 = 1\underbrace{0 \cdots 0}_{2n} - 2\underbrace{0 \cdots 0}_{n} + 1 = 1\underbrace{0 \cdots 0}_{n}\underbrace{0 \cdots 0}_{n-1}1 - 2\underbrace{0 \cdots 0}_{n} = \underbrace{9 \cdots 9}_{n-1}8\underbrace{0 \cdots 0}_{n-1}1$$

When $m = 3$, we have:

$$9^3 = 729,\ \underbrace{9 \cdots 9^3}_{n} = \Big(1\underbrace{00 \cdots 0}_{n} - 1\Big)^3 = 1\underbrace{0 \cdots 0}_{3n} - 3\underbrace{0 \cdots 0}_{2n} + 3\underbrace{0 \cdots 0}_{n} - 1 =$$

$$= 1\underbrace{0 \cdots 0}_{n}\underbrace{0 \cdots 0}_{n-1}3\underbrace{0 \cdots 0}_{n} - 3\underbrace{0 \cdots 0}_{n}\underbrace{0 \cdots 0}_{n}1 = \underbrace{9 \cdots 9}_{n-1}7\underbrace{0 \cdots 0}_{n-1}2\underbrace{9 \cdots 9}_{n-1}9.$$

When $m = 4$, we have:

$$9^4=6561,\ \underbrace{9\cdots9}_{n}{}^4=\left(1\underbrace{00\cdots0}_{n}-1\right)^4=1\underbrace{0\cdots0}_{4n}-4\underbrace{0\cdots0}_{3n}+6\underbrace{0\cdots0}_{2n}-4\underbrace{0\cdots0}_{n}+1=$$

$$=1\underbrace{0\cdots0}_{n}\underbrace{0\cdots0}_{n}6\underbrace{0\cdots0}_{n}\underbrace{0\cdots0}_{n}1-4\underbrace{0\cdots0}_{n}\underbrace{0\cdots0}_{n}4\underbrace{0\cdots0}_{n}=\underbrace{9\cdots9}_{n-1}6\underbrace{0\cdots0}_{n-1}5\underbrace{9\cdots9}_{n-1}6\underbrace{0\cdots0}_{n-1}1.$$

When $m=5$, we have:

$$9^5=59049,\ \underbrace{9\cdots9}_{n}{}^5=\left(1\underbrace{00\cdots0}_{n}-1\right)^5=1\underbrace{0\cdots0}_{5n}-5\underbrace{0\cdots0}_{4n}+10\underbrace{0\cdots0}_{3n}-10\underbrace{0\cdots0}_{2n}+5\underbrace{0\cdots0}_{n}-1=$$

$$=\underbrace{9\cdots9}_{n-1}5\underbrace{0\cdots0}_{n-1}9\underbrace{9\cdots9}_{n-1}0\underbrace{0\cdots0}_{n-1}4\underbrace{9\cdots9}_{n-1}9.$$

Observation: For $m\geq 6$ the formula is not true.

2.22.

We consider the set $A=\left\{\underbrace{9\cdots9}_{n}\cdot m\ /\ m\in N^*\right\}$ and $n\in N^*$, constant .

a) Compute the greatest number of $2n$ digits from the set A, which does not contain the digit 9

b) Compute the smallest number $2n+K$ digits which does not contain the digit 9. Discussion.

Solution

a) We'll try to find the greatest $m\in N^*$, $m=\overline{b_1\cdots b_n}$, which multiplied by $\underbrace{9\cdots9}_{n}$ gives us a product of $2n$ digits with all its digits different of 9.

$$\underbrace{9\cdots9}_{n}\cdot m=\left(1\underbrace{0\cdots0}_{n}-1\right)\cdot m=\overline{m\underbrace{0\cdots0}_{n}}-m.$$

We compute the difference : $m=\overline{b_1\cdots b_n\underbrace{0\cdots0}_{n}}-\overline{b_1\cdots b_n}=?$

If there exist $b_j=9, j\in\{1,2,...,n-1\}$, then: it there is at least a non zero digit at the end of b_j, by subtraction it will result $b_j=9$ in the solution, if all the $b_h=0$ with $h\in\{j+1,...,n\}$ by subtraction in the solution it will exist at least one digit equal to 9 in one of the places $j+1,...,n$.

The next case is $\underbrace{8\cdots8}_{n}9$. By doing the difference (that is a multiplication $\underbrace{9\cdots9}_{n}\underbrace{8\cdots8}_{n}9$) we will obtain the greatest number of $2n$ digits of the set A which does not contain the digit 9, which is $\underbrace{8\cdots8}_{n}\underbrace{1\cdots1}_{n}$.

b) The number m would be $n+K$ digits.

1) The case $1\leq K\leq n$. We prove that $m=1\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n-1}2$ is the smallest number of $n+K$ digits which will have the required property.

We cannot have any zero among the last digits of the number m because it would result, by multiplication, at least one digit 9 in the product; the last digit non zero of m is different of 1 (for the same reason); the other digits of m can be equal to zero, only the first would have the value of minimum 1 because $K \le n$, then $K-1 \le n-1$.

$$\underbrace{9\cdots9}_{n}\cdot 1\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n-1}2 = \underbrace{1\cdots1}_{n-K+1}\underbrace{1\cdots1}_{K-1}\underbrace{8\cdots8}_{n}$$

(here we wrote directly that $\underbrace{9\cdots9}_{n}\cdot\underbrace{1\cdots1}_{n-1}2 = \underbrace{1\cdots1}_{n}\underbrace{8\cdots8}_{n}$)

(We utilize the property that the smallest number of $2n$ digits of A, which does not contain the digit 9 is $\underbrace{1\cdots1}_{n}\underbrace{8\cdots8}_{n}$, and that the correspondent m is $\cdot\underbrace{1\cdots1}_{n-1}2$).

2) The case $K \ge n+1$. Momentarily, we cannot write $m = 1\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n-1}2$ because $K-1 > n-1$ and the result of the multiplication contains digits 9:

$$\underbrace{9\cdots9}_{n}\cdot 1\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n-1}2 = \underbrace{1\cdots1}_{n}\underbrace{8\cdots8}_{n} \tag{1}$$

$$\underbrace{9\cdots9}_{n} = \underbrace{9\cdots9}_{n}\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n}\underbrace{8\cdots8}_{n}$$

We're looking to find the smallest $m \in N^*$, of $n+K$ digits, which will have the requested property. The last n digits of m will be also $\underbrace{1\cdots1}_{n-1}2$. The first will be also 1.

Among the unknown digits we cannot have more than $n-1$ consecutive zeros because of (1). Because n is also small it is possible to attach $n-1$ consecutive zeros after the first digit, then a digit 1 (the minimum not null), other $n-1$ consecutive zeros and again a digit 1, etc. Therefore: $m = \underbrace{1\underbrace{0\cdots0}_{n-1}1\underbrace{0\cdots0}_{n-1}\cdots1\underbrace{0\cdots0}_{n-1}1\underbrace{0\cdots0}_{p}}_{K\ digits}\underbrace{1\cdots1}_{n-1}2$

Then, the number will be $\underbrace{9\cdots9}_{n}\cdot m = 1\underbrace{0\cdots0}_{K-1}\underbrace{1\cdots1}_{n-p-1}0\underbrace{1\cdots1}_{p}\underbrace{8\cdots8}_{n}$ with $p = K - n\cdot\left[\dfrac{K}{n}\right]-1$, where $[X]$ represents the integer part of X.

This is the multiplication:

$$\underbrace{9\cdots9}_{n}\cdot\underbrace{1\underbrace{0\cdots0}_{n-1}1\underbrace{0\cdots0}_{n-1}\cdots1\underbrace{0\cdots0}_{n-1}1\underbrace{0\cdots0}_{p}}_{K}\underbrace{1\cdots1}_{n-1}2 = \underbrace{1\cdots1}_{n-p}0\underbrace{1\cdots1}_{p}\underbrace{8\cdots8}_{n}\underbrace{9\cdots9}_{p}\underbrace{9\cdots9}_{n-p}\underbrace{9\cdots9}_{n}\cdots\underbrace{9\cdots9}_{n} =$$

$$= \underbrace{1\underbrace{0\cdots0}_{n}\cdots\underbrace{0\cdots0}_{n}\underbrace{0\cdots0}_{p}}_{K}\underbrace{1\cdots1}_{n-p-1}0\underbrace{1\cdots1}_{p}\underbrace{8\cdots8}_{n}$$

2.23.

If $x, y \in N$, then there exist $z \in N$ such that $\overline{10x}\cdot\overline{10y} = \overline{10z}$.

Generalize this result for the case of any number of zeros between 1 and x and between 1 and y.

Solution

Let's consider $x = \overline{a_1 ... a_n}$, $0 \le a_i \le 9$, $i \in \{1,2,...,n\}, n \in N^*$ and $y = \overline{b_1 ... b_m}$, $0 \le b_j \le 9$, $j \in \{1,2,...,m\}, m \in N^*$.

We do the multiplication:

$$\overline{10a_1...a_n} \times$$
$$\overline{10b_1...b_m}$$

$n+2$ digits · ⋯ the multiplication by b_m

....

$n+2$ digits · ⋯ the multiplication by b_1

$n+2$ digits 1 0 · · the multiplication by 1

$= 1$ · · ·

We noted by "·" a natural digit between 0 and 9.

Then, the first digit of the product is 1.

Generalization: if $x, y \in N$, then there is a $z \in N$ such that $\overline{1\underbrace{0\cdots0}_{s\ digits}x} \cdot \overline{1\underbrace{0\cdots0}_{t\ digits}y} = \overline{1\underbrace{0\cdots0}_{u\ digits}z}$,

Where we have $u = \inf(s,t) - 1$.

The proof of this generalization follows the proof of the first part of this problem.

2.24.

We consider a numeric base b, and p a simple divisor such that $\left(p, \frac{b}{p}\right) = 1$. Then $\forall n \in N^*,\ \exists A_n = \overline{a_1...a_n}$ written in base b which is divisible by p^n, with $a_i \in \{1,2,...,|p|\}, 1 \le i \le n$.

Solution

We will apply the recurrence reasoning for $n \in N^*$.

For $n = 1$ we have $\exists A_1 = |p|$ which is divisible by p^1. (We observe that, because $p \mid b$, it results that $b = Kp,\ K \in Z$; $1 = \left(p, \frac{b}{p}\right) = (p, K)$, also, all the digits of the numbers in the base b belong to the set $M_b = \{0,1,2,...,|p|,\ |p|+1,....,n-1\}$, and these are represented by one single symbol (for example, if $b > 10$, then the digits $10, 11, \ldots$ are noted by $A, B, \ldots$). Then $|p| \in M_p = \{1,2,...,|p|\}$ and it is formed by a digit in base b; ($p \mid b \Rightarrow |p| \le |b| = b$).

We suppose that the property is true for n, that is $\exists\, A_n = \overline{a_1 ... a_n}$, written in the base b, which is divisible by p^n, where $a_i \in M_p$, $1 \le i \le n$. We'll show that it is true for $n+1$.

Let $A_{n+1} = \overline{xa_1 ... a_n}$ with $x \in M_p$ written in base b. We determine a x for which A_{n+1} is divisible by p^{n+1} (it is sufficient to prove that there is such an x).

$A_{n+1} = x \cdot b^n + \overline{a_1 ... a_n} = x \cdot K^n \cdot p^n + A_n = p^n (K^n \cdot x + t)$, where $A_n = tp^n$, $t \in Z$ (from the recurrence hypothesis).

$\exists\, x \in M_p$ such that $K^n x + t \equiv 0 (\text{mod } p) \Leftrightarrow K^n \cdot x \equiv -t (\text{mod } p)$.

Because $\left(p, \dfrac{b}{p}\right) = 1 = (p, K)$ we have $\left(p, K^n\right) = 1$. Then there exists the inverse of the element K^n in rapport to the module p. The above congruence becomes: $x \equiv -t\left(K^n\right)^{-1} (\text{mod } p)$ and we chose the smallest x not null, that is: $x \in M_p$.

(There exists $x \in M_p$, because M_p constitute a complete systems of residues modulo p.)

2.25.

Let $n, m \in N^*$. We note $a_n^{(m)} = m^{m^{\cdot^{\cdot^{m}}}}$ with n digits m, and $b_n^{(m)} = \underbrace{\overline{m...m}}_{n}^{\underbrace{\overline{m...m}}_{n}}$. For each n and m compare $a_n^{(m)}$ with $b_n^{(m)}$. Discussion. (All numbers are written in base 10.)

Solution

In the precedent conditions, we have:

Lemma 1. $\forall n \in N^*,\ \forall m \ge 2\ \ m^{4n} > \underbrace{\overline{m...m}}_{n}$

Proof: We use the recurrence method for $n \in N^*$.

The case $n = 1$ implies $m^4 > m$ which is true. We assume that the property is true for n and we'll prove it for $n+1$:

$$m^{4(n+1)} = m^{4n} \cdot m^4 > \underbrace{\overline{m...m}}_{n} \cdot m^4 \ge \underbrace{\overline{m...m}}_{n} \cdot 16 = \underbrace{\overline{m...m0}}_{n+1} + \underbrace{\overline{m...m}}_{n} \cdot 6 > \underbrace{\overline{m...m0}}_{n+1} + m = \underbrace{\overline{m...mm}}_{n+1}.$$

Lemma 2. $\forall n \ge 3, \forall m \in N^*,\ b_n^{(m)} > 4^{(n+1)} \underbrace{\overline{m...m}}_{n+1}$

Proof: $\underbrace{\overline{m...m}}_{n}^{\,2} = \underbrace{\overline{m...m}}_{n} \cdot \underbrace{\overline{m...m}}_{n} > \underbrace{\overline{m...mm}}_{n+1}$ because $n \ge 3$.

$\underbrace{\overline{m...m}}_{n} > n + 1 \ge 4$ because $n \ge 3$.

$$b_n^{(m)} = \underbrace{\overline{m...m}}_{n}^{\underbrace{\overline{m...m}}_{n}} > \underbrace{\overline{m...m}}_{n}^{\,4} > 4^{(n+1)} \cdot \underbrace{\overline{m...mm}}_{n+1}.$$

Lemma 3: If there exists $n_0 \in N^*,\ n_0 \geq 3$, such that $a_{n_0}^{(m)} > b_{n_0}^{(m)}$ then $\forall m \geq 2, \forall n \geq n_0$ we have $a_n^{(m)} > b_n^{(m)}$.

Proof: use the recurrence method for $n \geq n_0$.

The case $n = n_0$ is true by hypothesis. We assume that the property is true for n, and we'll prove it for $n+1$:

$$a_{n+1}^{(m)} = m^{a_n^{(m)}} > m^{b_n^{(m)}} > m^{4(n+1)} \cdot \underbrace{\overline{m...m}}_{n+1} > \underbrace{\overline{m...m}}_{n+1}^{\overbrace{\overline{m...m}}^{n+1}} = b_n^{(m)}.$$

To prove these inequalities we will use the hypothesis from the recurrence from Lemma 2, respectively Lemma 1.

Lemma 4: $\forall m \geq 6,\ a_3^{(m)} > b_3^{(m)}$.

Proof. Because $m \geq 6$ and because of the results from the Lemmas 1 and 2, it results that:

$$m^m = m^2 \cdot m^{m-2} > 4 \cdot 3 \cdot m^{m-2} > 4 \cdot 3 \cdot 6^4 > 4 \cdot 3 \cdot \overline{mmm}$$

$$a_3^{(m)} = m^{m^m} > m^{4 \cdot 3 \cdot \overline{mmm}} = \left(m^{4 \cdot 3}\right)^{\overline{mmm}} > \overline{mmm}^{\overline{mmm}} = b_3^{(m)}.$$

We have: $a_n^{(m)} = m^{m^{\cdot^{\cdot^{m}}}}$ with n digits of m, $b_n^{(m)} = \underbrace{\overline{m...m}}_{n}^{\overbrace{\overline{m...m}}^{n}}$.

Case $m = 1$. $a_1^{(1)} = 1 = b_1^{(1)}$.

$$a_n^{(1)} = 1 < \underbrace{\overline{1...1}}_{n+1}^{\overbrace{\overline{1...1}}^{n+1}} = b_n^{(1)}, \forall n \geq 2$$

Case $m = 2$. $a_2^{(2)} = 2^2 < 22^{22} = b_2^{(2)}$

$$a_3^{(2)} = 2^4 < 222^{222} = b_3^{(2)}$$

$$a_4^{(2)} = 2^{16} < 2222^{2222} = b_4^{(2)}$$

$$a_5^{(2)} = 2^{65536} < 2^{3 \cdot 22222} < 22222^{22222} = b_5^{(2)}$$

Using Lemma 1, we obtain

$$2^{65536} > 2^5 \cdot 2^{4 \cdot 6} > 4 \cdot 7 \cdot 222222$$

Then

$$a_6^{(2)} = 2^{2^{65536}} > 2^{4 \cdot 7 \cdot 222222} = \left(2^{4 \cdot 7}\right)^{222222} > 222222^{222222} = b_6^{(2)}$$

From the Lemma 3 it results that $a_n^{(2)} > b_n^{(2)}, \forall n \geq 6$.

Case $m = 3$ $a_1^{(3)} = 3 < 3^3 = b_1^{(3)}$

$$a_2^{(3)} = 3^3 < 33^{33} = b_2^{(3)}$$

$$a_3^{(3)} = 3^{27} < 333^{333} = b_3^{(3)}$$

Using Lemma 1 we obtain $3^{27} > 3^3 \cdot 3^{4 \cdot 4} ?16 \cdot 3333$

Then

$$a_4^{(3)} = 3^{3^{27}} > 3^{16 \cdot 3333} = \left(3^{4 \cdot 4}\right)^{3333} > 3333^{3333} = b_4^{(3)}$$

From Lemma 3 it results $a_n^{(3)} > b_n^{(3)}, \forall n \geq 4$

Case $m = 4$. $a_1^{(4)} = 4 < 4^4 = b_1^{(4)}$

$$a_2^{(4)} = 4^4 < 44^{44} = b_2^{(4)}$$

$$a_3^{(4)} = 4^{4^4} = 4^{256} < 444^{444} = b_3^{(4)}$$

From Lemma 1 it results $4^{256} > 4^2 \cdot 4^{4\cdot 4} > 4 \cdot 4 \cdot 4444$.

Then $a_4^{(4)} = 4^{4^{256}} > 4^{4\cdot 4\cdot 4444} = \left(4^{4\cdot 4}\right)^{4444} > 4444^{4444} = b_4^{(4)}$.

From Lemma 3 it results:

$$\forall n \geq 4, a_n^{(4)} > b_n^{(4)}$$

Case $m = 5$. $a_1^{(5)} = 5 < 5^5 = b_1^{(5)}$

$$a_2^{(5)} = 5^5 < 55^{55} = b_2^{(5)}$$

$$a_3^{(5)} = 5^{5^5} = 5^{3125} = \left(5^5\right)^{625} = 3125^{625} > 555^{555} = b_3^{(5)}.$$

From Lemma 3 we have $\forall n \geq 3, a_n^{(5)} > b_n^{(5)}$.

Case $m = 6$. $a_1^{(m)} = m < m^m = b_1^{(m)}$

$$a_2^{(m)} = m^m < \overline{mm}^{\overline{mm}} = b_2^{(m)}$$

From Lemma 4 it results: $a_3^{(m)} > b_3^{(m)}$

And from Lemma 3 we have $a_n^{(m)} > b_n^{(m)}$, $\forall n \geq 3$ and the problem is solved

MATHEMATICAL LOGIC

3.26.

Consider $P, Q_i \ 1 \le i \le n$, logical propositions. Prove that the logical proposition

$$"\bigvee_{i=1}^{n}(P \wedge Q_i) \Rightarrow \bigwedge_{i=1}^{n}(P \vee Q_i)"$$

is always true

Solution

A logical proposition $"A \Rightarrow B"$ is false only when $A = 1$ (true) and $B = 0$ (false). We'll prove that this situation does not exist.

If $"\bigvee_{i=1}^{n}(P \wedge Q_i)" = 1$, then $\exists i_0 \in \{1, \ldots, n\}$ such that $P \wedge Q_i = 1$, that is $P = 1$ and $Q_{i_0} = 1$.

Then:

$P \vee Q_i = 1, \ \forall i \in \{1, \ldots, n\}$ since $P = 1$,

Then

$$"\bigwedge_{i=1}^{n}(P \vee Q_i) = 1" \neq 0$$

($\wedge$ means "and", $\vee$ means "or").

3.27.

Show that if the logical propositions "$A_1 \Rightarrow A_2$" and "$B_1 \Rightarrow B_2$" are true, then the logical propositions "$A_1 \wedge B_1 \Rightarrow A_2 \wedge B_2$" and "$A_1 \vee B_1 \Rightarrow A_2 \vee B_2$" are also true.

Solution

A_1	A_2	B_1	B_2	$A_1 \Rightarrow A$	$B_1 \Rightarrow B_2$	$A_1 \wedge B_1$	$A_2 \wedge B_2$	$A_1 \wedge B_1 \Rightarrow A_2 \wedge B_2$	$A_1 \vee B_1$	$A_2 \vee B_2$	$A_1 \vee B_1 \Rightarrow A_2 \vee B_2$
0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	1	0	1	0	0	1	0	1	1
0	0	1	0	1	0	0	0	1	1	0	0
1	0	0	0	1	0	0	1	1	1	0	0
0	0	1	1	1	1	0	0	1	1	1	1
0	1	1	0	1	0	0	0	1	1	1	1
1	1	0	0	1	1	0	0	1	1	1	1
0	1	0	1	1	1	0	1	1	0	1	1
1	0	1	0	0	0	1	0	0	1	0	0
1	0	0	1	0	1	0	0	1	1	1	1
0	1	1	1	1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	0	0	1	1	1
1	1	0	1	1	1	0	1	1	1	1	1
1	0	1	1	0	1	1	0	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12

We note with “1” the true and with “0” the false. Immediately we observe that "$A_1 \Rightarrow A_2$" and "$B_1 \Rightarrow B_2$" are true in the same time, it results that "$A_1 \wedge B_1 \Rightarrow A_2 \wedge B_2$" and "$A_1 \vee B_1 \Rightarrow A_2 \vee B_2$" are true in the same time.

TRIGONOMETRY

4.28.

Prove the following formulae of transformation of the products of functions in sums:

1) $\cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_n = \frac{1}{2^{n-1}} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \tau_n} \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_n\alpha_n)$

2)

a) $\sin\alpha_2 \cdots \sin\alpha_{2p} = \frac{(-1)^p}{2^{2p-1}} \sum_{\varepsilon_1,\ldots,\varepsilon_{2p} \in \tau_{2p}} (-1)^k \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_{2p}\alpha_{2p})$

b) $\sin\alpha_2 \cdots \sin\alpha_{2p+1} = \frac{(-1)^p}{2^{2p}} \sum_{\varepsilon_1,\ldots,\varepsilon_{2p+1} \in \tau_{2p+1}} (-1)^k \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_{2p+1}\alpha_{2p+1})$

where

$\tau_m = \bigcup_{k=0}^{\left[\frac{m}{2}\right]} (\varepsilon_1,\ldots,\varepsilon_m) / \varepsilon_{i_1} = \varepsilon_{i_2} = \ldots = \varepsilon_{i_k} = -1$ and $\varepsilon_j = 1$ for

$j \notin \{i_1,\ldots,i_k\} - \{(\varepsilon_1,\ldots,\varepsilon_m)/(-\varepsilon_1,\ldots,-\varepsilon_m) \in \tau_m\}$.

Solution

The set τ_m contains all the m-lepts $(\varepsilon_1,\ldots,\varepsilon_m)$ which have the components $\varepsilon_i = \pm 1$ arranged in all possible orders, but such that $(\varepsilon_1,\ldots,\varepsilon_m) \in \tau_m$, then $(-\varepsilon_1,\ldots,-\varepsilon_m) \in \tau_m$. Therefore τ_m has in total $(C_m^0 + C_m^1 + \ldots + C_m^m) : 2 = 2^{m-1}$ elements, by C_m^k, $0 \le k \le m$ we represent the numbers of m-lepts such that the k components are equal to -1, and the rest of $m-k$ are equal to +1.

1) We'll make the prove using the recurrence method on n.

The case $n=1$ is evident. We suppose that the equality is true for n, and then prove it for $n+1$:

$$(\cos\alpha_1 - \cos\alpha_n)\cos\alpha_{n+1} = \frac{1}{2^{n-1}} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\tau_n} \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_n\alpha_n)\cos\alpha_{n+1} =$$

$$= \frac{1}{2^n} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\tau_n} \cos\left[(\varepsilon_1\alpha_1 + \ldots + \varepsilon_n\alpha_n + \alpha_{n+1}) + \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_n\alpha_n + \varepsilon_{n+1} - \alpha_{n+1})\right] =$$

$$= \frac{1}{2^n} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\tau_n} \cos(\varepsilon_1\alpha_1 + \ldots + \varepsilon_{n+1}\alpha_{n+1})$$

2)

a) We apply the recurrence rational for $p \in N^*$.

If p=1 we have $\sin\alpha_1 \sin\alpha_2 = \frac{(-1)^1}{2}[\cos(\alpha_1+\alpha_2) - \cos(-\alpha_1+\alpha_2)]$ which is true.

We suppose that the equality is true for p, we we'll prove for $p+1$:

$(\sin\alpha_1 \ldots \sin\alpha_{2p})\sin\alpha_{2p+1}\sin\alpha_{2p+2} =$

$$=\frac{(-1)^p}{2^{2p-1}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{j_p})\in\tau_{2p}}(-1)^k\cos\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}\right)\sin\alpha_{2p+1}\cos\alpha_{2p+2}=$$

$$=\frac{(-1)^{p-1}}{2^{2p}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p})\in\tau_{2p}}(-1)^k\cos\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}\right)\left[\cos(\alpha_{2p+1}+\alpha_{2p+2})-\cos(-\alpha_{2p+1}+\alpha_{2p+2})\right]=$$

$$\frac{(-1)^{p+1}}{2^{2p+1}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p})\in\tau_{2p}}(-1)^k\left[\cos\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}+\alpha_{2p+1}+\alpha_{2p+2}\right)+\right.$$

$$+\cos\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}-\alpha_{2p+1}-\alpha_{2p+2}\right)-$$

$$\left.-\cos(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}+\alpha_{2p+1}-\alpha_{2p+2})-\cos(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}+\alpha_{2p+1}-\alpha_{2p+2})\right]=$$

$$=\frac{(-1)^{p+1}}{2^{2p+1}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p+2})\in\tau_{2p+2}}(-1)^k\cos(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p+2}\alpha_{2p+2}).$$

(We can prove easily the relations:

$\tau_{m+1}=\left\{(\varepsilon_1,\ldots,\varepsilon_m,-1),(\varepsilon_1,\ldots,\varepsilon_m,1)\mid(\varepsilon_1,\ldots,\varepsilon_m)\in\tau_m\right\}$ and

$\tau_{m+2}=\left\{(\varepsilon_1,\ldots,\varepsilon_m,-1,-1),(\varepsilon_1,\ldots,\varepsilon_m,-1,1),(\varepsilon_1,\ldots,\varepsilon_m,1,-1),(\varepsilon_1,\ldots,\varepsilon_m,1,1)\text{ such that }(\varepsilon_1,\ldots,\varepsilon_m)\in\tau_m\right\}$

(We can also generalize).

b) The first method: by recurrence for $p\in N^*$ (similarly with above reasoning.

The second method:

$$(\sin\alpha_1\ldots.\sin\alpha_{2p})\sin\alpha_{2p+1}=\frac{(-1)^p}{2^{2p-1}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p})\in\tau_{2p}}(-1)^k\cos\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}\right)\sin\alpha_{2p+1}=$$

$$=\frac{(-1)^p}{2^{2p}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p})\in\tau_{2p}}(-1)^k\left[\sin\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}+\alpha_{2p+1}\right)+\sin\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p}\alpha_{2p}-\alpha_{2p+1}\right)\right]=$$

$$=\frac{(-1)^p}{2^{2p}}\sum_{(\varepsilon_1,\ldots,\varepsilon_{2p+1})\in\tau_{2p+1}}(-1)^k\sin\left(\varepsilon_1\alpha_1+\ldots+\varepsilon_{2p+1}\alpha_{2p+1}\right).$$

4.29.

Let $P(x)=2x^2-1$. Prove that for $n>2$ we have:

$$\sin 2^n x=2^{n-1}2x=\prod_{i=1}^{n-1}\underbrace{p(\ldots p(p}_{i\ times}(\cos x))\ldots)$$

Solution

a) We will prove by recurrence for $n\in N^*$ we have:

$$\sin 2^n x=2^n\sin x\cos x\cos 2^1 x\cos 2^2 x\ldots\cos 2^{n-1}x \qquad (1)$$

In the case $n=1$ the property is evident

We suppose the equality true for n, and we prove that it is true also for $n+1$.

$\sin 2 \cdot 2^n ... = 2\sin 2^n x \cos 2^n x ... 2^{n+1} \sin x \cos x ... \cos 2^{n-1} x \cos 2^n x$

(we used the hypothesis of the recurrence)

b) We'll prove by recurrence that for $i \in N^*$ we have:

$$\cos 2^i x = p(...p)(p(\cos x))...) \tag{2}$$

In the case $i = 1$ we have $\cos 2x = 2\cos^2 x - 1 = p(\cos x)$.

If we suppose the equality true for i we'll prove that it is true also for $i+1$:

$$\cos 2 \cdot 2^i x = 2\cos^2 2^i x - 1 = p(\cos 2^i x) = p2x\cos 2^i x = p(\underbrace{p(...p)(p}_{i\ times}\cos x))...)$$

Substituting (2) in (1) for all $\cos 2^i x$, it will result the equality that we need.

4.30.

Let $s, n \in N^*$, K_i, P_i rational with $1 \le i \le n$, and the continuous functions $f_i, g_i : R^s \to R$ for $1 \le i \le n$.

a) Find a method for solving the equation:

$$\sin^{K_1} f_1(x_1,...,x_s)\cos^{P_1} g_1(x_1,...,x_s) + ... + \sin^{K_n} f_n(x_1,...,x_s)...\cos^{P_n} g_n(x_1,...,x_s) = n$$

b) Find the necessary and sufficient condition that the equation from a) is equivalent to the following system of equations:

$$\begin{cases} \sin^{K_1} f_1(x_1,...,x_s) + ... + \sin^{K_n} f_n(x_1,...,x_s) = n \\ \cos^{P_1} g_1(x_1,...,x_s) + ... + \cos^{P_n} g_n(x_1,...,x_s) = n \end{cases}$$

<u>Solution</u>

a) The right side of the equation is a sum of n terms, each belonging to $[-1,1]$. Then each term must be equal to 1, because if not we have $s < n$. Then the equation is equivalent to the system:

$\sin^{K_i} f_i(x_1,...,x_s)\cos^{P_i} g_i(x_1,...,x_s) = 1$,

which is equivalent to

$$\begin{cases} \sin^{K_i} f_i(x_1,...,x_s) = 1 \\ \cos^{P_i} g_i(x_1,...,x_s) = 1 \end{cases} \tag{1'}$$

or

$$\begin{cases} \sin^{K_i} f_i(x_1,...,x_s) = -1 \\ \cos^{P_i} g_i(x_1,...,x_s) = -1 \end{cases} \tag{1''}$$

with $i \in \{1,...,n\}$;

which are resolved normally, we obtain then an algebraic system

b) The system from b) is also equivalent with the system from (1'). Because the equations from a) are equivalent to the system from b), therefore with (1'), we must eliminate the case (1'').

Then, if $K_i = \frac{r_i}{t_i}$, $P_i = \frac{u_i}{v_i}$, r_i, t_i, u_i, v_i are integers, $1 \le i \le n$, then for $\forall i \in \{1,...,n\}$, there exist at least one integer in $\{r_i, t_i, u_i, v_i\}$ which is even.

GEOMETRY

5.31.

We design the projections M_i of a point M on the sides A_iA_{i+1} of a polygon $A_1...A_n$. Show that $\|M_1A_1\|^2+...+\|M_nA_n\|^2=\|M_1A_2\|^2+...+\|M_{n-1}A_n\|^2+\|M_nA_1\|^2$.

Solution

For all i we have: $\|M_iA_i\|^2-\|M_iA_{i+1}\|^2=\|MA_i\|^2-\|MA_{i+1}\|^2$. From which:

$\sum_i(\|M_iA_i\|^2-\|M_iA_{i+1}\|^2)=\sum_i(\|MA_i\|^2-\|MA_{i+1}\|^2)=0$.

5.32.

On a line we have the following points $A_1,A_2,...,A_n$ in this order.

Let $n_1=\left[\frac{n}{2}\right]$ and $n_2=\left[\frac{n+1}{2}\right]$. Prove that

$$\sum_{i=1}^{n_2}\|A_iA_{n+1}\|=\sum_{j=1}^{n_1}\|A_jA_{n+1-j}\|$$

Solution

a) $n=2K\Rightarrow n_1=n_2=K$

We make the notation: $\|A_iA_{i+1}\|=x_i$, $1\le i\le n-1$ (1)

Our relation becomes:

$$\sum_{i=1}^{k}\|A_iA_{k+1}\|=\sum_{i=1}^{k}\|A_iA_{2k+1-i}\|$$

From (1) we have:

$$\sum_{i=1}^{k}(x_i+x_{i+1}+...+x_{i+k})=\sum_{i=1}^{k}(x_i+x_{i+1}+...+x_{2k-1})$$

The left side is equal to:

$$x_1+x_2+x_3+...+x_k+$$
$$x_2+x_3+...+x_k+x_{k+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_k+x_{k+1}+...+x_{2k-1}$$

which is equal to

$x_1+2x_2+3x_3+...+kx_k+(k-1)x_{k+1}+(k-2)x_{k+2}+...+x_{2k-1}$ (2)

The side from the right is equal to

$x_1 + x_2 + x_3 + ... + x_k + x_{k+1} + x_{2k-2} + x_{2k-1} +$

$x_2 + x_3 + ... + x_k + x_{k+1} + ... + x_{2k-2} +$

....

Which ia also equal to (2)

b)

$$n = 2k + 1 \Rightarrow n_1 = k$$

$$n_2 + k + 1$$

The proof is the same.

5.33.

Let ABC an arbitrary triangle and O the center of its inscribed triangle. On one of the sides, BC, we take n points $A_1,...,A_n$, in this order, such that the lines $AA_1,...,AA_n$ divide the angle BAC in $n+1$ equal parts. In a similar mode we proceed for the sides CA and AB on which we take the points $B_1,...,B_n$ respectively $C_1,...,C_n$.

Prove that the point O belongs to the geometrical figure determined by the intersection of the lines AA_i, BB_i, and CC_i, $i \in \{1,2,...,n\}$.

Solution

a) If $\frac{n+1}{2} = \left[\frac{n+1}{2}\right] = i$ then AA_i, BB_i, and CC_i are the bisectrics of the angles A, B, and C because these divide the angles into two equal parts. Then O is thir intersections.

b) If $\frac{n+1}{2} \neq i$, then AA_i, BB_i, and CC_i are not anymore bisectrics. These intersect each other in pairs forming a triangle that is in the interior of the triangle ABC. We obtain the small triangle from the figure (1).

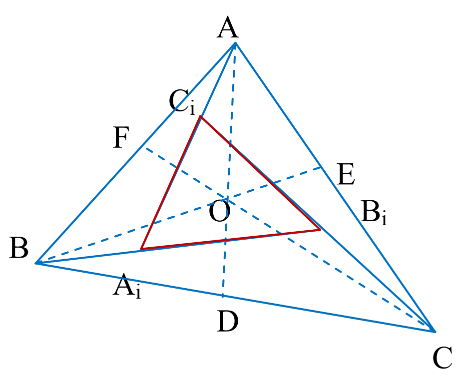


Fig. (1)

Let AD, BE, CF be the bisectrics of the angles A, B, C. These can be on the left side of the lines AA_i, BB_i, and CC_i (as we look at the sides starting from A to A_i) or to the right. In figure (1) we have the case where the lines are on the left side. We will have the same proof for other cases).

Because AD is on the left side of AA_i and $O \in AD$, it results that $O \in \triangle AA_iC$. Because BE is on the left side of BB_i and that $O \in BE$ we have that $O \in \triangle BB_iA$. The same, $O \in \triangle CC_iB$. Then $O \in \triangle AA_iC \cap \triangle BB_iA \cap \triangle CC_iB$.

5.34.

Given n lines that intersect two by two and are not on the same plane three by three, prove that these lines pass through the same point.

Solution

We'll consider the case $n = 3$.

The lines d_1 and d_2 intersect in M; d_3 intersects d_1 in M' and d_2 in M''. If $M' \neq M$ and $M'' \neq M$, then the three lines are on the same plane, which is absurd. Therefore $M' \equiv M'' \equiv M$.

The case $n > 3$ is reduced to the previous case.

Among the n lines we choose arbitrary three, that satisfy the conclusion. Among these three lines we take two arbitrary ones and one among the $n-3$ left lines. We obtain three lines that pass through the same point, which is also M.

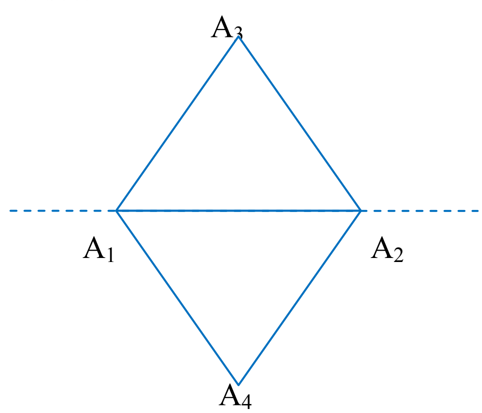
The rational continues the same until we finish with all the lines.

5.35.

Let n points $A_1, \ldots, A_n$ in a plane, $n \geq m \geq 3$, such that m of these points form a regular polygon. Prove that $n = m$.

Solution

1) Case $m > 3$. Let $m-1$ points in this plane. We add a new point and we construct a regular polygon with m sides.



Each regular polygon can be inscribed in a circle. We consider for beginning that the $m-1$ points are placed on the circumference of a circle. Evidently, the other point (which has been added) belongs to the same circle, and it is well determined, because the circle is divided in equal arcs. But, through $m-1 \geq 3$ points passes only one circle. Then to $m-1$ points we can add one single point to form a regular polygon (of m sides).

More, the number of points that have the property from above cannot be larger than m; but, also, it cannot be less than m either because we cannot form a regular polygon with m sides. From here we have that $n = m$

2) The case $m = 3$. Then $m-1 = 2$. Taking two distinct points, to form an equilateral triangle, we can find: let one point in a semi-plane, let one point on another semi-plane (the semi-planes determined by the line that unite the two points and divide the plane in two parts).

If $\triangle A_1A_2A_3$ and $\triangle A_1A_2A_4$ are equilateral, then $\triangle A_1A_3A_4$ will not be equilateral.

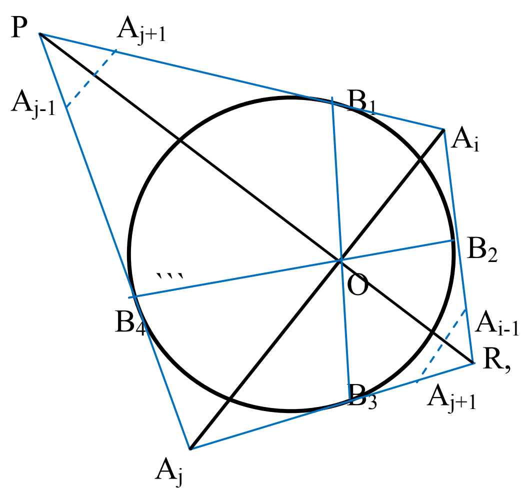
The proof, now becomes similar to the case 1.

5.36.

We consider a polygon (which has at least 4 sides) circumscribed to a circle, and D the set of the diagonals and of the lines that connect the contact points of two non adjacent sides. Then D contains at least three concurrent lines.

Solution

Let n be the number of sides. If $n = 4$, then the two diagonals and the two lines that connect the points of contact of the two non adjoined sides are concurrent (in conformity to the Newton's theorem)



The case $n > 4$ will resume to anterior case: we consider the arbitrary polygon $A_1 \ldots A_n$ (see the figure) circumscribed to a circle and we select two segments A_i, A_j, $i \neq j$, such that

$$A_jA_{j-1} \cap A_iA_{i+1} = P,\ A_jA_{j+1} \cap A_iA_{i-1} = R$$

Let B_h, $h \in \{1,2,3,4\}$ be the contact points of the quadrilateral PA_jRA_i with the circle in the center O.

Due to Newton's theorem, the lines A_iA_j, B_1B_3, B_2B_4 are concurrent.

5.37.

In a triangle ABC let AA', BB', CC' be the ceviene that intersect in the point P. Compute the minimum value of the expressions:

$$E(P) = \frac{\|AP\|}{\|PA'\|} + \frac{\|BP\|}{\|PB'\|} + \frac{\|CP\|}{\|PC'\|}$$

and

$$F(P) = \frac{\|AP\|}{\|PA'\|} \cdot \frac{\|BP\|}{\|PB'\|} \cdot \frac{\|CP\|}{\|PC'\|}$$

where

$$A' \in [BC],\ B' \in [CA],\ C' \in [AB].$$

Solution

We apply the Van Aubel theorem three times for triangle ABC, and we have:

(1) $$\frac{\|AP\|}{\|PA'\|} = \frac{\|AC'\|}{\|CB'\|} + \frac{\|AB'\|}{\|B'C\|}$$

(2) $$\frac{\|BP\|}{\|PB'\|} = \frac{\|BA'\|}{\|A'C\|} + \frac{\|BC'\|}{\|C'A\|}$$

(3) $$\frac{\|CP\|}{\|PC'\|} = \frac{\|CA'\|}{\|A'B\|} + \frac{\|CB'\|}{\|B'A\|}$$

If we add these three relations and if we make the following notations

$$\frac{\|AC'\|}{\|CB'\|} = X > 0\ ,\frac{\|AB'\|}{\|B'C\|} = Y > 0,\quad \frac{\|BA'\|}{\|A'C\|} = Z > 0$$

then we obtain:

$$E(P) = \left(X + \frac{1}{X}\right) + \left(Y + \frac{1}{Y}\right) + \left(Z + \frac{1}{Z}\right) \geq 2 + 2 + 2 = 6.$$

The minimum value will be obtained when X=Y=Z=1, that is when P will be the center of gravity of the triangle.

Multiplying the three relations will find that

$$F(P) = \left(X + \frac{1}{X}\right) + \left(Y + \frac{1}{Y}\right) + \left(Z + \frac{1}{Z}\right) + \frac{YZ}{X} + \frac{X}{YZ} \geq 8$$

5.38.

If the points A_1, B_1, C_1 divide the sides BC, CA, respectively AB of a triangle in the rapport k, determine the minimum of the following expressions:

$$\|AA_1\|^2 + \|BB_1\|^2 + \|CC_1\|^2$$

Solution

We suppose that $k > 0$, because we work with distances.

$$\|BA_1\| = k\|BC\|; \ \|CB_1\| = k\|CA\|; \ \|AC_1\| = k\|AB\|.$$

We apply three times the Stewart theorem in the triangle ABC, with the segments AA_1, BB_1 respectively CC_1:

$$\|AB\|^2 \cdot \|BC\|(1-k) + \|AC\|^2 \cdot \|AC\|^2 \cdot k - \|AA_1\|^2 \cdot \|BC\|^2 = \|BC\|^3 (1-k)k$$

where:

(1) $\|AA_1\|^2 = (1-k)\ \|AB\|^2 + k\|AC\|^2 - (1-k)k\|BC\|^2$

(2) $\|BB_1\|^2 = (1-k)\ \|BC\|^2 + k\|BA\|^2 - (1-k)k\|AC\|^2$

(3) $\|CC_1\|^2 = (1-k)\ \|CA\|^2 + k\|CB\|^2 - (1-k)k\|AB\|^2$

By adding these three equalities we find:

$$\|AA_1\|^2 + \|BB_1\|^2 + \|CC_1\|^2 = \left(k^2 - k + 1\right)\left(\|AB\|^2 + \|BC\|^2 + \|CA\|^2\right),$$

Which take the minimum value when $k = \frac{1}{2}$, that is the case when the three lines from the problem hypothesis are the medians of the triangle.

The minimum is $\frac{3}{4}\left(\|AB\|^2 + \|BC\|^2 + \|CA\|^2\right)$.

5.39.

In the triangle ABC we construct the concurrent lines AA_1, BB_1, CC_1 such that $AB_1^2 + B_1C^2 + C_1A^2 = AB_1^2 + BC_1^2 + C_1A^2$ and one of them is the median

Prove that the other two lines are the same medians when the triangle ABC is isosceles.

Solution

Suppose that AA_1 is the median, without diminishing the problem's generality, then $A_1B = A_1C$, and the relation from the hypothesis becomes:

$$B_1C^2 + C_1A^2 = AB_1^2 + BC_1^2 \qquad (1)$$

From the concurrency of the lines AA_1, BB_1, CC_1 and from the Menelaus' theorem it results that

$$\frac{AB_1}{B_1C} = \frac{AC_1}{C_1B} \qquad (2)$$

We make the notation: $\frac{AB_1}{B_1C} = k,\ \ k > 0$, from which we have:

$$B_1C^2 + k^2C_1B^2 = k^2BC_1^2 + BC_1^2.$$

Consequently $\left(k^2 - 1\right)C_1B^2 - BC_1^2 = 0$, and from here we find $k = 1$, or $C_1B = B_1C$.

If k=1 then $AB_1 = B_1C$ and $AC_1 = C_1B$ then consequently BB_1, CC_1 are medians .

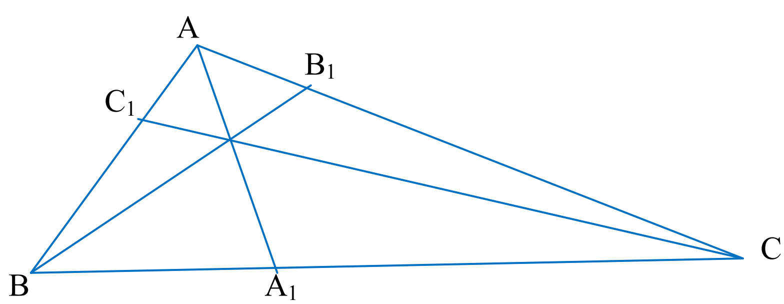
If $C_1B = B_1C$ from (2) it results that $AB_1 = AC_1$, consequently $AB = AC$, and the triangle ABC is isosceles.

5.40.

In a triangle we construct the ceviane AA_1, BB_1, CC_1 that intersect in a point P. Prove that

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A}$$

<u>Solution</u>



In the triangle ABC we apply the Ceva theorem

$$AC_1 \cdot BA_1 \cdot CB_1 = -AB_1 \cdot CA_1 \cdot BC_1 \qquad (1)$$

In the triangle AA_1B cut by the transversal AA_1, we apply also the Menelaus' theorem:

$$AC_1 \cdot BC \cdot A_1P = AP \cdot A_1C \cdot BC_1 \qquad (2)$$

In the triangle BB_1C cut by the transversal AA_1, we apply also the Menelaus' theorem:

$$BA_1 \cdot CA \cdot B_1P = BP \cdot B_1A \cdot CA_1 \qquad (3)$$

We apply one more time the Menelaus' theorem in the triangle CC_1A cut by the transversal BB_1:

$$AB \cdot C_1P \cdot CB_1 = AB_1 \cdot CP \cdot C_1B \qquad (4)$$

We divide each relation (2), (3) and (4) by the relation (1), and we obtain:

$$\frac{PA}{PA_1} = \frac{BC}{BA_1} \cdot \frac{B_1A}{B_1C} \qquad (5)$$

$$\frac{PB}{PB_1} = \frac{CA}{CB_1} \cdot \frac{C_1B}{C_1A} \qquad (6)$$

$$\frac{PC}{PC_1} = \frac{AB}{AC_1} \cdot \frac{A_1C}{A_1B} \qquad (7)$$

We'll multiply (5) by (^) and by (7), and we obtain

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A} \cdot \frac{AB_1 \cdot BC_1 \cdot CA_1}{A_1B \cdot B_1C \cdot C_1A},$$

but the last fraction is equal to 1 in conformity to the Ceva's theorem.

5.41.

Let the triangle ABC which has all the angles acute and we consider $A'B'C'$ the triangle formed by the legs of its heights.

In which conditions the following expression is maximum?

$$\|A'B'\| \cdot \|B'C'\| = \|B'C'\| \cdot \|C'A'\| + \|C'A'\| \cdot \|A'B'\|$$

Solution

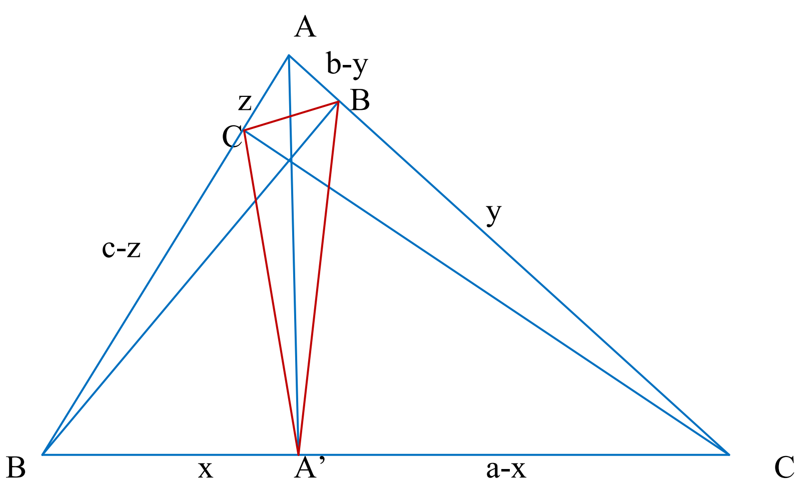
We have

$$\triangle ABC \sim \triangle A'B'C' \sim \triangle AB'C' \sim \triangle A'BC' \qquad (1)$$

We note:

$$\|BA'\| = x, \|CB'\| = y, \|AC'\| = z$$

It results that:



$$\|A'C\| = a - x, \|B'A\| = b - y, \|C'B\| = c - z$$

$$\sphericalangle BAC = \sphericalangle B'A'C = \sphericalangle BA'C'$$
$$\sphericalangle ABC = \sphericalangle AB'C' = \sphericalangle A'B'C$$
$$\sphericalangle BCA = \sphericalangle BC'A' = \sphericalangle B'C'A$$

Qualities it results relation (1).

$$\triangle A'BC' \sim \triangle A'B'C \Rightarrow \frac{\|A'C'\|}{a-x} = \frac{x}{\|A'B'\|} \qquad (2)$$

$$\triangle A'BC' \sim \triangle AB'C' \Rightarrow \frac{\|A'C'\|}{z} = \frac{c-z}{\|B'C'\|} \qquad (3)$$

$$\triangle AB'C' \sim \triangle A'B'C \Rightarrow \frac{\|B'C'\|}{y} = \frac{b-y}{\|A'B'\|} \qquad (4)$$

From the relations(2), (3) and (4) we conclude that the sum of the products from the hypothesis is equal to:

$$x(a-x)+y(b-y)+z(c-z)=\frac{1}{4}(a^2+b^2+c^2)-(x-\frac{a}{2})^2-(y-\frac{b}{2})^2-(z-\frac{c}{2})^2$$

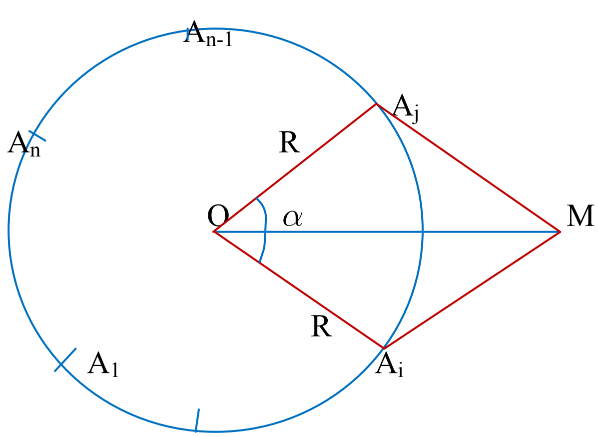
which reaches its maximum when $x=\frac{a}{2}, y=\frac{b}{2}, z=\frac{c}{2}$ that is when the heights' legs fall in the middle of the sides, therefore the $\triangle ABC$ is equilateral. The maximum of the expression is $\frac{1}{4}\left(a^2+b^2+c^2\right)$.

5.42.

Let's consider $A_1,...,A_n$ n distinct points on the circumference of a circle with the center O and radius R.

Sow that there exist two points A_i and A_j such that $\left\|\overrightarrow{OA_i}+\overrightarrow{OA_j}\right\| \geq 2R\cos\frac{180^\circ}{n}$.

Solution



A2

Because

$$S = \sphericalangle A_1OA_2 + \sphericalangle A_2OA_3 + ... + \sphericalangle A_{n-1}OA_n + \sphericalangle A_nOA_1 = 360° \text{ and}$$

$$\forall i \in \{1,2,...,n\}, \sphericalangle A_iOA_{i+1} > 0°,$$

It results that there exist at least one angle $\sphericalangle A_iOA_j \leq \frac{360°}{n}$ (if not it results that)

$S > \frac{360°}{n} \cdot n = 360°$).

$\overrightarrow{OA_i} + \overrightarrow{OA_j} = \overrightarrow{OM} \Rightarrow \left\|\overrightarrow{OA_i} + \overrightarrow{OA_j}\right\| = \left\|\overrightarrow{OM}\right\|$.

The quadrilateral $OA_iM + A_j$ is a rhomb.

While α is very small, $\left\|\overrightarrow{OM}\right\|$ is very big.

Because $\alpha \leq \frac{360°}{n}$, it results that

$\left\|\overrightarrow{OM}\right\| = 2R\cos\frac{\alpha}{2} \geq 2R\cos\frac{180°}{n}$.

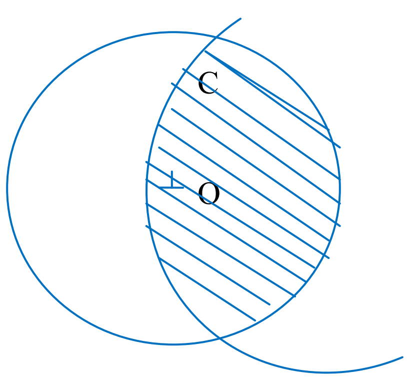
5.43.

Determine the maximum number of points which can be found on the circumference of a circle, such that the distance between two arbitrary points is greater or equal to the circle's radius.

Solution

The side of a regular hexagon inscribed in a circle has the same dimension as the radius of the respective circle. Therefore there are at least 7 points on a circle that have the property from the hypothesis, one point in the center of the circle and 6 points on the circumference such that the six points constitute the vertexes of the regular hexagon inscribed in the circle.

The selected 7 points are taken in an optimal way. For example, if we want to construct the set of point that have the property from the problem's hypothesis it would not be at all optimal of taking the first point different from the center of the circle, and not on the circumference.



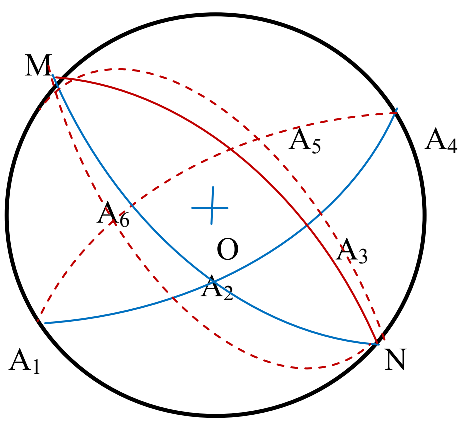
Therefore in the geometric figure included here, taking C_1 in the interior of the circle and different from the center, then in the shaded portion (which is a circle of the same radius like the initial circle and with the center in C_1, intersects the other one) it is not possible to take any other point. Therefore the best is to have the shaded portion as small as possible. It will result that C_1 must be on the circumference. From this will result that the other points will be: 5 on the circumference such that the 6 points will constitute a regular hexagon and the other in the center of the circle. Therefore we have constructed 7 points.

5.44.

How many points we can find in a sphere (and on its surface), such that the distance between any two of them to be greater or equal with the radius.

Solution

We consider the large circles of the sphere, determined by the plane A_1OA_4, where O is the center of the sphere.



On its circumference we take the points $A_1, A_2, ..., A_6$ such that we get a regular hexagon - therefore, the distance $\|A_iA_j\| \geq$ than the ray of the sphere, for which $i \neq j$. We construct a plane A_2A_5MNO perpendicular on the plane A_1OA_4 which cuts the sphere by the circle A_2A_5MN. On its circumference we take also 6 points which constitute a regular hexagon. Then, we construct the third big circle of the sphere, determined by A_3, A_6, M, N. The same, on the circumference of this last circle we take 6 points, which are the vertexes of a regular hexagon, among which are the points A_3, A_6. Etc.

We have in total 6+4+0=10 points, and if we add the center of the sphere we obtain 11 points which keep the property from the hypothesis.

This method of constructing the points is the optimal one. If we start the construction of the points, for example taking a point A which does not belong to surface of the sphere, then the sphere of center A and having the same ray will intersect a large zone of the initial sphere, but with the condition is that this zone is the smallest possible. Then A belongs to the surface. And the demonstration continues in the same way.

5.45.

Given n distinct points in a plane, connected two by two through a line:
a) What is the maximum number of lines which we can construct with these points?
b) If only m points $1 \le m \le n$, are collinear, how many distinct lines are there?
c) Prove that we cannot have $\frac{(n-2)(n+1)}{2}$ lines, regardless of how we arrange the n distinct points.

Solution

Let $A_1,...,A_n$ the n distinct points.

a) If they are three to three non-collineart, then we can form all possible lines A_iA_j with $i<j$ and $(i,j)\in\{1,...,n\}$, therefore $\frac{n(n-1)}{2}$ lines.

b) If $A_1,...,A_m$ are collinear, then the lines A_hA_k with $h<k$ and $(h,k)\in\{1,...,n\}^2$ are the same, and constitute a single line. Then it remains:

$$\frac{n(n-1)}{2}-\frac{m(m-1)}{2}+1=\frac{n^2-m^2-n+m+2}{2}$$

distinct lines.

c) $\frac{(n-2)(n+1)}{2}=\frac{n(n-1)}{2}-1$.

If the n points are three to three non-collinear, we saw that we have: $\frac{n(n-1)}{2}$ distinct lines. If m points are collinear, we have $\frac{m(m-1)}{2}-1$ lines less. But $\frac{m(m-1)}{2}-1 \Leftrightarrow \frac{m(m-1)}{2}-1-m-4=0$ which does not have a natural solution.

For example, if we have 3 collinear points, we eliminate 2 lines from the total of $\frac{n(n-1)}{2}$, but not a line as it should.

5.46.

Given n distinct points in a plane, three to three non collinear $A_1,...,A_n$, find the locus of the points $M \ne A_i$, $1\le i\le n$, such that it doesn't matter which line that passes through M and which doesn't contain any point A_i ($1 \le i \le n$) divides the plane in two semi-planes that contain the $\left[\frac{n}{2}\right]$ and the other $\left[\frac{n+1}{2}\right]$ points.

Solution

We note Π the plane that contains the points $A_1,...,A_n$ and let α be the locus we are looking for.

1) If $n=1$, then obvious $\alpha = \Pi - \{A_1\}$.

2) If $n=2$, $\alpha = [A_1, A_2] - \{A_1, A_2\}$, where $[A_1, A_2]$ represents the segment of line which unite the points A_1, A_2.

3) $n > 2$.

a) $n = 2k$. Let d_1 the line that passes through A_1 and through a point A_{s_1}, $2 \le s_1 \le n$, such that on a side and the other side of the line d_1 we find to be $k-1$ points $A_2,...,A_{s_1-1}, A_{s_1+1},...,A_n$. We have $\left[\frac{2k}{2}\right] = \left[\frac{2k+1}{2}\right] = k$. Obviously α is included in $\left[A_1, A_{s_1}\right]$. We proceed the same for all the points A_i, $1 \le i \le n$, and we find that $\alpha = \bigcap_{i=1}^{n}\left[A_i, A_{s_1}\right]$. Then, if all these segments intersect in one point, then that point would be $\alpha 1$ if not $\alpha = \Phi$.

b) $n = 2k+1$. We have $\left[\frac{n}{2}\right] = k$ and $\left[\frac{n+1}{2}\right] = k+1$. For A_1 we construct the triangle $A_1A_{u_1}A_{v_1}$, where A_{u_1} is such that the line $A_1A_{u_1}$ divides the plane in two semi-planes, one containing $k-1$, the other k points among the points A_i; in the same time that A_{v_1} is such that the line $A_1A_{v_1}$ divides also the plane in two semi-planes, one containing k and the other $k-1$ points among the points A_i. Evidently $\alpha = \subseteq \triangle A_1A_{u_1}A_{v_1}$. We continue the rational for all the points A_i, $1 \le i \le n$, and we find that $\alpha = \bigcap_{i=1}^{n} \triangle A_iA_{u_1}A_{v_1}$.

5.47.

Prove that a sphere cannot be included in the union of two spheres whose rays are strictly smaller than that of the sphere itself.

Solution

Let's S be the sphere, C the big circle, r the ray of the sphere (implicitly r is the ray of the circle C).

By reduction ad absurdum, let S_1 and S_2 the spheres that comprise this sphere and such that they are strictly inferior to S.

We note C_1 (respectively C_2) the ray of the sphere S_1 (respectively S_2), [implicitly r_1 (respectively r_2), is the ray of circle C_1 (respectively C_2)].

We intersect S with S_1.

a) $S \cap S_1 = \{P\}$ (one common point)

Let O the center of the sphere S. We construct a plane Π that contains the ray OP.

$\Pi \cap S = C$, $C \cap S_1 = \{P\}$, then $S_2 \supset C - \{P\}$, that means $r_2 \geq r$. Contradiction.

b) $S \cap S_1 = C_{SS_1}$ (a circle). It results that its ray $r_{C_{SS_1}} \leq \min\{r_1, r\} = r_1 < r$.

Then there will exist a big circle C of the sphere S which has the property that $C \cap S_1 = \Phi$. Then $S_2 \supset C$, and $r_2 \geq r$, which is absurd.

c) The case $S_1 \subset C$ and the surface of S_1 does not intersect the surface of S, then it will exist a big circle C of the sphere S such that $C \cap S_1 = \Phi$. Then $S_2 \supset C$, and then $r_2 \geq r$, which is also absurd.

d) The same prove when $S \cap S_1 = \Phi$ and $S_1 \not\subset S$.

CALCULUS

6.48.

Let's consider the numbers $k_1,...,k_p$ that form an arithmetic sequence. Prove that if $a_1,...,a_n$ form an arithmetic sequence (respectively geometric) then $a_{k_1},...,a_{k_m}$ constitute an arithmetic sequence (respectively geometric).

Solution

$k_i = k_1 + (i-1)r_1$, $1 \le i \le p$ and r_1 is the ratio of this arithmetic sequence.

a) When $a_1,...,a_n$ form an arithmetic sequence, then

$$a_{k_i} = a_{k_1} + \left(k_i - k_1\right)r_1 = a_{k_1} + rr_1(i-1),$$

where $1 \le i \le p$ and r is the ratio of the sequence $a_1,...,a_n$. Then $a_{k_1},...,a_{k_p}$ constitute also an arithmetic sequence of ratio rr_1.

b) When $a_1,...,a_n$ constitute a geometric sequence, with the ratio q, then

$a_{k_i} = a_{k_1} q^{k_i - k_1} = a_{k_1}\left(q^{r_1}\right)^{i-1}$, where $1 \le i \le p$. Then $a_{k_1},...,a_{k_p}$ constitute a geometric sequence with the ratio q^{r_1}.

6.49.

Let x_n and y_n natural sequences such that $x_n = ay_n$, $\forall n \in N^*$, and $a \ne 1$. From the arithmetic progression $b_1, b_2, ...$ we will eliminate the terms of rank x_n, $n \in N^*$. Prove that among the remaining terms there exists an arithmetic sub-progression.

Solution

We observe that if the naturals numbers $i_1,...,i_s$ constitute an arithmetic progression then $b_{i_1},...,b_{i_s}$ has the same property, since

$$2b_{i_j} = 2\left[b_1 + (i_j - 1)r\right] = 2b_1 + \left(2i_j - 2\right)r = 2b_1 + \left(i_{j-1} + i_{j+1} - 2\right)r = \left(b_1 + \left(i_{j-1} - 1\right)r\right) + \left(b_1 + \left(i_{j+1} - 1\right)r\right) = b_{i_{j-1}} + b_{i}$$

$$\begin{aligned} 2b_{i_j} &= 2\left[b_1 + (i_j - 1)r\right] = 2b_1 + \left(2i_j - 2\right)r = \\ &= 2b_1 + \left(i_{j-1} + i_{j+1} - 2\right)r = \left(b_1 + \left(i_{j-1} - 1\right)r\right) + \left(b_1 + \left(i_{j+1} - 1\right)r\right) = b_{i_{j-1}} + b_{i_{j+1}} \end{aligned}$$

(We used $2i_j = i_{j-1} + i_{j+1}$). Then we can replace $b_1, b_2, ...$ by $1, 2, ...$.

We must construct an arithmetic progression $a_1, a_2, ..$ such that $a_{i+1} \ne x_n$, $\forall (i,n) \in N \times N^*$.

A) The case $a = 0$ is trivial

B) Let $a \ne 0$.

$a_{i+1} = a_1 + ir$, where r is the ratio. Then $a_1 = ?,\ r = ?$ such that

$a_1 + ir \ne ay_n$, $\forall (i,n) \in N \times N^*$ (1)

From here we have $i \neq \frac{ay_n - a_1}{r}$. Because $i \in N$, we put the condition: $(ay_n - a_1)\frac{1}{r} \in Z$.

We take $r = a \geq 2$ (because $a \neq 0$, $a \neq 1$ and x_n and y_n are the natural sequences, then $a \in N$ or $a_1 = a - 1$. It results that $i \neq \frac{ay_n - a + 1}{a} = y_n - 1 + \frac{1}{a} \in N$ and the relation (1) is verified.

6.50.

Prove that $(f_1 \circ .... \circ f_n)' = \prod_{i=1}^{n} f_i^{'} \circ f_{i+1} \circ ... \circ f_n$.

Solution

For $i = 1$ we have an immediate result.

We suppose that the equality is true for $i = n - 1$. Then

$$(f_1 \circ .... \circ f_n)' = (f_1 \circ (f_2 \circ ... \circ f_n))' = (f_1 ' \circ (f_2 \circ ... \circ f_n)) \cdot (f_2 \circ ... \circ f_n)' =$$

$$= (f_1 ' \circ f_2 \circ ... \circ f_n) \cdot \prod_{i=2}^{n} f_i^{'} \circ f_{i+1} \circ ... \circ f_n = \prod_{i=1}^{n} f_i^{'} \circ f_{i+1} \circ ... \circ f_n .$$

6.51.

Given the sets $\Phi \neq A \subset M_1 \subset M_2$, where M_1 is everywhere dense in M_2, and $\inf A \cong$ in M_2. Then there exist an $\inf A$ in M_1 if and only if $\alpha \leq M_1$. The same question for $\sup A$.

Solution

The sufficiency.

If $\inf A = \alpha$ in M_2 and $\alpha \in M_1 \subset M_2$, then evidently $\inf A = \alpha$ in M_1.

The necessity:

Let $\alpha' = \inf A$ in M_1. Then $\alpha' \in M_1$. We know that $\inf A = \alpha$ in M_2 by hypothesis.

1) If $\alpha' > \alpha$, then $\inf A = \alpha' \neq \alpha$ in M_2. Contradiction.

2) If $\alpha' < \alpha$, because M_1 is everywhere dense in M_2, it results that there exists $\gamma \in M_1$ such that $\alpha' < \gamma < \alpha$. If $\gamma \in A$, then α is not equal to $\inf A$ in M_2 (contradiction); then $\gamma \in A$. If there exist $\gamma' \in (\alpha', \gamma)$ such that $\gamma' \in A$, then $\alpha' \neq \inf A$ in M_1. Contradiction.

Therefore:

3) $\alpha' = \alpha$, that is $\alpha \in M_1$.

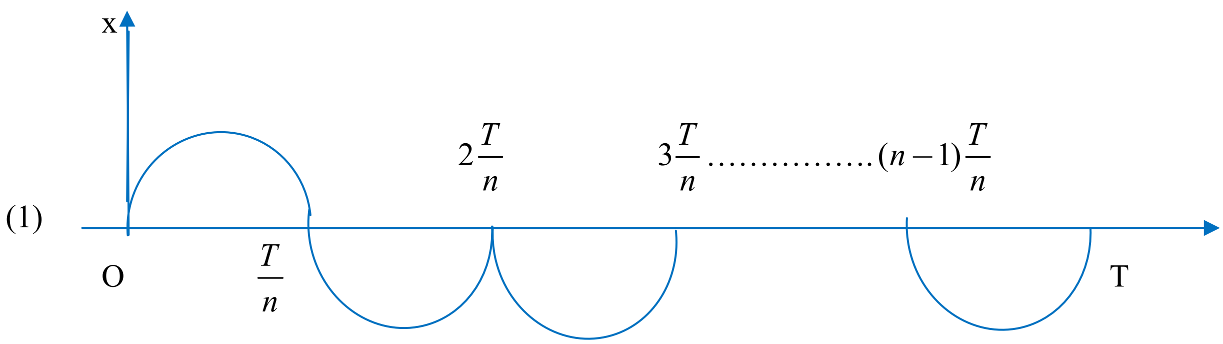
The proof for $\sup A$ is done the same way.

6.52.

Show that, given a natural number $n > 0$ and a natural number $T > 0$, there exists a function $f : R \to R$ with a period T and such that $|f|$ has the period $\frac{T}{n}$. In this case, if f is continue prove that f is null in at least $n-1$ points in the interval of length T.

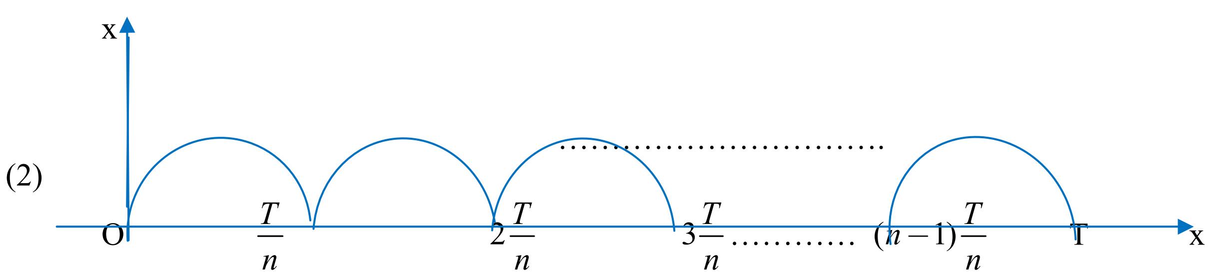
Solution

Let's consider $n \in N^*,\ T \in R_+^*$ and the function $f : R \to R$, of a period T, which has the following graphic representation.:



where all n semi-circles of (1) are equal among them.

The function $|f| : R \to R$ will have the following representation:



Then its period is $\frac{T}{n}$

Evidently, there exist an infinity of such functions, because we can replace the semi-circles of (1) by other curves as long as the property from the hypothesis will be satisfied.

We must prove the second point:

The case $n = 1$ is banal. Let's consider $n > 1$. Let k the number of points for which f is null in the interval of length T. But k is non-null, because otherwise it would result that

$|f| = f$ or $|f| = -f$ on R, that means that $|f|$ has the period T. Therefore $k \geq 1$.

$f(x) = 0 \Leftrightarrow |f|(x) = 0$.

(3) Because $|f|(x)$ is null more than one times (we have $k \geq 1$), it results that $|f|$ is null in an interval of length $\frac{T}{n}$ or in the interior of that interval or at one extremity of the interval. But in an interval of length T there are n intervals of length $\frac{T}{n}$. Therefore $|f|$ will be zero on at least $n-1$ times in an interval of length T, and therefore knowing (3) it results the last question of the problem.

6.53.

Let's consider the positive functions $f_1,..., f_n$ on an interval I such that they vary on the same direction on this interval.

Then $f_1 \cdot ... \cdot f_n$ varies on the same direction on the interval I.

Solution

We consider that all functions f_i are increasing. (Analog prove if all f_i are decreasing)

We will use the recurrence rational.

For $i = 2$. Let $x_1 < x_2$.

$$\frac{(f_1 f_2)(x_2) - (f_1 f_2)(x_1)}{x_2 - x_1} = f_1(x_1)\frac{f_2(x_2) - f_2(x_1)}{x_2 - x_1} + f_2(x_2)\frac{f_1(x_2) - f_1(x_1)}{x_2 - x_1} \geq 0$$

Therefore, $f_1 f_2$ is a an increasing function on I.

We suppose that $f_1 \cdot ... \cdot f_{n-1}$ is increasing, then $f_1 \cdot ... \cdot f_{n-1} \cdot f_n$ is increasing, because we can note $f_1 \cdot ... \cdot f_{n-1} = g$ which is positive and increasing by the recurrence's hypothesis, and $f_1 \cdot ... \cdot f_{n-1} \cdot f_n = gf_n$ which is increasing concludes the proof for $i = 2$.

6.54.

Let n a natural number and not null.

a) Determine the functions $f : R \to R$, odd, derivable $2n$ times, such that the derivative order $2n$ are non negative.

2)Determine the functions $g : R \to R$, even, derivable $2n-1$ times, such that the derivative of the order $2n-1$ are not negative.

Solution

a) $f(x)=-f(-x),\ \forall x\in R$. It results that $f^{(2n)}(x)=\left(-1\right)^{2n+1}f^{(2n)}(-x)$.

But $f^{(2n)}(x)\geq 0$ and $f^{(2n)}(-x)\leq 0$, which implies that $f^{(2n)}(x)=0$ on R.

By integration $2n$ times, we obtain:

$$f(x)=a_{2n-1}x^{2n-1}+a_{2n-2}x^{2n-2}+...+a_1x+a_0$$

with $a_i\in R$, $0\leq i\leq 2n-1$.

Since f is odd, it results:

$$a_{2n-1}x^{2n-1}+a_{2n-2}x^{2n-2}+...+a_1x+a_0=a_{2n-1}x^{2n-1}-a_{2n-2}x^{2n-2}+...+(-1)^{i+1}a_ix^i+a_1x^1-a_0.$$

We obtain: $a_{2n-2}=a_{2n-4}=...=a_0$. Therefore $f(x)=a_{2n-1}x^{2n-1}+a_{2n-3}x^{2n-3}+...+a_1x$

b) $g(x)=g(-x)$, $\forall x\in R$. It results that $g^{(2n-1)}(x)=(-1)^{(2n-1)}g^{(2n-1)}(-x)$, which implies that $g^{(2n-1)}(x)=0$ on R.

By integrating $2n$ times, we have $g(x)=b_{2n-2}x^{2n-2}+b_{2n-3}x^{2n-3}+...+b_1x+b_0$ with $b_i\in R,\ \ 0\leq i\leq 2n-2$.

Because g is even, it results that:

$$g(x)=b_{2n-2}x^{2n-2}+b_{2n-3}x^{2n-3}+...+b_1x+b_0=$$
$$=b_{2n-2}x^{2n-2}-b_{2n-3}x^{2n-3}+...+(-1)^ib_ix^i+...+(-1)b_1x-b_0.$$

We obtain: $b_{2n-3}=b_{2n-5}=...=b_1=0$, then $g(x)=b_{2n-2}x^{2n-2}+b_{2n-4}x^{2n-4}+...+b_2x^2+b_0$.

6.55.

A function $f:R\rightarrow R$ admits a symmetry center if and only if it exist two real constants a,b such that the function $g(x)=f(x+a)-b$ is odd.

In these conditions, the symmetry center is for coordinates $\left(a,b\right)$.

Solution

The necessity.

Let $C(\alpha,\beta)$ the center of symmetry. We raise $a=\alpha,b=\beta$. We execute a translation of axes, by moving the origin in $C(a,b)$. The formulae of the change of the reference system CXY to $CX'Y'$ are

$$\begin{cases}x'=x-a\\y'=y-b\end{cases}\quad\Leftrightarrow\quad\begin{cases}x=x'+a\\y=y'+b\end{cases}$$

Then $y=f(x)$ becomes $y'+b=f(x'+a)$ m where $y'=f(x+a)-b$. We make the notation $g(x'=f(x'+a)-b,\ \ g:R\rightarrow R$. The function g admits a center of symmetry, which is the same with the axes origin. Then g is odd.

The sufficiency.

g being odd, it results that g admits the axes origin as center of symmetry.
We execute the axes' translation, moving the origin in $C''(-a,-b)$.
The formulae of the change of the reference system CXY to $CX''Y''$ are

$$\begin{cases} x'' = x + a \\ y'' = y + b \end{cases} \Leftrightarrow \begin{cases} x = x'' - a \\ y = y'' - b \end{cases}$$

Then $y = g(x) = f(x+a) - b$ becomes $y'' - b = f(x'' - a + a) - b$, that is $y'' = f(x'')$.
Because g admits the symmetry center $O(o,o)$ in the reference system CXY, this implies then that f admits the symmetry center $O(a,b)$ in the reference system $O''X''Y''$.

6.56.

In an system of orthogonal axes, the function $f : R \to R$ has an axis of symmetry if and only if there exists a real constant such that the function $g(x) = f(x+a)$, is even.

In this condition, the symmetry axis has the equation $x = a$.

Solution

The necessity.

Let $x = \alpha$ be the symmetry axis of function f. We set $x = a$. We execute a translation of axes, by moving the origin in $O'(a,0)$. The equations of change from the (OXY) to $(O'X'Y')$ are:

$$\begin{cases} x' = x - a \\ y' = y \end{cases} \Leftrightarrow \begin{cases} x = x' + a \\ y = y' \end{cases}$$

Then $y = f(x)$ becomes $y' = f(x'+a) = g(x')$, $g : R \to R$. f admits as axis of symmetry the line $x = a$. Il results that g admits as axis of symmetry the line $x' = 0$ (in $(O'X'Y')$), that is the axis $O'Y'$.

Therefore g is even.

The sufficiency:

Because g is even, we have that g admits the axis OY as axis of symmetry.

We'll execute a translation of axes, by moving the origin in $O''(-a,0)$. The movement of (OXY) in $(O''X''Y'')$ is done by:

$$\begin{cases} x'' = x + a \\ y' = y \end{cases} \Leftrightarrow \begin{cases} x = x'' - a \\ y = y'' \end{cases}$$

Then $y = g(x) = f(x+a)$ becomes $y'' = f(x'')$. g admits as axis of symmetry the line $x = 0$, from which it results that f admits as axis of symmetry the line $x'' = a$.

6.57.

We consider the continuous functions $A, B, C: I \to R$, where I is an interval on R, with $A(x) \le C(x) \le B(x)$, $\forall x \in I$. Let's consider $x_1, x_2 \in I$, $x_1 < x_2$ and $A(x_1) = f(x_1)$, $B(x_2) = f(x_2)$, where f is a continuous function on $[x_1, x_2]$.

Prove that there exist $x_3 \in [x_1, x_2]$ such that $C(x_3) = f(x_3)$.

Solution

We'll use the absurd method.

By reduction ad absurdum we suppose that the conclusion is not true, then $\forall x \in [x_1, x_2]$ we have that $C(x) \ne f(x)$, or $C(x) - f(x) \ne 0$. C and f being continuous on $[x_1, x_2]$, it results that $C - f$ is continuous on $[x_1, x_2]$. Then: $\forall x \in [x_1, x_2]$, $C(x) - f(x) > 0$, or $\forall x \in [x_1, x_2]$, $C(x) - f(x) < 0$.

We consider the first situation (the second will be similar).

Because $A(x) \le C(x) \le B(x)$ on I, we have: $A(x) - f(x) \le C(x) - f(x) \le B(x) - f(x)$ on $[x_1, x_2]$. Therefore $C(x_2) - f(x_2) \le B(x_2) - f(x_2) = 0$, which is a contradiction.

Then we have the conclusion tha there exist $x_3 \in [x_1, x_2]$ such that $C(x_3) = f(x_3)$ is true.

6.58.

Find the real numbers $a, b, c,$ such that

$$\lim_{x\to\infty} \frac{a\left(2x^3 - x^2\right) + b\left(x^3 + 5x^2 - 1\right) + c\left(3x^3 - x^2\right)}{a\left(5x^4 - x\right) + b\left(-x^4\right) + c\left(4x^4 + 1\right) + 2x^2 + 5x} = 1$$

Solution

We can write:

$$\lim_{x\to\infty} \frac{(2a + b - 3c)x^3 + (-a + 5b - c)x^2 - b}{(5a - b + 4c)x^4 + 2x^2 + (-a + 5)x + c} = 1 \qquad (1)$$

If $5a - b + 4c \ne 0$ then the limit (1) is equal to $0 \ne 1$.

Then $5a - b + 4c = 0$. It results that $2a + b - 3c = 0$, because if not, the limit (1) would be equal to $\pm\alpha \ne 1$. Then:

$$\lim_{x\to\infty} \frac{(-a + 5b - c)x^2 - b}{2x^2 + (-a + 5)x + c} = 1,$$

from where $\dfrac{-a + 5b - c}{2} = 1$.

and the real numbers a, b, c verify the system:

$$\begin{cases} 5a-b+4c \\ 2a+b-3c=0 \\ -a+5b-c=2 \end{cases}$$

When resolved we fund the values:

$$a=\frac{2}{109},\ \ b=\frac{46}{109},\ \ c=\frac{14}{109}.$$

6.59.

Given the natural numbers a_i, b_j between 0 and 9, with $a_i \neq 9 \neq b_m$ and $a_1+1=x_1$, $9-a_i=y_i$, $i=\overline{2,n}$, and $9-b_j=z_j$, $j=\overline{1,m}$. Compute:

$$\lim_{p\to\infty}\left[\overline{0,x_1}-\left(\sum_{i=2}^{n}\overline{0,\underbrace{0...0}_{i}y_i}+\sum_{k=0}^{p}\sum_{j=1}^{m}\overline{0,\underbrace{0........0}_{n-1+j+k-m}z_j}\right)\right]$$

Solution

We note

$$\alpha=\sum_{i=2}^{n}\overline{0,\underbrace{0...0}_{i}y_i}=\overline{0,0y_2...y_n}$$

and

$$\beta(p)=\sum_{k=0}^{p}\sum_{j=1}^{m}\overline{0,\underbrace{0........0}_{n-1+j+k-m}z_j}=0,\underbrace{0....0}_{n}\underbrace{z_1....z_m}_{1}\underbrace{....}_{......}\underbrace{z_1....z_m}_{p}$$

If we take $t=b_m+1$, we have:

$$\gamma_p=\overline{0,x_1}-(\alpha+\beta(p))=\overline{0,a_1...a_n\underbrace{b_1...b_m}_{1}...\underbrace{b_1...b_m}_{p-1}\underbrace{b_1...b_m}_{p}t} \qquad (1)$$

We show that $\gamma_p \underset{p\to\infty}{\to} \overline{0,a_1...a_n(b_1...b_m)}=\gamma_0$.

Let $\varepsilon>0$, $\exists p_0=p(\varepsilon)\in N$, p_0 being the smallest natural number which has the property :

$$p_0>-\frac{\lg\varepsilon\cdot 10^n}{m},$$

such that

$$\forall p\geq p_0,\ p\in N,\ \left|\gamma_p-\gamma_0\right|=10^{-n-pm}<10^{-n}\cdot 10^{\lg\varepsilon\cdot 10^n}=\varepsilon,$$

and it results (1).

6.60.

We consider the functions $f_1, f_2: R\to R$ such that $\lim\limits_{p\to\infty}\left|f_1(x)\right|=\infty$ and $\lim\limits_{x\to\infty}\left|f_2(x)\right|=a$.

Show that $\lim_{x\to\infty}\left(f_1(x)\cdot\left[\frac{1}{f_2(x)}\right]\right)$ exists and compute this limit , where $|\alpha|$ represents the integer part of α.

Solution

Then

$$\lim_{x\to\infty} f_1(x)=+\infty \text{ or } -\infty \qquad (1)$$

We note $f(x)=f_1(x)\left[\frac{1}{f_2(x)}\right]$;

Discussion:

A) If $a=0$, then

$\lim_{x\to\infty}\frac{1}{f_2(x)}=+\infty$ or $-\infty$ or it doesn't exist and the same $\lim_{x\to\infty}\left[\frac{1}{f_2(x)}\right]=+\infty$ or $-\infty$ or it doesn't exist.. From here $\lim_{x\to\infty} f(x)=+\infty$ or $-\infty$ (conform to (1) and A) or we cannot conclude anything.

B) $a\le 1$ and $a\ne 0$ then $\left[\frac{1}{a}\right]\ne 0$. From where, also, $\lim_{x\to\infty} f(x)=+\infty$ or $-\infty$ (conform (1) and B), that is conform to (1) and the sign of $\left[\frac{1}{a}\right]$.

C) $a>1\Rightarrow\left[\frac{1}{a}\right]=0$. We have $f_2(x)\underset{x\to\infty}{\to} a>1$. Then it exists $\alpha>0$ such that $a=1+\alpha$. Let $V_\alpha=(a-\alpha,a+\alpha)$ a neighborhood of a ; $1\notin V_\alpha$. Let a sequence $x_n\to\infty$, then $f_2(x_n)\to a$ (conform to the limit's definition.). $f_2(x_n)$ is a real sequence which tends to a . It results that outside of V_α we find at most a limited number of terms of the sequence $f_2(x_n)$ if and only if it exists at most a limited number of terms that have the property $f_2(x_i)\le 1$. Therefore, the majority if terms are found in V_α, therefore it exists $n_\alpha\in N^*$ such that $\forall i>n_\alpha$, $f_2(x_i)>1$.

Therefore the sequence $\left(\left[\frac{1}{f_2(x_n)}\right]\right)_{n\in N^*}$ is the following:

$$\left[\frac{1}{f_2(x_1)}\right],...,\left[\frac{1}{f_2(x_{n_\alpha})}\right],0,0,0,..$$

Therefore , with the exception of a limited number of null terms, this sequence is the null sequence.

It results that the sequence $(f_2(x_n))_{n\in N^*}$ is the following:

$$f_1(x_1)\cdot\left[\frac{1}{f_2(x_1)}\right],...,f_1(x_{n_\alpha})\cdot\left[\frac{1}{f_2(x_{n_\alpha})}\right],0,0,0,...$$

Then, also, with the exception of a limited number of null terms, this new sequence is the constant, null sequence.

In conclusion: $\lim\limits_{x\to\infty} f(x)=0$

6.61.

Compute without using l'Hospital theorem)

$$\lim_{x\to x_0}\prod_{i=1}^{n}\frac{\sqrt[r_i]{f_i(x)}-\sqrt[r_i]{f_i(x_0)}}{\sqrt[s_i]{f_i(x)}-\sqrt[s_i]{f_i(x_0)}}$$

with the conditions that the anterior solutions exist. Generalization.

<u>Solution</u>

We multiply each fraction by the conjugate of the numerator and denominator.

Let $1\le i\le n$, we note:

$$A=\sqrt[r_i]{f_i(x)}$$

$$B=\sqrt[r_i]{f_i(x_0)}$$

$$C=\sqrt[s_i]{f_i(x)}$$

$$D=\sqrt[s_i]{f_i(x_0)}$$

We have:

$$\frac{A-B}{C-D}=\frac{(A-B)\left(A^{r_i-1}+A^{r_i-2}B^1+\ldots+A^1B^{r_i-2}+B^{r_i-1}\right)\left(C^{s_i-1}+A^{s_i-2}D^1+\ldots+C^1D^{s_i-2}+D^{s_i-1}\right)}{(C-D)\left(A^{r_i-1}+A^{r_i-2}B^1+\ldots+A^1B^{r_i-2}+B^{r_i-1}\right)\left(C^{s_i-1}+C^{s_i-2}D^1+\ldots+C^1C^{s_i-2}+D^{s_i-1}\right)}=$$

$$=\frac{\cancel{A^{r_i}-B^{r_i}}}{\cancel{C^{s_i}-D^{s_i}}}\cdot\frac{\left(C^{s_i-1}+C^{s_i-2}D^1+\ldots+D^{s_i-1}\right)}{\left(A^{r_i-1}+A^{r_i-2}B^1+\ldots+B^{r_i-1}\right)}\xrightarrow[x\to x_0]{}\frac{s_i}{r_i}\cdot\frac{\left(\sqrt[s_i]{f_i(x_0)}\right)^{s_i-1}}{\left(\sqrt[r_i]{f_i(x_0)}\right)^{r_i-1}}=\frac{s_i}{r_i}\cdot\left(f_i(x_0)\right)^{\frac{s_i-r_i}{s_ir_i}}$$

From where the limit from the hypothesis will be equal to:

$$\prod_{i=1}^{n}\frac{s_i}{r_i}\cdot\left(f_i(x_0)\right)^{\frac{s_i-r_i}{s_ir_i}}$$

Generalization:

$$\lim_{x\to x_0}\frac{\prod\limits_{i=1}^{n}\dfrac{\sqrt[r_i]{f_i(x)}-\sqrt[r_i]{f_i(x_0)}}{\sqrt[s_i]{f_i(x)}-\sqrt[s_i]{f_i(x_0)}}}{\prod\limits_{j=1}^{m}\dfrac{\sqrt[r_i]{f_i(x)}-\sqrt[r_i]{f_i(x_0)}}{\sqrt[s_j]{f_i(x)}-\sqrt[s_j]{f_i(x_0)}}}=\begin{cases}\prod\limits_{i=0}^{n}\dfrac{s_i}{r_i}\cdot\left(f_i(x_0)\right)^{\frac{s_i-r_i}{s_ir_i}}\ , & if\ \ n=m\\ 0 & if\ \ n>m\\ does\ not\ exist & if\ \ n<m\end{cases}$$

in the conditions in which the anterior solutions exist.

If $n<m$ the limit does not exist because the denominator is equal to zero, then the numerator is not null; from where the two lateral limits are different.

Remark:

If there exist at least a constant function f_{i_0}, $1 \le i_0 \le \min\{m,n\}$, then the limit does not exist.

The same thing for the limit from the problem's enounce.

ALGEBRA

7.62.

Compare the sets:

$$A=\left\{X \mid X=\sum_{i=1}^{n} a_i K_i + a,\ \text{with}\left(K_1,..,K_n\right)\in Z^n,\ \text{and}\, a_1,...,a_n \text{are integer constants such that}\left(a_1,...,a_n\right)=1\right\}$$

$$B=\left\{X \mid X=\sum_{j=1}^{m} b_j K_j + b,\ \text{with}\left(K_1,..,K_m\right)\in Z^m,\ \text{and}\, b_1,...,b_m,b \text{ are integer constants}\right\}$$

Solution

First of all we prove that $A=Z$.

It is obvious that $A\subseteq Z$, and also $Z\subseteq A$ because

$$\forall Z\in Z,\exists\left(K_1^0,...,K_n^0\right)\in Z^n : \sum_{1}^{n} a_i K_i^0 + a = Z$$

Because the greatest common divisor $\left(a_1,...,a_n\right)=1$ and 1 divides $Z-a$.

We see immediately that $B\subseteq Z$.

If $\left(b_1,...,b_m\right)$=1 then $B=Z=A$, otherwise $B\subset A$ and $B\neq A$.

7.63.

Given a prime number p and M a set of p consecutive natural numbers, prove that M cannot be divided in two disjoint subsets M_1, M_2, with $M_1\cup M_2=M$, as the product of the numbers in M_1 are equal to the product of the numbers in M_2.

Solution

Because M contains p consecutive natural numbers, then M constitute a complete system of residues modulo p. Then:

$$\exists n_0\in M : n_0=\mathcal{M}_p\ \forall n\in M-\left\{n_0\right\},\ n\neq \mathcal{M}_p$$

We consider that $n_0\in M_1$ (the same proof for the contraire cae).

The product of the numbers in M_1 is divisible by p, but the product of numbers from M_2 is not divisible by p, because p is a prime number and because it does not exist any element in M_2 which is a multiple of p. From here, the product from M_1 cannot be equal to the product from M_2.

7.64.

Let's consider M a set which contains m natural numbers, $m \geq 2$, and $n \in \mathbb{N}^*$, $n < m$. We note: $K = \left[\frac{m-1}{n}\right] + 1$, where $[X]$ represents the integer part of X.

Prove that there exists a subset M' of M, such that:

a) M' contains at least K elements

b) The difference between two random elements of M' is a multiple of n.

Solution

The set M has the form: $M = \{a_1, ..., a_m\}$, where all $a_i \in \mathbb{N}$. $\forall i \in \{1, ..., m\}, a_i = \mathcal{M}_{n + r_i}$ with $r_i \in \{0, 1, 2, ..., n-1\}$.

Let $m = q \cdot n + r,\ q \in \mathbb{N},\ 0 \leq r \leq n-1$. Because $n < m$, it results that $q \in \mathbb{N}^*$. We construct the set $M' \subseteq M$. Because the condition b) are achieved it must that M' contains only elements from M which divided by n will give the same residue.

The residues obtained from the division by n are: 0,1,…,n-1. We have n equivalence classes modulo n.

The problem is reduced to prove that there exists a class which contains at least K elements. M' will be exactly this class.

1) The case when $r = 0$. Then $K = \left[\frac{m-1}{n}\right] + 1 = \left[\frac{q : n - 1}{n}\right] + 1 = q$.

If $m = q : n$ elements of M are equally distributed in the n classes, then each class contains q elements, and the problem is solved.

In the contrary case, there exists at least a class which contains at least q elements and also, in this case the problem is resolved.

2) The case when $r \neq 0$. Then $K = \left[\frac{m-1}{n}\right] + 1 = \left[\frac{qn + r - 1}{n}\right] + 1 = q + 1$.

The $m = qn + r$ elements of M are distributed randomly in the n classes, it results that there exist at least a class which contains at least $q+1$ elements (if not it would result that there would be a maximum of qn elements $< n$), and the problem is completely solved.

7.65.

Let's consider the homogeneous polynomials $P_n(x, y)$ and $Q_n(x, y)$ of n degree in x, y. If $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ then $\frac{P_n(a_1, b_1)}{Q_n(a_1, b_1)} = \frac{P_n(a_2, b_2)}{Q_n(a_2, b_2)}$

Solution

Let

$$P_n(x,y)=\alpha_0 x^n+\alpha_1 x^{n-1}y^1+...+\alpha_{n-1}xy^{n-1}+\alpha_n y^n$$

and

$$Q_n(x,y)=\beta_0 x^n+\beta_1 x^{n-1}y^1+...+\beta_{n-1}xy^{n-1}+\beta_n y^n$$

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}\Rightarrow\frac{a_1}{a_2}=\frac{b_1}{b_2}\Rightarrow\frac{a_1^n}{a_2^n}=\frac{b_1^n}{b_2^n}=\frac{a_1^{n-i}b_1^i}{a_2^{n-1}b_2^i}=\frac{\alpha_i a_1^{n-i}b_1^i}{\alpha_i a_2^{n-1}b_2^i}$$

$0\le i\le n$.

Since the sum of the numerators divided by the sum of the denominators is equal with each rapport, we have:

$$\frac{\alpha_0 a_1^n+\alpha_1 a_1^{n-1}b_1^1+...+\alpha_n b_1^n}{\alpha_0 a_2^n+\alpha_1 a_1^{n-1}b_2^1+...+\alpha_n b_2^n}=\frac{a_1^n}{a_2^n}$$

The same way, we obtain:

$$\frac{\beta_0 a_1^n+\beta_1 a_1^{n-1}b_1^1+...+\beta_n b_1^n}{\beta_0 a_2^n+\beta_1 a_1^{n-1}b_2^1+...+\beta_n b_2^n}=\frac{a_1^n}{a_2^n},$$

Then:

$$\frac{P_n(a_1,b_1)}{P_n(a_2,b_2)}=\frac{Q_n(a_1,b_1)}{Q_n(a_2,b_2)}$$

Therefore, the conclusion.

7.66.

Let's consider a natural number

$p\ge 2$ and a sequence such that $a_1=1,\ a_{n+1}=pa_n+1,\ \ \forall n\in\mathbb{N}^*$. Prove that $\forall K\in\mathbb{N}^*$, K can be uniquely expressed as follows: $K=t_1a_{n_1}+...+t_1a_{n_1}$ with $1\le t_i\le p-1$ for $i\in\{1,2,...,l-1\}$ and $1\le t_i\le p$ and $n_1>...>n_1$.

Solution

We deduct immediately that $a_n=\dfrac{p^n-1}{p-1}$ which is a sequence of natural numbers, strictly ascending, unlimited. Then: $\exists n_1\in\mathbb{N}^*$ such that $a_{n_1}\le k<a_{n_1+1}=pa_{n_1}+1$.

From here k can be written uniquely as follows:

$k=t_1a_{n_1}+r_1\le pa_{n_1}$, with $0\le r<a_{n_1}$.

If $r_1=0$, it results that $t_1=\dfrac{k}{a_{n_1}}$ and $1\le t_1\le\left[\dfrac{a_{n_1}p}{a_{n_1}}\right]=p$

If $r_1\ne 0$, it results that there exists $n_2\in\mathbb{N}^*$ such that $a_{n_2}\le r_1<a_{n_2+1}=pa_{n_1}+1\Rightarrow r_1=t_2a_{n_2}+r_2$

$$a_{n_2} \le r_1 < a_{n_1} \Rightarrow n_2 < n_1;\ t_1 = \frac{k - r_1}{a_{n_1}} \le \frac{pa_{n_1} - 1}{\frac{a_{n_1}}{a_{n_1}}} < p;$$

Then $1 \le t_1 \le p - 1$.

And k is written uniquely $k = t_1 a_{n_1} + t_2 a_{n_2} + r_2$, and the process continue. After a limited number of steps we reach $r_1 = 0$. We will have the same prove for $n_i > n_{i+1}, i \in \{1,...,1-1\}$ and $1 \le t_i \le p-1$, $i \in \{1,...,1-1\}$ and $1 \le t_1 \le p$, and the problem is solved.

7.67.

If $a_1,...,a_n,b$ are positive real numbers with $b \le a_1 + ... + a_n$ and $\alpha \notin \{-a_1,...,-a_n,-b\}$, then $\frac{b}{\alpha+b} \le \frac{a_1}{\alpha+a_1} + ... + \frac{a_n}{\alpha+a_n}$.

Solution

We will use the recurrence rational for $n \in \mathbb{N}^*$.

For $n = 1$, if $b \le a_1$, we have $\frac{b}{\alpha+b} - \frac{a_1}{\alpha+a_1} \le 0 \Leftrightarrow \frac{\alpha(b-a_1)}{(\alpha+b)(\alpha+a_1)} \le 0$; which is true, and taking into account the hypothesis/

We suppose that the inequality is true for all the values smaller or equal to n.

We will prove it for $n+1$: $b \le (a_1 + ... + a_n) + a_{n+1}$ and in conformity to the hypothesis of the recurrence we have: $\frac{b}{\alpha+b} \le \frac{(a_1+...+a_n)}{\alpha+(a_1+...+a_n)} + \frac{a_{n+1}}{\alpha+a_{n+1}}$, but $a_1 + ... + a_n \le a_1 + ... + a_n$, then if we apply one more time the hypothesis of the recurrence, we obtain:

$$\frac{b}{\alpha+b} \le \frac{a_1}{\alpha+a_1} + \frac{a_2}{\alpha+a_2} + ... + \frac{a_n}{\alpha+a_n} + \frac{a_{n+1}}{\alpha+a_{n+1}}$$

7.68.

Let's consider the expression: $E(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, $(x,y) \in \mathbb{R}^2$, with A,B,C,D,E,F real, and $A^2 + C^2 \ne 0$. Find a necessary and sufficient condition for $E(x,y)$ to admit an extreme.

Solution

Because $A^2 + C^2 \ne 0$, it results that at least A or C is not null, let it be A; (the result will be similar if $C \ne 0$).

We suppose that $E(x,y)$ admits an extreme.

$$E(x,y)=A\left(x^2+\frac{B}{A}xy+\frac{D}{A}x\right)+Cy^2+Ey+F=$$

$$=A\left(x+\frac{B}{2A}y+\frac{D}{2A}\right)^2-\frac{B^2}{4A}y^2-\frac{D^2}{4A}-\frac{2BD}{4A}y+Cy^2+Ey+F=$$

$$=A\left(x+\frac{B}{2A}y+\frac{D}{2A}\right)^2+\frac{4AC-B^2}{4A}y^2+\frac{4AE-2BD}{4A}y^2+\left(F-\frac{D^2}{4A}\right)$$

$\dfrac{4AC-B^2}{4A}\neq 0$, because if not $E(x,y)$ would not have extreme.

We have

$$E(x,y)=A\left(x+\frac{B}{2A}y+\frac{D}{2A}\right)^2+\frac{4AC-B^2}{4A}\cdot\left[y^2+2\frac{2AE-BD}{4AC-B^2}y+\left(\frac{2AE-BD}{4AC-B^2}\right)^2\right]-$$

$$-\frac{4AC-B^2}{4A}\cdot\left(\frac{2AE-BD}{4AC-B^2}\right)^2+\left(F-\frac{D^2}{4A}\right)=$$

$$=A^2\left(x+\frac{B}{2A}y+\frac{D}{2A}\right)^2+\frac{4AC-B^2}{4A}\left(y+\frac{2AE-BD}{4AC-B^2}\right)^2+F-\frac{D^2}{4A}-\frac{(2AE-BD)^2}{4A(4AC-B^2)} \qquad (1)$$

Evidently $E(x,y)$ admits only one extreme if A and $\dfrac{4AC-B^2}{4A}$ have the same sign, that is if $A\dfrac{4AC-B^2}{4A}>0$, from where $4AC-B^2>0$, which constitute a necessary condition for $E(x,y)$ to admit un extreme (as long as the two are positive and have a minimum, if not a maximum). But $4AC-B^2>0$ constitute also a sufficient condition, because $E(x,y)$ can be written as (1) and A and $\dfrac{4AC-B^2}{4A}$ have the same sign.

7.69.

Let's consider the integers $A=\overline{a_1...a_{2n}}+\overline{a_{2n}...a_1}$ and $B=\overline{a_1...a_n}-\overline{a_n...a_1}$ written in base $b\in\mathbb{N}^*-\{1\}$. Show that A is divisible by $b+1$ and B is divisible by $b-1$.

Solution

We'll compute the criteria of divisibility by $b+1$ and by $b-1$ in base b.

$b^i\equiv(-1)^i\,(\bmod\, b+1)$ and $b^i\equiv 1\,(\bmod\, b-1)$, $\mathrm{i}\in\mathbb{N}$.

Because

$$A\equiv\left(a_{2n}-a_{2n-1}+...+(-1)^k a_k+...+a_2-a_1\right)+$$

$+\left(a_1 - a_2 + ... + (-1)^k a_k + ... + a_{2n-1} - a_{2n}\right) \equiv 0\left(\mathrm{mod}\, b+1\right).$

Similarly:

$B \equiv \left(a_n - a_{n-1} + ... + a_1\right) - \left(a_1 + a_2 + ... + a_n\right) \equiv 0\left(\mathrm{mod}\, b+1\right).$

7.70.

Let's consider a natural number a and p a non- null integer. Determine the number of elements of the set $M = \left\{a, \overline{aa}, ..., \underbrace{\overline{a...a}}_{n}\right\}$ which are divisible by p. Discussion.

Solution

Let a be written in base 10 in the form: $a = \overline{a_1 ... a_s}$, with $a_i \in \mathbb{N}$, $1 \le i \le s$, $s \in \mathbb{N}^*$.An element $\alpha \in M$ is $\alpha = \underbrace{\overline{a...a}}_{n}$ with $1 \le n \le m$, n integer.

$\alpha = a \cdot 10^{(n-1)s} + a \cdot 10^{(n-2)s} + ... + a \cdot 10^{1s} + a = a\left(1 + 10^s + ... + 10^{(n-1)s}\right) = a\dfrac{10^{sn} - 1}{10^s - 1}.$

We must find the n for which α divides p.

Let $d = (a, p)$. Then $a = da_1$ and $p = dp_1$ and $(a_1, p_1) = 1$. Then we must determine n such that $\cancel{d}a_1 \dfrac{10^{sn} - 1}{10^s - 1} \mid \cancel{d}p_1$.

1) If $(p, 10) = 1$, let's consider δ the oreder of the class of residues of 10 in rapport to mod $p_1\left(10^s - 1\right)$. We have also: $10^\delta - 1 \equiv 0\left(\mathrm{mod}\, p_1\left(10^s - 1\right)\right)$, then $10^\delta - 1 \equiv 0\left(\mathrm{mod}\left(10^s - 1\right)\right)$ and it results that $\delta = ks$, $k \in \mathbb{N}^*$. Then it exists exactly $\left[\dfrac{m}{k}\right] = \left[\dfrac{ms}{\delta}\right]$ elements in M which are divisible by p.

7.71.

Prove that $\forall k \in \mathbb{N}^* - \{1\}$ there exist an infinity of natural numbers whose property is that they admit exactly k positive divisors.

Solution

Any natural number A is written in the following form:

$A = p_1^{\alpha_1} ... p_s^{\alpha_s}$, (obviously $A \notin \{0,1\}$) where p_i are prime numbers, $i \in \{1, ..., s\}$ and $p_i \ne p_j$ for $i \ne j$, $\alpha_i \in \mathbb{N}^*$ with $i \in \{1, ..., s\}$, $s \in \mathbb{N}^*$ (which is the canonic form of a number, and which is unique).

Re note by $d(A)$ the number of positive divisors of A.

Then

$$d(A)=\prod_{i=1}^{n}(\alpha_i+1) \qquad (1)$$

Our problem is reduced to prove that the equation $\prod_{i=1}^{n}(\alpha_i+1)=k$ has solutions in $(\mathbb{N}^*)^s$ (the unknown are $\alpha_1,...\alpha_s$ and s). We take $s=1\in\mathbb{N}^*$ and $\alpha_1=k-1\in\mathbb{N}^*$ and we obtain $k-1+1=k$. From all the numbers $n=p^{k-1}$, with a random prime number, has exactly k positive divisors (we have an infinity of numbers n because we have an infinity of prime numbers p).

We can see that equation (1) has other solutions . For example, if $k=k_1,...k_t$ with all $k_i\in\mathbb{N}^*-\{1\}$, we have the infinite solution, $s=t$, $\alpha_1=k_1-1,...,\alpha_t=k_t-1$ from where $n=p_1,...,p_t$ and where all p_j are the different prime numbers.

7.72.

Knowing that a_i, $i\in\{1,2,...,n\}$, satisfying the conditions of existence for all n logarithms, solve the equation:

$$\log_{a_1}\log_{a_2}...\log_{a_n}x=b$$

Solution

$$\log_{a_1}\left(\log_{a_2}\log_{a_3}...\log_{a_n}x\right)=b\Leftrightarrow\log_{a_2}\log_{a_3}...\log_{a_n}x=a_1$$

Where

$$\log_{a_2}\left(\log_{a_3}...\log_{a_n}x\right)=a_1^b\Leftrightarrow\log_{a_3}\log_{a_4}...\log_{a_n}x=a_2^{a_1^b}$$

Where

$$\log_{a_3}\left(\log_{a_4}...\log_{a_n}x\right)=a_2^{a_1^b}\Leftrightarrow\log_{a_4}\log_{a_5}...\log_{a_n}x=a_3^{a_2^{a_1^b}}$$

………………………………………………………………………………

$$\Leftrightarrow\log_{a_{n-1}}\log_{a_n}x=a_{n-2}^{\cdot^{a_1 b}}$$

$$\Leftrightarrow\log_{a_n}x=a_{n-1}^{\cdot^{a_1 b}}$$

$$\Leftrightarrow x=a_n^{\cdot^{a_1 b}}$$

Which is the solution of the problem.

7.73.

If $a,b\in\mathbb{R}_+^*$, $b\neq 1$ and $\forall\alpha\in\mathbb{Q}$, $a\neq b^\alpha$, then $\log_b a\notin\mathbb{Q}$, and reciprocal.

Solution

First of all we observe that this problem is the result of several particular problems; for example "show that $\log_{21} 10$ is not a rational number", etc.

The proof will be done using the absurd rational.

We suppose that $\log_b a \frac{m}{n} \in \mathbb{Q}$, with least common divisor $(m,n)=1$.

It results that $b^{\frac{m}{n}} = a$. By raining this equality to the n th power, we have $b^m = a^n$, with $m,n \in \mathbb{Q}$, $a^n \neq b^{n\alpha}$, where $n\alpha \in \mathbb{Q}$ (because $a>0$, $b>0$),

Then, if we cannot write it as a rational power of b ,we would not be able to write a^n any more as a rational power of b.

And, it results that $a^n \neq b^m$, $\forall m \in \mathbb{Q}$. Contradiction.

Therefore $\log a \in \mathbb{Q}$.

Reciprocal:

If $\log_b a \notin \mathbb{Q}$, then obviously $a \neq b^\alpha$, $\forall \alpha \in \mathbb{Q}$, because if not it will result that $\log_b b^{\alpha\alpha} = \alpha \in \mathbb{Q}$.

And the problem is completely proved.

7.74.

Let $s \neq 0$ a natural number. Determine the natural numbers n that verify the propriety $\sqrt[s]{n}$ divides n. (we note $[x]$ the integer part of x).

Solution

$\forall n \in \mathbb{N} \quad \exists p \in \mathbb{N}:\ p^s \leq n < (p+1)^s = p^s + C_s^1 p^{s-1} + \ldots + C_s^{s-1} p^1 + 1$.

From here n can be written: $n = p^s + k$ with $0 \leq k < C_s^1 p^{s-1} + \ldots + C_s^{s-1} p^1 + 1$ and $k \in \mathbb{N}$. It results $\left[\sqrt[s]{n}\right] = p$. Because $\left[\sqrt[s]{n}\right]$ divides n, we obtain that p divides k. Then $k = \alpha : p$, with $\alpha \in \mathbb{N}$, and $0 \leq k \leq (p+1)^s - (p^s + 1)$ from where:

$$0 \leq \alpha \leq \frac{(p+1)^s - (p^s+1)}{p} \in \mathbb{N}$$

Therefore the natural numbers that have the property from the hypothesis are:

$n = p^s + \alpha p$, with the property $\alpha \in \mathbb{N}$ and $0 \leq \alpha \leq \dfrac{(p+1)^s - (p^s+1)}{p}$, $p \in \mathbb{N}^*$, and also the trivial solution $n=0$, because 0 divides 0.

7.75.

Let p a natural number, $p \geq 2$ and the function:

$$\beta_p(x) = \left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \dots,\ x \in \mathbb{Z}_+ .$$

Show that if $x = \left(\overline{a_n \dots a_1 a_0}\right)_p$ then:

$$\beta_p(x) = \frac{1}{p-1}\left[x - \left(a_0 + a_1 + \dots + a_n\right)\right]$$

Solution

$$x = \left(\overline{a_n \dots a_1 a_0}\right)_p = a_n p^n + \dots + a_1 p^1 + a_0$$

$$\left[\frac{x}{p}\right] = a_n p^{n-1} + a_{n-1} p^{n-2} + \dots + a_1 p^1 + a_1 \text{ because } \left[\frac{a_0}{p}\right] = 0$$

$$\left[\frac{x}{p^2}\right] = a_n p^{n-2} + a_{n-1} p^{n-3} + \dots + a_2 \text{ because } 0 \leq a_1 p^1 + a_0 < p^2$$

…………………………………………………………………………………..

$$\left[\frac{x}{p^n}\right] = a_n$$

$$\left[\frac{x}{p^m}\right] = 0 \text{ for all } m \text{ natural and } m \geq n+1 .$$

Therefore

$$\beta_p(x) = a_n\left(1 + \dots + p^{n-1}\right) + a_{n-1}\left(1 + \dots + p^{n-2}\right) + \dots + a_2\left(1 + p\right) + a_1 =$$

$$= a_n\left(1 + \dots + p^{n-1}\right) + a_{n-1}\left(1 + \dots + p^{n-2}\right) + \dots + a_2\left(1 + p\right) + a_1 =$$

$$= a_n \frac{p^n - 1}{p-1} + \dots + a_2 \frac{p-1}{p-1} + a_1 \frac{p-1}{p-1} =$$

$$= \left(a_n p^n + \dots + a_1 p^1 - a_n - \dots - a_1\right)\frac{1}{p-1} = \frac{1}{p-1}\left[x - \left(a_0 + a_1 + \dots + a_n\right)\right].$$

7.76.

Prove the inequality:

$$\left(\frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_1^2}\right) + \left(\frac{a_2^2}{a_3^2} + \frac{a_3^2}{a_2^2}\right) + \dots + \left(\frac{a_n^2}{a_1^2} + \frac{a_1^2}{a_n^2}\right) \geq$$

$$\geq \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)^2 + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right)^2 + \dots + \left(\frac{a_n}{a_1} + \frac{a_1}{a_n}\right)^2$$

where

$a_1,...,a_n \in \mathbb{R}\setminus\{0\},\ \ n\geq 2$

Solution

It is sufficient to prove that: $\left(\frac{a_1^2}{a_2^2}+\frac{a_2^2}{a_1^2}\right)\geq\left(\frac{a}{b}+\frac{b}{a}\right)^2$, where $a,b\in\mathbb{R}\setminus\{0\}$.

After computing the powers, passing all the terms to the right side and executing all the reductions of all similar terms it results that:

$$\frac{a^4}{b^4}+\frac{b^4}{a^4}-\frac{a^2}{b^2}-\frac{b^2}{a^2}\geq 0$$

We note $\frac{a^2}{b^2}=u$. We have

$$u^2+\frac{1}{u^2}-u-\frac{1}{u}\geq 0.$$

$$\left(u+\frac{1}{u}\right)^2-\left(u+\frac{1}{u}\right)-2\geq 0,\ u=\frac{a^2}{b^2}>0$$

We note $t=u+\frac{1}{u}\geq 2$; we have $t^2-t-2\geq 0$, that is $(t+1)(t-2)\geq 0$, inequality which is true for $t\geq 2$.

Therefore each parenthesis from the right side, squared, is greater or equal to the correspondent parenthesis in the left side squared.

7.77.

Show that if $a_1+a_2+...+a_n=a_1...a_n,\ \ (a_i\neq 0,\ i\in\{1,2,...,n\})$, then

$$a_1\left(\frac{1}{a_2}+\frac{1}{a_3}+...+\frac{1}{a_n}\right)+a_2\left(\frac{1}{a_3}+\frac{1}{a_n}+...+\frac{1}{a_n}+\frac{1}{a_1}\right)+...+a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_{n-1}}\right)+n=$$

$$=\sum_{i=1}^{n}a_1...a_{i-1}a_{i+1}...a_n.$$

Solution

The left side of the equality can be written:

$$\left(\frac{a_1}{a_2}+\frac{a_1}{a_3}+...+\frac{a_1}{a_n}\right)+\left(\frac{a_2}{a_3}+\frac{a_2}{a_n}+...+\frac{a_2}{a_n}+\frac{a_2}{a_1}\right)+...+\left(\frac{a_n}{a_1}+\frac{a_n}{a_2}+...+\frac{a_n}{a_{n-1}}\right)+n=$$

$$=\left(\frac{a_2}{a_1}+\frac{a_3}{a_1}+...+\frac{a_n}{a_1}\right)+\left(\frac{a_1}{a_2}+\frac{a_3}{a_2}+...+\frac{a_n}{a_2}\right)+...+\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+\frac{a_{n-1}}{a_n}\right)+n=$$

$$=\frac{a_1...a_n-a_1}{a_1}+\frac{a_1...a_n-a_2}{a_2}+...+\frac{a_1...a_n-a_n}{a_n}+n=$$

$$= \sum_{i=1}^{n} a_1 ... a_{i-1} a_{i+1} ... a_n - n + n = \sum_{i=1}^{n} a_1 ... a_{i-1} a_{i+1} ... a_n .$$

7.78.

Let the integer numbers a, b, a_i, b_j with $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., m\}$. Show that if x_i, y_j (with $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., m\}$) are integer numbers, the expression:

$$E(x_1, ..., x_n, y_1, ..., y_m) = \frac{\sum_{i=1}^{n} a_i x_i + a}{\sum_{j=1}^{m} b_j y_j + b}$$

has integer values if and only if the greatest common divisor of the numbers $a_1, ..., a_n, b_1, ..., b_n$, b divides a .

Solution

The expression $E(x_1, ..., x_n, y_1, ..., y_m)$ will have integer values if and only if the equality $E(x_1, ..., x_n, y_1, ..., y_m) = t$ admits integer solutions, where the unknown are $x_1, ..., x_n, y_1, ..., y_m, t$. This equation is equivalent to:

$$\sum_{i=1}^{n} a_i x_i - \sum_{j=1}^{m} b_j y_j t - bt = -a \tag{1}$$

$$\Leftrightarrow \sum_{i=1}^{n} a_i x_i - \sum_{j=1}^{m} b_j t_j - bt = -a$$

with

$$t_j = y_j, t, j \in \{1, ..., m\}. \tag{2}$$

Necessity

If this equation admits solutions in $\mathbb{Z}$, then it results that the greatest common divisor of the numbers $a_1, ..., a_n, b_1, ..., b_m, b$ divide a .

Sufficiency

1) The case $b \neq 0$. Because the greatest common divisor of $a_1, ..., a_n, b_1, ..., b_m, b$ divide b , it results that the equation (2) admits solutions in $\mathbb{Z}$, then it exist $t \in \mathbb{Z}$ such that $E(x_1, ..., x_n, y_1, ..., y_m) = t$.

2) The case $b = 0$. We note $d_1 = (a_1, ..., a_n)$ and $d_2 = (b_1, ..., b_m)$. From which $D = (a_1, ..., a_n, b_1, ..., b_m, b) = (d_1, d_2)$ divides a . The equation (1) becomes:

$$\sum_{i=1}^{n} a_i x_i - d_2 \left(\sum_{j=1}^{m} b'_j y_j \right) \cdot t = -a \tag{3}$$

where $b_j = d_2 b'_j$, $1 \leq j \leq m$.

Because $(b'_1, ..., b'_m) = 1$, it results that there exist $(y_1^0, ..., y_m^0) \in \mathbb{Z}^m$ such that

$\sum_{j=1}^{m} b'_j \, y_j^0 = 1$. From here we have:

$$\sum_{i=1}^{n} a_i x_i - d_2 t = -a \qquad (4)$$

Because $D = (d_1, d_2) = (a_1, ..., a_n, d_2)$ divides a, it results that exists t in $\mathbb{Z}$ such that $E(x_1, ..., x_n, y_1, ..., y_m) = t$.

7.79.

Let $a_i \in \mathbb{R}, \ i \in \{1, 2, ..., n\}$ and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{1 \le i_1 \le .. \le i_k \le n} \prod_{j=1}^{k} (x - ax_j)$, k natural odd number.

Show that whatever are the values $a_1, ..., a_n$ the function does not have the same sign on the whole real axis.

Solution

Let $m = \min_i \{a_i\}$ and $M = \max_i \{a_i\}$. We have $m \le M$ (the sign + does not bother us).

For $x \in (-\infty, m)$ we have $x - a_i < 0, \ x - a_i < 0, \ \forall i \in \{1, 2, ..., n\}$ and $\prod_{j=1}^{k} \left(x - a_{i_j}\right) < 0$ because k is odd and all factors of the product are negative.

Then $f(x) < 0, \ \forall x \in (-\infty, m)$

The same: $f(x) > 0, \ x \in (M, \infty)$.

And the signs are different and $\forall a_1, ..., a_n$ in $\mathbb{R}$ the function is both negative and positive.

7.80.

If $\alpha_1, ..., \alpha_n$ are natural numbers not null, show that

$$1 + n \sum_{k=1}^{n} \sum_{1 \le i_1 \le .. \le i_k \le n} \alpha_{i_1} ... \alpha_{i_k} = \prod_{h=1}^{n} (\alpha_h + 1)$$

Solution

Let's consider the natural number $a = p_1^{\alpha_1} ... p_n^{\alpha_n}$, where $p_1, ..., p_n$ are distinct numbers two by two.

We'll determine the number of positive divisors D. We know that: $D=\prod_{h=1}^{n}\left(\alpha_h+1\right)$.

Then, we apply another method to compute D: we write all the divisors (k represents the number of different prime factors, of divisor d):

$k=1$. We have $\alpha_1+...+\alpha_n$ are the divisors of $p_1^1,...,p_n^{\alpha_1},...,p_n^1,...,p_n^1,...,p_n^n$;

$k=2$. We have $\sum_{1\le i_1\le i_2\le n}\alpha_{i_1}\alpha_{i_2}$ divisors:

$$\begin{cases} p_1p_2^1,...,p_1p_2^{\alpha_2} \\ .... \\ p_1^{\alpha_1}p_2^1,...,p_1^{\alpha_1}p_2^{\alpha_2} \end{cases} \qquad \text{(are } \alpha_1\alpha_2 \text{ divisors)}$$

....

$$\begin{cases} p_{n-1}p_n^1,...,p_{n-1}p_n^{\alpha_n} \\ .... \\ p_{n-1}^{\alpha_{n-1}}p_n^1,...,p_{n-1}^{\alpha_{n-1}}p_n^{\alpha_n} \end{cases} \qquad \text{(are } \alpha_{n-1}\alpha_n \text{ divisors)}$$

....

$k=l$. We follow the same process, we have $\sum_{1\le i_1\le..\le i_l\le n}\alpha_{i_1}\alpha_{i_2}...\alpha_{i_l}$ divisors.

Because k can take the l values until n, it results that $D=1+\sum_{k=1}^{n}\sum_{1\le i_1\le..\le i_k\le n}\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}$ (we added 1 because, if $k=0$ we have the divisor 1 for a).

7.81.

Determine the natural number of n digits such that the following expression $\frac{\overline{x_1...x_n}}{\sum_{h=0}^{m-1}x_{i_h+1}x_{i_h+2}...x_{i_{h+1}}}$ is maximum, knowing that $i_0=0,i_1,...,i_{m-2},i_{m-1}=n$ are fixed, and that all the numbers are written in base b.

Solution

$\overline{x_{i_h+1}x_{i_h+2}...x_{i_{h+1}}}=a_{h+1}$, $0\le h\le m-1$. In this case the rapport becomes:

$$R=\frac{a_1b^{n-i_1}+a_2b^{n-i_1-i_2}+...+a_m}{a_1+a_2+...+a_m}$$

We note $c_j=b^n-\sum_{h=1}^{j}i_h$, $0\le j\le m$. Then $R=\frac{\sum_1^m a_jc_j}{\sum_1^m a_j}=1+\frac{\sum_1^{m-1}a_j\left(c_j-1\right)}{\sum_1^m a_j}$

Which is maximum when $a_m = 0$ (because $c_m = 1$).

Then $R_{\max} = 1 + \frac{\sum_1^{m-1} a_j (c_j - 1)}{\sum_1^{m-1} a_j} = 1 + (c_{m-1} - 1) + \frac{\sum_1^{m-1} a_j (c_j - 1 - c_{m-1} + 1)}{\sum_1^{m-1} a_j} =$

$$= c_{m-1} + \frac{\sum_1^{m-2} a_j (c_j - c_{m-1})}{\sum_1^{m-1} a_j},$$

which is maximum when $a_{m-1} = 0$. Then $R_{\max} = c_{m-1} + \frac{\sum_1^{m-2} a_j (c_j - c_{m-1})}{\sum_1^{m-1} a_j}$ and the process continues.

After a limited number of steps, we have:

$$R_{\max} = c_3 + \frac{a_1(c_1 - c_3) + a_2(c_2 - c_3)}{a_1 + a_2} = c_3 + (c_2 - c_3) + \frac{a_1(c_1 - c_2)}{a_1 + a_2}$$

Which is maxim when $a_2 = 0$. Therefore, $R_{\max} = c_2 + \frac{a_1(c_1 - c_2)}{a_1} = c_1 = b^{n-i_1}$.

And the numbers we

Re a looking for are: $x_1 x_2 ... x_{i_1} \underbrace{0...0}_{n-i_1}$ written in base b.

7.82.

We consider all residue modulo m classes $C_1, C_2, ..., C_k$, prime with m, and $a_i \in C_i,\ 1 \le i \le k$. Prove that $m \mid \sum_{1 \le i_1 < .. < i_s \le k} a_{i_1} ... a_{i_s}$

Solution

It is obvious $k = \varphi(m)$, where φ is the Euler's indicator

1) $m = 0 \Rightarrow \varphi(m) = 2 \Rightarrow s = 1$

$C_1 = \{-1\}; C_2 = \{+1\}$ and then $a_1 = -1, a_2 = +1$, the sum $S = -1 + 1$ and 0 divides 0.

2) $m = \pm 1 \Rightarrow \varphi(m) = 1$ and $\pm 1 \mid s$, for any $S \in \mathbb{Z}$.

3) $|m| \ge 2$. We note $S_j = \sum_{1 \le i_1 < .. < i_j \le k} a_{i_1} ... a_{i_j},\ 1 \le j \le k$

(A) We construct $f(x) = \prod_{i=1}^{k} (x - a_i) = \sum_{j=0}^{k} (-1)^{k-j} x^{k-j} s_j$ taking $s_0 = 1$.

Propriety 1: If $(a,m)=1$, then $(m-a,m)=1$.

Property 2: If $m\neq\pm1$, then $\varphi(m)=\mathrm{M}_2$ (Their proof is banal)

Then $k=2k_1$, $k_1\in\mathbb{Z}$, and the set $\{a_1,,...,a_{k_1}\ a_{k_1+1},...,a_{2k_1}\}\equiv\{a_1,,...,a_{k_1}\ \text{-}a_{k_1},...,-a_1\}$ and

(B) $f(x)=\prod_{i=1}^{k_1}\left(\left(x-a_i\right)\left(x+a_i\right)\right)(\bmod m)$

If we compare (A) and (B), we observe that for s odd, $0<s\leq k$, we have $(-1)^{k-s}S_s\equiv 0(\bmod m)$, which is equivalent to $m\,|\,S_s$.

(We used the property 3: If $a\in C_{i_0}$, then $-a\in C_{j_0}$, $i_0\neq j_0$. Then the set $\{a_1,,...,a_{k_1}\ \text{-}a_{k_1},...,-a_1\}$ contains exactly an element of each of the $2k_1$ classes of prime residues of m modulo m.

7.83.

Let φ a permutation on the set $\{1,2,...,n\}$. Then $\frac{1}{n}\cdot\sum_{h=1}^{n}\left|\varphi(h)-h\right|\leq\frac{n-1}{2}+\frac{1}{n}\left[\frac{n}{2}\right]$.

Solution

For the permutation $\omega=\begin{pmatrix}1 & 2 & ... & n-1 & n\\ n & n-1 & ... & 2 & 1\end{pmatrix}$ we have:

$$\sum_{h=1}^{n}\left|\omega(h)-h\right|=2\left[(n-1)+(n-3)+(n-5)+...\right]=$$

$$=2\sum_{k=1}^{\left[\frac{n}{2}\right]}(n-2k+1)....=2\left[\frac{n}{2}\right]\left(n-\left[\frac{n}{2}\right]\right)=\frac{n(n-1)}{2}+\left[\frac{n}{2}\right].$$

We prove now using the recurrence method for $n\in\mathbb{N}$, $n\geq2$, that the sum:

$S=\sum_{h=1}^{n}\left(n-\left[\frac{n}{2}\right]\right)$ gets the maximum value when $\varphi=\omega$.

For $n=2$ and 3 we can verify it very easy .

We suppose that the property is true for the values less than $n+2$. We'll prove that it is true for $n+2$:

$$\omega=\begin{pmatrix}1 & 2 & ... & n+1 & n+2\\ n+2 & n+1 & ... & 2 & 1\end{pmatrix}.$$

Knowing that $\omega'=\begin{pmatrix}2 & ... & n+1\\ n+1 & ... & 2\end{pmatrix}$ is a permutation of n elements and for which S will have the same value as for the permutation $\omega''=\begin{pmatrix}1 & ... & n\\ n & ... & 1\end{pmatrix}$, in other words the maximum (ω''

was obtained from ω' by reducing each element by one unit) in conformity to the hypothesis of the recurrence.

The permutation of two elements $\eta = \begin{pmatrix} 1 & \dots & n+2 \\ n+2 & \dots & 1 \end{pmatrix}$ gives the maximum value for S (in conformity with the hypothesis of the recurrence). But ω is obtained starting with ω' and η:

$$\omega(h) = \begin{cases} \omega'(h), \text{ if } h \notin \{1, n+2\} \\ \eta(h), \quad \text{otherwise} \end{cases}$$

7.84.

Let p an integer number $p \geq 2$ and $a_i^{(k)} \in \mathbb{R}$, where $i \in \{1,2,...,n\}$, $k \in \{1,2,...,m\}$.

Then $\left(\sum_{i=1}^{n} \left(\sum_{k=1}^{m} a_i^{(k)} \right)^2 \right)^{\frac{1}{p}} \leq \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \left(a_i^{(k)} \right)^2 \right)^{\frac{1}{p}}$.

Solution

First of all we prove that

$$\left(\sum_{i=1}^{n} \left(a_i^{(1)} + a_i^{(2)} \right)^2 \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \left(a_i^{(1)} \right)^2 \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \left(a_i^{(2)} \right)^2 \right)^{\frac{1}{p}} \tag{1}$$

We can compute the power of p of this inequality because both sides are positive.

We have:

$$\cancel{\sum_{i=1}^{n} \left(a_i^{(1)} \right)^2} + \cancel{\sum_{i=1}^{n} \left(a_i^{(2)} \right)^2} + 2\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} \leq \cancel{\sum_{i=1}^{n} \left(a_i^{(1)} \right)^2} + \cancel{\sum_{i=1}^{n} \left(a_i^{(2)} \right)^2} + \sum_{k=1}^{n-1} c_p^k \alpha^{p=k} \beta^{p=k}$$

where $\alpha = \left(\sum_{i=1}^{n} \left(a_i^{(1)} \right)^2 \right)^{\frac{1}{p}}$ and $\beta = \left(\sum_{i=1}^{n} \left(a_i^{(2)} \right)^2 \right)^{\frac{1}{p}}$.

(A) If $p = 2k$, then

$$C_p^k \alpha^k \beta^k \geq 2(\alpha\beta)^k = 2\left(\left(\sum_{i=1}^{n} \left(a_i^{(1)} \right)^2 \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left(a_i^{(2)} \right)^2 \right)^{\frac{1}{p}} \right)^k \geq 2\left(\left(\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} \right)^{\frac{2}{p}} \right)^k \geq$$

$$\geq 2\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)}$$

We used the inequality Cauchy-Buniakouski-Schwarts.

(B) Let $p = 2k+1$.

a) $\alpha \leq \beta$. It results:

$$C_p^{k+1} \alpha^k \beta^{k+1} = C_p^{k+1} \alpha^k \beta^k \beta^{\frac{1}{2}} \beta^{\frac{1}{2}} \geq C_p^{k+1} \beta^{k+\frac{1}{2}} \alpha^{k+\frac{1}{2}} = C_p^{k+1} (\alpha\beta)^{k+\frac{1}{2}} \geq$$

$$\geq 2\left(\left(\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)}\right)^{\frac{2}{p}}\right)^{k+\frac{1}{2}} \geq 2\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)}$$

b) $\alpha > \beta$. It results that

$$C_p^k \alpha^{k+1}\beta^k = C_p^k \alpha^{k+\frac{1}{2}} > C_p^k \alpha^{k+\frac{1}{2}}\beta^{k+\frac{1}{2}} > 2(\alpha\beta)^{k+\frac{1}{2}} \geq 2\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} .$$

Now, from (1), using the absurd reasoning, we obtain what we're looking for.
The case $m = 2$ is verified.
We suppose that the property is true for the values smaller than m.
We prove for m.

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_i^{(k)}\right)^2\right)^{\frac{1}{p}} = \left(\left(\sum_{i=1}^{n} a_i^{(1)} + \sum_{i=1}^{n} a_i^{(k)}\right)^2\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}\left(a_i^{(1)}\right)^2\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}\left(\sum_{k=2}^{m} a_i^{(k)}\right)^2\right)^{\frac{1}{p}} \leq \sum_{k=1}^{m}\left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^2\right)^{\frac{1}{p}}$$

7.86.

Prove the inequality: $n! > 2^{n-1}\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!$

Solution

a) $n = 2k$

$$n! = (2\cdot 4\cdot 6\cdot\cdot 2k)\cdot[1\cdot 3\cdot 5\cdot\cdot(2k-1)] = 2^k\cdot[1\cdot 3\cdot 5\cdot\cdot k]\cdot[1\cdot 3\cdot 5\cdot\cdot(2k-1)] =$$

$$= 2\cdot k![1\cdot 3\cdot 5\cdot\cdot(2k-1)] > 2^k\cdot k!2^{k-1}\cdot(k-1)! = 2^{2k-1}\cdot(k-1)!\cdot k!$$

b) $n = 2k+1$

$$n! = 2\cdot 4\cdot 6\cdot\cdot 2k\cdot 1\cdot 3\cdot 5\cdot\cdot(2k-1)\cdot(2k+1) = 2^k\cdot k!\cdot 1\cdot 3\cdot 5\cdot\cdot(2k+1) >$$

$$> 2^k\cdot k!2\cdot 4\cdot\cdot 2k = 2^k\cdot k!\cdot 2k\cdot k! = 2^{2k}\cdot k!k!$$

From these two results we conclude that: $n! > 2^{n-1}\cdot\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!$

7.87.

Prove the inequality:

$$\sum_{i=0}^{m}\sum_{j=0}^{2^i-1}\left(\left[\frac{n-j}{2^i}\right]-1\right)\cdot\prod_{h=0}^{2^{m+1}-1}\left[\frac{n-h}{2^{m+1}}\right]!n! > 2$$

Solution

(1) $n! > 2^{n-1} \cdot \left[\frac{n}{2}\right]! \cdot \left[\frac{n-1}{2}\right]!$

(See the anterior problems)
We can easily prove that

(2) $$\left[\frac{\left[\frac{n-a}{2^p}\right]}{2}\right] = \left[\frac{n-a}{2^{p+1}}\right]$$

with $0 \le a < 2^p$

(3) $$\left[\frac{\left[\frac{n-a}{2^p}\right]-1}{2}\right] = \left[\frac{n-a-2^p}{2^{p+1}}\right]$$

with $0 \le a < 2^p$,

(For this we put $n = 2^{p+1} \cdot k + \alpha$ and $0 \le \alpha < 2^{p+1}$.
Then we use the recurrence method to prove the inequality from enounce. We consider this inequality as a mathematical proposition which depends of m, then $P(m)$. We apply the recurrence on m.

For $m = 0$ we obtain the inequality (1) which is true. We suppose that $P(m)$ is true. We must prove that $P(m+1)$ is true.

From (1), (2), and (3) it results

(4) if $0 \le h < 2^{m-1}$ then $\left[\frac{n-h}{2^{m+1}}\right]! > 2^{\left[\frac{n-h}{2^{m+1}}\right]-1} \cdot \left[\frac{n-h}{2^{m+2}}\right]! \left[\frac{n-h-2^{m+1}}{2^{m+2}}\right]!$

For each h natural, $0 \le h < 2^{n-1}$ we rapport (4) in $P(m)$, and then we execute all computations.

We'll find exactly $P(m+1)$.

Remark: To generalize this inequality we replace 2 by a random natural p, $2 \le p < n$, and we follow a similar method. We'll find that the writing is more complicated and the inequality will be less fine.

7.88.

Having $a_{ij} \in [0,1]$, $p_{ij} > 0$, $1 \le i \le n$, $1 \le j \le m$; $n, m \in \mathbb{N}^*$. Prove that

$$\prod_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^{p_{ij}} + \prod_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij}^{p_{ij}} - \sum_{j=k+1}^{m} a_{ij}^{p_{ij}} \right) \le m^n + (2k-m)^n$$

Where k is a natural number

Solution:

We note

$$P_1 = \prod_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^{p_{ij}} = \prod_{i=1}^{n}\left(a_{i1}^{p_{i1}} + a_{i2}^{p_{i2}} + ... + a_{im}^{p_{im}}\right) = \left(a_{11}^{p_{11}} + a_{12}^{p_{12}} + ... + a_{1m}^{p_{1m}}\right)...\left(a_{n1}^{p_{n1}} + a_{n2}^{p_{n2}} + ... + a_{nm}^{p_{nm}}\right)$$

If all the multiplications are done, one can see P_1 is an algebraic sum which contains m^n terms and each of them has the sign +.

It is noted

$$P_2 = \prod_{i=1}^{n}\left(\sum_{j=1}^{k} a_{ij}^{p_{ij}} - \sum_{j=k+1}^{m} a_{ij}^{p_{ij}}\right) = \prod_{i=1}^{n}\left(a_{i1}^{p_{i1}} + ... + a_{ik}^{p_{ik}} - a_{ik+1}^{p_{ik+1}} - ... - a_{im}^{p_{im}}\right) =$$

$$= \left(a_{11}^{p_{11}} + ... + a_{1k}^{p_{1k}} - a_{1k+1}^{p_{1k+1}} - ... - a_{1m}^{p_{1m}}\right)...\left(a_{n1}^{p_{n1}} + ... + a_{nk}^{p_{nk}} - a_{nk+1}^{p_{nk+1}} - ... - a_{nm}^{p_{nm}}\right)$$

Also, when all the multiplications are done it is obtained an algebraic sum which contains m^n terms, some have the sign +, others -. The terms of P_2 are equal two by two to the terms of P_1 in absolute value. We note $P = P_1 + P_2$

In P all the negative terms of p will be reduced, because each has a positive corresponding. (The null terms of P_2 which have the sign – will be reduced with the null terms of P_1 which have the sign + and which have the same absolute value.

Therefore, without affecting the generality, it is considered that the null terms of P_1 and P_2 are positive or negative, in function of the sign + or – which are found in front of them.)

Thus, P will be equal to two times the sum of all the positive terms of P_2. A positive term of P_2 has the following form: $a_{i_1 j_1}^{p_{i_1 j_1}} ....a_{i_m j_m}^{p_{i_m j_m}} \in [0,1]$.

It results that P P is inferior or equal to two times the number of positive terms of P_2 (the equality will be true when all $a_{ij} = 1$).

Let's consider the sequence

$$b_1 = k, b_{n+1} = (2k-m)b_n + (m-k)m^n$$

By the recurrence method it is possible to prove that b_n will be calculated exactly the number of positive terms of P_2.

Because we are only interested in the sign of the terms, we convine to designate by $+$ a positive term, and by – a negative term.

The case $n=1$ implies $P_2^1 = \underbrace{+....+}_{k}\cdot\underbrace{-...-}_{m-k}$, therefore the number of the positive terms is k and $b_1 = k$. It is supposed that the property is true for n, and we have to prove it for $n+1$. For n, we have

$$P_2^{(n)} = \left(\underbrace{+....+}_{b_n}\cdot\underbrace{-...-}_{m^n - b_n}\right)$$

where b_n represents the number of positive terms of $P_2^{(n)}$. For $n+1$ we have

$$P_2^{(n)+1} = \left(\underbrace{+....+}_{b_n}\cdot\underbrace{-...-}_{m^n - b_n}\right)\cdot\left(\underbrace{+....+}_{k}\cdot\underbrace{-...-}_{m-k}\right)$$

The number of positive terms, in this case, will be:

$k \cdot b_n + (m-k)(m^n - b_n) = (k-m+k)b_n + (m-k)bm^n = (2k-m)b_n + (m-k)m^n = b_{n+1}$

But b_n is a linear recurrent and homogenous sequence.

From which it results that $b_n = \frac{1}{-2}\left(m^n + (2k-m)^n\right)$, therefore $P \le m^n + (2k-m)^n$.

7.89.

Let's consider a polynomial $P(x)$ of $r < n-1$ degree which for the distinct numbers $x_1,...,x_n$ takes the values $y_1,...,y_n$. For $1 \le i \le n$ we consider the equations

$$x^{n-1} - s_{i,1}x^{n-2} + s_{i,2}x^{n-3} + ... + (-1)s_{i,n-1} = 0$$

which have the solutions $x_1,...,x_{i-1},x_{i+1},...,x_n$

Then

$$\sum_{i=1}^{n} y_i s_{i,h} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} = 0$$

with $0 \le h \le n-r-2$, where by convention is noted $s_{i,0} = 1$.

Solution

The polynomial which, for the distinct numbers $x_1,...,x_n$ takes the values, respectively, $y_1,...,y_n$ is

$$P(x) = y_1 \frac{(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_2)(x_1-x_3)...(x_1-x_n)} + y_2 \frac{(x-x_2)(x-x_3)...(x-x_n)}{(x_2-x_2)(x_2-x_3)...(x_2-x_n)} + ... +$$

$$+ y_2 \frac{(x-x_1)(x-x_2)...(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})},$$

and this is the one with the smallest degree having this property (according to Lagrange).

The degree of $P(x) = r < n-1$, this implies that the coefficients of $x^{n-1}, x^{n-2},...,x^{r+1}$ are null. But the coefficients of x^k, with $r+1 \le k \le n-1$, are exactly the expressions:

$$(-1)^{n-k-1} \sum_{i=1}^{n} s_{i,n-k-1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}$$

When $r+1 \le k \le n-1$, we have $0 \le n-k-1 \le n-r-2$ and it is noted $h = n-k-1$.

$s_{i,h}$ is the sum of all the products of h factors ($h \neq 0$), which it is formed with the numbers $x_1,...,x_{i-1},x_{i+1},...,n_n$ (that is the h-th sum of the relation of Viète, applied to the equation from the problem).

7.90.

Let's consider the polynomial with integer coefficients $P(x)=\sum_{i=0}^{n} a_i x^i$. Prove that for $p,q\in\mathbb{Z}$, if $P\left(\frac{p}{q}\right)\neq 0$, then $\left|P\left(\frac{p}{q}\right)\right|>\frac{1}{|q^m|}$, $m\in\mathbb{N}^*$, $m\geq n$.

Solution

$$P\left(\frac{p}{q}\right)=\left|a_n p^n + a_{n-1}p^{n-1}q^1+...+a_1p^1q^{n-1}+a_0q^n\right|\frac{1}{|q^n|}\geq$$

$$\geq\frac{1}{|q^m|}\left|a_n p^n + a_{n-1}p^{n-1}q\ +...+a_1p^1q^{n-1}+a_0q^n\right|\geq\frac{1}{|q^m|}$$

because: $m\geq n$ implies that

$$\frac{1}{|q^n|}\geq\frac{1}{|q^m|}$$

and $a_n p^n + a_{n-1}p^{n-1}q+...+a_1p^1q^{n-1}+a_0q^n\in\mathbb{Z}-\{0\}$ and its absolute value is ≥ 1.

7.91.

Prove that $\sum_{s_1+...+s_p=k} C_{n_1}^{s_1}C_{n_2}^{s_2}....C_{n_p}^{s_p}=C_{n_1+...n_p}^{k}$

Solution

We have

$$(1+x)^{n_1}(1+x)^{n_2}...(1+x)^{n_p}=(1+x)^{n_1+...n_p}$$

The coefficient x^k from the right side is $C_{n_1+...n_p}^{k}$.

The coefficient x^k from the left side is $\sum_{s_1+...+s_p=k} C_{n_1}^{s_1}C_{n_2}^{s_2}....C_{n_p}^{s_p}$

From this observation it results the equality.

7.92.

Let's consider $k,m,n\in\mathbb{N}^*$ and $a_j\in\mathbb{C},\ j=\overline{i,m}$. If $a_j^{2kn}+a_j^{(2k-1)n}+....+a_j^n+1=0$ for $j=\overline{i,m}$, compute:

$$E(a_1,...,a_m)=(a_1,...,a_m)^{pn}+\frac{1}{(a_1,...,a_m)^{pn}}+\sum_{1\leq h\leq m-1}\ \sum_{(i_1,...,i_j)\in C_m^h}\left(\frac{a_{i_1}...a_{i_h}}{a_{i_{h+1}}...a_{i_m}}\right)^{pn}$$

knowing that:

$$C_m^h=\left\{(i_1,...,i_p)\in\{1,2,...,m\}^h\ /\ i_s\neq i_t\ for\ s\neq t\right\}$$

Solution

We'll note $a_j^n = y_j,\ j = \overline{1,m}$. Because $y_j \neq 1$, by multiplying eache equality from the enounce, respectively by $y_j - 1$ we obtain:

$$0 = (y_j - 1)\left(y_j^{2k} + y_j^{2k-1} + \ldots + y_j + 1\right) = y_j^{2k+1} - 1.$$

Then, because $j \in \overline{i,m}$ we have

$$y_j = \cos\frac{p_j\pi}{\frac{2k+1}{2k+1} + \ldots + \frac{1}{2k+1}} + i\sin\frac{p_j\pi}{2k+1},\ p_j = 1,2,\ldots 2k.$$

We will prove by recurrence in function of $m \in \mathbb{N}^*$, that:

$$E(a_1,\ldots,a_m) = \left(a_1^{pn} + \frac{1}{a_1^{pn}}\right)\ldots\left(a_m^{pn} + \frac{1}{a_m^{pn}}\right).$$

For $m = 1$ we have $E(a_1) = a_1^{pn} + \frac{1}{a_1^{pn}}$. We assume the property true for m, we must prove for $m+1$, that $E(a_1,\ldots,a_m,a_{m+1}) = E(a_1,\ldots,a_m)\left(a_{m+1}^{pn} + \frac{1}{a_{m+1}^{pn}}\right)$..

Then

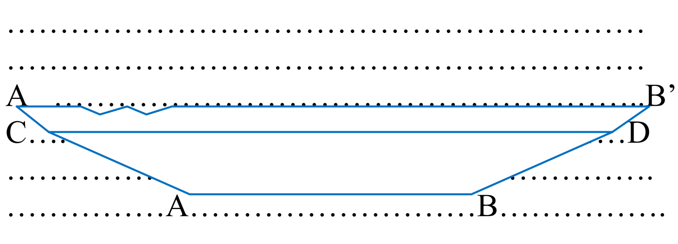
$$E(a_1,\ldots,a_m,a_{m+1}) = 2^m \cos\frac{p_1\pi}{2k+1}p\ldots\cos\frac{p_m\pi}{2k+1}p,$$

$(p_1,\ldots,p_m) = \{1,2,\ldots 2k\}^m$

7.93.

In a plane we consider the set of points whose coordinates are integers. Having the natural numbers n,m,p with $p \geq 4$. Prove that there exists a polygon with p sides which has n points on its border and m points in the interior.
Generalize in the space.

Solution:



The proof is done by construction.

We draw the segment $[AB]$ which contains exactly n points. On the line located above of A' and AB, we draw the segment $[CD]$ which contains exactly m points.

It is possible to designate the segments $[AA']$ and $[BB']$ (see the figure nearby) such that they do not pass through any point, and also A', B' are between the line of CD and the line situated above this; and the segment CD (that is the m points) is located in the interior of the quadrilateral $AA'B'B$.

$p \geq 4 \Rightarrow p-3 \geq 1$. Because the polygon has p sides, by uniting the points A', B' with a line polygonal which contains $p-3$ sides and which is situated between the line of CD and the line above this, without touching any point.

Generalization.

In the Euclidean space $\mathbb{R}^3$, one considers the set of points whose coordinates are integers. If n, m, p are natural numbers, $p \geq 5$, then there exists a polyhedral with p faces which contains n points on its border and m points in the interior.

The proof is also done by construction: one considers the segments $[AB]$ and $[CD]$ with their previous properties, but $[AA']$ and $[BB']$ are replaced by planes which keep the same properties. Then one constructs two planes which pass through the previous planes, keeping of course the required conditions.

At the end, the polygonal line $A'B'$ will be replaced by a series of $p-4$ planes which will be constructed in a similar way.

7.94.

Determine the set A defined by:

a) $102 \in A$

b) if $x \in A$ then $\overline{1 \times 2} \in A$

c) the elements of A are only obtained by the utilization of the rules a) and b) for a limited number of times.

Solution:

We show that $A = M$

where

$$M = \left\{ \overline{\underbrace{1...1}_{n} \circ \underbrace{2...2}_{n}} \,/\, n \in \mathbb{N}^* \right\}$$

First of all, we show that $A \supset M$

One utilizes the reasoning by recurrence for $n \in \mathbb{N}^*$, to show that $\overline{\underbrace{1...1}_{n} \circ \underbrace{2...2}_{n}} \in A$. For $n = 1$ one has $102 \in A$ according to the rule a). One assumes that the property is true for n, then $\overline{\underbrace{1...1}_{n} \circ \underbrace{2...2}_{n}} \in A$ and it will result that also $\overline{1\underbrace{1...1}_{n} \circ \underbrace{2...2}_{n}2} \in A$ according to the rule b)

We show that $A \subset M$.
Let's consider $x \in A$. If one only applies the rule a) it will result that $x = 102 \in M$. One cannot apply the rule a) but one time. Now the rule b) cannot be applied if the rule a was applied. If one applies b) one time, one obtains $x = 11022$. By recurrence one proves that if one applies the rule b) n times, then

$$x = \underbrace{1...1}_{n+1} \circ \underbrace{2...2}_{n+1} \in A$$

but

$$\underbrace{1...1}_{n+1} \circ \underbrace{2...2}_{n+1} \in M$$

Therefore $A \subset M$, from which $A = M$.

7.95.

One constructs the set B such that
a) the elements 0, 9 and 1 belong to B
b) if $x, y \in B$ then $|x - y|$ and $xy \in B$
c) all the elements of B are obtained by the utilization of the rules a) and b) for a limited number of times.

Find B.

Solution:
We prove that $B = M$, where

$$M = \left\{0,\overline{x_1...x_p} \mid 0 \le x_i \le 9,\ i \in \{1,...p\},\ p \in \mathbb{N}^*,\ p < +\infty\right\} \cup \{1\}$$

"$\supset$"
First of all, we show that $\{0; 0,1; 0,2; \ldots 0,9; 1\} \subset B$

$0,9$ et $0,9 \in B \Rightarrow |0,9 - 0,9| = 0 \in B$

$0,9$ et $1 \in B \Rightarrow |0,9 - 1| = 0,1 \in B$

$0,9$ et $0,1 \in B \Rightarrow |0,9 - 0,1| = 0,8 \in B$

$0,8$ et $0,1 \in B \Rightarrow |0,8 - 0,1| = 0,7 \in B$

……………………………………..

$0,3$ et $0,1 \in B \Rightarrow |0,3 - 0,1| = 0,2 \in B$

If $y \in \{0,1,2,....,9\}$ then $\overline{0,\underbrace{0....0}_{i}y} = \underbrace{0,1...0,1}_{i \text{ times}}\overline{0,y} \in B$
because one obtains by the utilization of the rules a) and b).

Let's consider $x \in M$; if $x = 1$ one has $1 \in B$ by the rule a); if $x \neq 1$ one has $0 \le x < 1$ if $x = 0$ one has $0 \in B$. It remains therefore the case $x = \overline{0, x_1...x_p}$ with

$0 \le x_i \le 9,\ i \in \{1,...,p\},\ p \in \mathbb{N}^*,\ p < +\infty$ and $x \neq 0$;

without diminishing the generality one assumes $x_p \neq 0$.

α) If $x_i \neq 9$ then $\overline{0,x_1...x_p} = \overline{0,w_1} - \overline{0,0y_2} - ... - \overline{0,\underbrace{0...0}_{p-2}y_{p-1}}\overline{0,\underbrace{0...0}_{p-1}z_p}$

with $w_1 = x_1 + 1,\ y_j = 9 - x_j,\ -2 \leq j \leq p-1$ and $z_p = 10 - x_p$

Of course $1 \leq w_1 \leq 9$ since $0 \leq x_1 \leq 8$; one has $0 \leq y_j \leq 9$, $2 \leq j \leq p-1$ since $0 \leq x_j \leq 9,\ 2 \leq j \leq p-1;\ 1 \leq z_p \leq 9 \qquad 1 \leq z_p \leq 9$ since $1 \leq x_p \leq 9$.

One has: $\overline{0,w_1} - \overline{0,0y_2} + \overline{0,00y_3} + \overline{0,\underbrace{0...0}_{p-2}y_{p-1}} + \overline{0,\underbrace{0...0}_{p-1}z_p} = \overline{0,w_1} - \overline{0,0y_2...y_{p-1}z_p}$

Performing the subtraction:

$$\overline{0,w_1 0...00} - \overline{0,0y_2...y_{p-1}z_p} = \overline{0,x_1x_2...x_{p-1}x_p}$$

because $10 - z_p = x_p,\ \ w_1 - 1 = x_1$ and $9 - y_j = x_j,\ 2 \leq j \leq p-1$

$\overline{0,w_1}$ and $\overline{0,0y_2} \in B \Rightarrow \left|\overline{0,w_1} - \overline{0,0y_2}\right| \in B$

$\overline{0,w_1} - \overline{0,0y_2}$ and $\overline{0,0y_3} \in B \Rightarrow \left|\left|\overline{0,w_1} - \overline{0,0y_2}\right| - \overline{0,0y_3}\right| \in B$

but the last absolute value is equal to $\overline{0,w_1} - \overline{0,0y_2} - \overline{0,00y_3}$

By recurrence it results that $x \in B$, since

$$x = \overline{0,w_1} - \overline{0,0y_2} - ... - \overline{0,\underbrace{0...00}_{p-2}y_{p-1}} - \overline{0,\underbrace{0...00}_{p-1}z_p} =$$

$$= \underbrace{1...1}_{p-1},\overline{0,w_1} - \overline{0,0y_2} - \overline{0,00y_3} - .... - \overline{0,\underbrace{0...0}_{p-2}y_{p-1}} - \overline{0,\underbrace{0...0}_{p-1}z_p}$$

which is obtained by the utilization of the rules a) and b) a limited number of times.

β) If $x_1 = 9$, one has $0,\underbrace{9...9}_{n} = \left|1 - 0,\underbrace{0...0}_{n-1}1\right| \in B$, $\forall n \in \mathbb{N}^*$

$$x = \overline{0,9x_2...x_p} = \left|0,\underbrace{9...9}_{p} - \overline{0,0t_2...t_p}\right|$$

where $t_j = 9 - x_j,\ 2 \leq j \leq p;\ 0 \leq t_j \leq 9$ since $0 \leq x_j \leq 9$ for $2 \leq j \leq p$

To show that $t = \overline{0,0t_2...t_p} \in B;\ \ 0 \leq t_j \leq 9,\ 2 \leq j < p$

it is sufficient to see that the first decimal digit of t is zero, therefore different of 9, therefore we'll use the case α)

$$t \in B \Rightarrow x \in B$$

"$\subset$"

0,9 and $1 \in M \subset [0,1]$; $\{0;0,1;0,2;...0,9;1\} \subset M$. The operations of the rule b) applied a limited number of times on the elements 0, 9 and 1 will also give elements of M, because: if $\alpha, \beta \in M$ it results that $\alpha\beta \in M$ and $|\alpha - \beta| \in M$ since $0 \leq |\alpha - \beta| \leq 1$ and $0 \leq \alpha\beta \leq 1$ and α, β having a limited number of decimals, then also $\alpha\beta$ and $|\alpha - \beta|$ will have a limited number of decimals.

7.96.

Let's consider the set $I=\left\{\frac{a}{b}\,|\,a,b\in\mathbb{Z},\ b\neq 0,\ (b,m)=1\right\}$ which is called the ring of m-integers, $m\in\mathbb{Z}$, fix.

It is said that $x\equiv y(\operatorname{mod} m)$ in I, with x,y of I, if and only if there exists $z\in I$ such that $z\in I$.

We consider the element r of I; find its class of equivalence modulo m in I.

Solution. 1)

The case $r\in\mathbb{Z}$

a) $m\notin\{-1,0,1\}$

We note with r the class we are looking for, M_m the set $m\mathbb{Z}$, and

$$M=\left\{\frac{mk_1+h}{mk_2+p}\ \mid k_1,k_2,h,p\in\mathbb{Z},\ 0\leq h,\ p<1/m,6(m,p)=1 \text{ and } h-pr=M_m\right\}$$

We show that $\hat{r}=M$

First of all we show that $\hat{r}\supset M$

Let's consider $x\in M$. Then there exist $k_1,k_2,h,p\in\mathbb{Z}$ with the properties written above.

1) $x\in I$, that is x is a m-integer, because from $(m,p)=1$ and $k_2\in\mathbb{Z}$ it results that $(mk_2+pm)=1$.

2) $x\equiv r(\operatorname{mod} m)$ in I, since there exists $\frac{\gamma_1}{\gamma_2}\in I$, $\gamma_1=k_1-rk_2+h_m-rp$, $\gamma_1\in\mathbb{Z}$ because $h-rp+M_m$ and $\gamma_2=mk_2+p$ (therefore $\frac{\gamma_1}{\gamma_2}$ is a m-integer because $(\gamma_2,m)=1$, such that: $\frac{\gamma_1}{\gamma_2}m=\frac{mk_1+h-mk_2r-rp}{mk_2+p}=x-r$. Therefore $x\in\hat{r}$.

We prove that $\hat{r}\subset M$

Let's consider $x\in\hat{r}$. Therefore $x=\frac{a}{b}$ and $b\neq 0$ as well as $(b,m)=1$ such that $x\equiv r(\operatorname{mod} m)$ in I. But b can be written $b=mk_2+p,\ 0\leq p<|m|,\ (p,m)=1$ and the same $a=mk_1+h,\ 0\leq h<|m|$ with $k_1,k_2,h,p\in\mathbb{Z}$.

$x\equiv r(\operatorname{mod} m)$ in I, implies that $m\,|\,x-r$ in I, therefore implies that there exists $\gamma=\frac{\gamma_1}{\gamma_2}\in I$ such that $m\gamma=x-r$ and $(\gamma_2,m)=1$.

We consider γ irreducible. It results $\frac{\gamma_1}{\gamma_2}=\frac{m(k_1-k_2r)+h-rp}{m(mk_2+p)}$. Because $(\gamma_1,\gamma_2)=1$ and $(\gamma_2,m)=1$ it results: $m\,|\,m(k_1-k_2r)+h-rp$ from which $m \nmid h\text{-}rp$ that is $x\in M$.

Remark 1.

For $m=0$ we have $I=\mathbb{Z}$ and for $m=\pm1$ we have $I=\mathbb{Q}$.

b) $m=0$. Each class of equivalence contains only an element therefore $\hat{r}=\{r\}$

c) $m=\pm1$. It exists only one class of equivalence, therefore $\hat{r}=I=\mathbb{Q}$.

2) The case $\hat{r}=I-\mathbb{Z}$: a) $m\notin\{-1,0,1\}$

Property 1.

We have $m\in\mathbb{Z}$, I the ring of m-integers and $\frac{r_1}{r_2}\in I$. Then $\exists a\in\mathbb{Z}$ such that $\frac{r_1}{r_2}\in I\Rightarrow(r_2,m)=1$ in I.

Proof.

$\frac{r_1}{r_2}\in I$ then $(r_2,m)=1$ $r_1,r_2\in\mathbb{Z}$.

Let's consider the Diophantine equation $xr_2+ym=r$ (the unknowns being x and y) which admits integer solutions, since $(r_2,m)=1$ and $1\,|\,r_1$. Let's consider $x=x_0\in\mathbb{Z}$ and $y=y_0\in\mathbb{Z}$ a particular solution. Taking $a=x_0$ and $\gamma_1=y_0$. We have $\frac{r_1}{r_2}\equiv x_0(\operatorname{mod} m)$ since $m\,|\left(\frac{r_1}{r_2}-x_0\right)$ because there exists $\gamma=\frac{\gamma_1}{\gamma_2}=\frac{y_0}{r_2}\in I$ such that $m\frac{y_0}{r_2}=\frac{r_1}{r_2}-x_0$. Therefore $x_0r_2+y_0m=r_1$.

Remark 2.

It exists an infinite number of integers $\alpha\in I$ such that $\frac{r_1}{r_2}=\alpha(\operatorname{mod} m)$ in I, with $m\neq0$

These numbers are, for example, particular solutions of the previous Diophantine equation. Therefore, for $r\cup I-\mathbb{Z}$, $\exists a\in\mathbb{Z}$ such that $r\equiv a_r(\operatorname{mod} m)$ in I and $\hat{r}=\hat{a}_r$ which can be determinate as in the case 1 a).

b) The sub-case $m=0$ does not exist because it would result $I=\mathbb{Z}$ and therefore $r\in\mathbb{Z}$.

c) $m=\pm1$. One has $p=I=\mathbb{Q}$.

7.97.

Let's consider the equation $a_1x_1^{m_1}+...+a_mx_i^{m_n}=b$ with $a_i,m_i\in\mathbb{N}^*$ for $i\in\{1,...,n\}$ and $b\in\mathbb{Z}$. Show that the equation has a limited number of natural numbers solutions.

Solution:

α) $b>0$. We note all $x_i^{m_i}=y_i$. The initial equation becomes

$$a_1 y_1 + ... + a_n y_n = b \qquad (1)$$

One can see that:

$0 \le y_1 \le \left[\frac{b}{a_1}\right]$ (otherwise one has: $a_1 y_1 > b$)

....

$0 \le y_n \le \left[\frac{b}{a_n}\right]$ (same explanation)

It results that: $0 \le \overline{\text{the number of natural solutions of the equation (1)}}$

$$(1) \le \prod_{i=1}^{n}\left(1+\left[\frac{b}{a_i}\right]\right) = M \text{ = finite number.}$$

$$x_i = \sqrt[m_i]{y_i}, \quad i \in \{1,...,n\} \qquad (2)$$

Thus, if the number of solutions of the equation (1) is limited, then also, from (2) will result that there exists a limited number of values for each x_i.

β) $b = 0$. Then, the only natural solution is the banal solution.

γ) $b < 0$. The equation does not admit any natural solution.

7.98.

Let's consider $a_1, b > 0$ for $i \in \{1,...,n\}$. Then, the equation $a_1^x + ... + a_n^x = b^x$, $n \ge 2$ admits in addition a solution in the set of real numbers.

Solution:

The equation can have no solution, or it can have at least one solution.

If the equation admits at least one solution, let it be $x_0 \in \mathbb{R}$ one of those. Therefore $a_1^{x_0} + ... + a_n^{x_0} = b^{x_0}$.

1) $x_0 > 0$. Let's consider $x > x_0$. It results that $x = x_0 + t$ with $t > 0$.

Let's consider $z = \max\{a_1,...,a_n\}$

$$a_1^x + ... + a_n^x = a_1^t a_1^{x_0} + ... + a_n^t a_n^{x_0} \le z^t\left(a_1^{x_0} + ... + a_n^{x_0}\right) = z^t b^{x_0}$$

If $z \ge b \Rightarrow z^{x_0} \ge b^{x_0} \Rightarrow a_1^{x_0} + ... + a_n^{x_0} > b^{x_0}$. From which $z < b$. Thus $a_1^x + ... + a_n^x < b^x$, $\forall x > x_0$.

Let's consider $x < x_0$, It results that $x = x_0 - t$ with $t > 0$

$$a_1^x + ... + a_n^x = a_1^{-t} a_1^{x_0} + ... + a_n^{-t} a_n^{x_0} > z^{-t}\left(a_1^{x_0} + ... + a_n^{x_0}\right) = z^{-t} b^{x_0}$$

Since $z < b \Rightarrow z^{-t} > b^{-t}$. Thus $a_1^x + ... + a_n^x > b^x$, $\forall x < x_0$ from which x_0 is the only solution of the equation.

2) $x_0 < 0$. One has: $a_1^{x_0} + ... + a_n^{x_0} = b^{x_0}$ where $\left(\frac{1}{a_1}\right)^{-x_0} + ... + \left(\frac{1}{a_n}\right)^{-x_0} = \left(\frac{1}{b}\right)^{-x_0}$,

with $-x_0 > 0$, therefore one has reduced this case to the first case.
3) The case $x_0 = 0$ is not possible, because it would result $a_1^0 + ... + a_n^0 = b^0$ where $n = 1$, but by hypotheses $n \geq 1$. Contradiction.

7.99.

Show that a congruence $\bmod m$, $m \neq 0$ which contains unknowns, admits a limited number of distinct solutions (two by two non-congruent)

Solution:

Each unknown cannot take but the values: $0,1,2,...,|m|-1$, that is at maximum $|m|$ values (a complete system of remainders modulo $|m|$). If the equation – a congruence containing n unknowns, then the maximum number of solutions will be $|m|^n < \infty$.

Observation:
When $m = 0$, the congruence becomes an equality (an equation) which can have an infinite number of solutions, for example $0x \equiv 0(\bmod 0)$.

7.100.

Solve the linear congruence $2x - 1 \equiv 1 - 6y(\bmod 4)$

Solution:

The congruence can be written: $2x + 6y - 2 \equiv 0(\bmod 4)$. From which $2x + 6y - 2 = 4k$ with $k \in \mathbb{Z}$ From where $x + 3y - 2k - 1 = 0$
(Remark: one cannot divide the congruence by 2 at beginning (one would obtain $x + 3y \equiv 0(\bmod 2)$), because solutions will be lost).

The module of the congruence gives the rest 4. One has: $x = -3y + 2k + 1$

where $x \equiv y + 2k + 1(\bmod 4)$.

One takes $(y,k) \in \{0,1,2,3\}^2$, therefore all the possibilities.

But it is sufficient to give to k the values 0 and 1, since: for $k = 3 \Rightarrow 2k \equiv 2 \cdot 1(\bmod 4)$ and for $k = 2 \Rightarrow 2k \equiv 2 \cdot 6(\bmod 4)$.

If one successively gives the values $(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(3,1)$ to the couple (y,k) one obtains for x, respectively the values $1,2,3,0,3,0,1,2$. But we are not interested in k. Therefore

$$\begin{cases} x \equiv 1(\bmod 4)(2)(3)(0)(3)(0)(1)(2) \\ y \equiv 1(\bmod 4)(1)(2)(3)(0)(1)(2)(3) \end{cases}$$

This shows that the number of solutions of this congruence is equal to $(2,6)4 = 8$.

7.101.

Let's consider $a_i \in \mathbb{Q}$, $i \in \{1,2,...,n\}$, $n \geq 2$ and $b \in \mathbb{Q}$. Show that the equation $\sum_{i=1}^{n} a_i x_i = b$ admits an infinite number of solutions in the set of natural numbers if and only if $(a_1,...,a_n)$ divides b and if there exists $(i_0, j_0) \in \{1,...,n\}^2$ such that $a_{i_0} \cdot a_{j_0} < 0$.
(One notes $a_1,...,a_n$ the greatest common divisor of $a_1,...,a_n$).

Solution:

If we put the coefficients of the equation with the same denominator we can eliminate the denominators and therefore we can say that all the a_i and b are integers.

Necessity.

Since the equation has solutions in $\mathbb{N}^n$, then it would also have in $\mathbb{Z}^n$ because $\mathbb{N}^n \subset \mathbb{Z}^n$. It results that $(a_1,...,a_n)$ divides b, according to a known theorem.

Let's suppose by absurd that all the terms of the equation have the same sign, for example positive (in the opposite case one multiplies the equation by -1):

- If $b < 0$ then the equation does not have any natural solution: contradiction.
- If $b \geq 0$, each unknown x_i cannot take values which are between 0 and $\left[\frac{b}{a_i}\right]$ (natural values), therefore a finite number of solutions; also contradiction. From which the supposition is false, therefore it is not true that the terms of the equation have the same sign.

Sufficiency.

Because $(a_1,...,a_n)$ divides b it results that the equation has solutions in $\mathbb{Z}^n$. By hypothesis, the equation has l has terms positives non nulls $1 \leq l < n$ and $k = n - l$ terms negatives non nulls. One has $1 \leq k \leq n-1 < n$. Then one writes:

$$\sum_{h=1}^{l} a_h x_h - \sum_{j=l+1}^{n} a_j^{'} x_j = b, \; 0 < a_j^{'} = -a_j$$

$j \in \{l+1,...,n\}$

(One has supposed the first l terms positives and the following k negatives. In the other cases one reorders the terms and (implicitly) one re-numerates them.)

Let's consider $0 < M = [a_1,...,a_n]$ the smallest common multiple of $a_1,...,a_n$. One notes $c_i = |M / a_i|$ and $i \in \{1,...,n\}$

If one notes $0 < p = [l,k]$ the smallest common multiple of l and k.

We note $l_1 = p / l$ and $k_1 = p / k$. If we note:

$$\begin{cases} x_h = l_1 c_h t + x_h^0, & 1 \le h \le l \\ x_j = k_1 c_j t + x_j^0, & l+1 \le j \le n \end{cases}$$

with $t \in \mathbb{N}$, $t \ge \max\limits_{h,j} \left\{ \left[\dfrac{-x_h^0}{l_1 c_h} \right], \left[\dfrac{-x_j^0}{k_1 c_j} \right] \right\} + 1$

where $[x]$ represents the integer part of x, and $\left(x_1^0,...,x_n^0\right)$ is a particular solution of the equation (it was shown at the beginning of this proof that there exist integer solutions), then one obtains an infinite number of natural solutions for our equation.

7.102.

Let's consider the linear equation with integer coefficients

$$\sum_{i=1}^{n} a_i x_i = b$$

a) If there exists $(i_0, j_0) \in \{1,...,n\}^2$ such that $a_{i_0} \equiv \pm 1 (\text{mod}\, a_{j_0})$ then the equation admits solutions in the set of integer numbers.
b) In this case, solve it.

Solution:

a) Let's consider $a_{i_0} \equiv \pm 1 (\text{mod}\, a_{j_0}) \Leftrightarrow \exists h_0 \in \mathbb{Z}:\ a_{i_0} - h_0 = \pm 1$. It results that $d \sim \left(a_{i_0}, a_{j_0}\right)$, therefore $d \nmid a_{i_0} - h_0 a_{j_0}$, then $d \mid \pm 1$, from where $d \sim 1$.
Because $\left(a_{i_0}, a_{j_0}\right) \sim 1$ one has $\left(a_1,...,a_n\right) \sim 1$, but $1 \mid b$. Therefore the equation admits integer solutions.

b) $\sum\limits_{i=1}^{n} a_i x_i = \sum\limits_{\substack{i \ne i_0 \\ i \ne j_0}} a_i x_i + \left(h_0 a_{j_0} \pm 1\right) x_{i_0} + a_{j_0} x_{j_0} = \sum\limits_{\substack{i \ne i_0 \\ i \ne j_0}} a_i x_i + a_{j_0}\left(x_{j_0} + h_0 x_{i_0}\right) \pm x_{i_0} =$

$$= \sum_{\substack{i \ne i_0 \\ i \ne j_0}} a_i x_i + a_{j_0} t \pm x_{i_0} = b$$

where $t = x_{j_0} + h_0 x_{i_0}$. It results:

$$\begin{cases} x_{i_0} = \pm\left(\sum\limits_{\substack{i\neq i_0\\ i\neq j_0}} a_i x_i + a_{j_0} t - b\right) \\ x_{j_0} = t - h_0 x_{i_0} = \pm\left(h_0 \sum\limits_{\substack{i\neq i_0\\ i\neq j_0}} a_i x_i + a_{j_0}\left(h_0 x_{i_0} \pm 1\right)t - h_0 b\right) \\ \text{with } x_i \in \mathbb{Z},\ i \notin \{i_0, j_0\},\ \text{and } t \in \mathbb{Z} \end{cases}$$

(parameters).

7.103.

It is given the system:

$$f_j\left(x_1,...,x_n,[x_1],...,[x_n],\{x_1\},...,\{x_n\}\right) = b_j \in \mathbb{Q},\ j = \overline{1,n}$$

where f_j are linear functions which have their coefficients in $\mathbb{Q}$ and $[x],\{x\}$ represent respectively the integer part and the fractional part of x.
Find a method for solving this system.

Solution:

(1) Writing $x_i = [x_i] + \{x_i\}$ with $[x_i] \in \mathbb{Z},\ 0 \le \{x_i\} < 1$, $i = \overline{1,n}$ one obtains (after the elimination of denominators) the equivalent system:

(2) $g_j\left([x_1],...,[x_n],\{x_1\},...,\{x_n\}\right) = c_j \in \mathbb{Z},\ j = \overline{1,n}$

where the g_j are now linear functions which have their coefficients in $\mathbb{Z}$

One solves this system considering that $\{x_1\},...,\{x_n\}$ are the unknowns. Since:

$$\sum_{i=1}^{n} \alpha_{ij}[x_i] + \sum_{j=1}^{n} \beta_{ij}\{x_j\} = c_i,\ \ i = \overline{1,n}$$

and because $\alpha_{ij}[x_i],\ \ c_i \in \mathbb{Z},\ i, j = \overline{1,n}$ it results:

$$\sum_{j=1}^{n} \beta_{ij}\{x_j\} \in \mathbb{Z},\ \ \forall i \in \overline{1,n}$$

One applies the method of the substitution. One calculates $\{x_{j_1}\}$ of an equation $1 \le j_1 \le n$, and replaces it in the other equations. It will remain a linear system of n-1 equations with n-1 unknowns.

We proceed in the same way with this new system. (If one obtains during this procedure an impossible equation then the system (2) does not admit solutions. Stop.). At the end one obtains:

$\delta\left\{x_{j_n}\right\}=h([x_1],...,[x_n])+k\in\mathbb{Z}$, then $\delta\left\{x_{j_n}\right\}\in\mathbb{Z}$. It results: $\left\{x_{j_n}\right\}=0$ where $\left\{x_{j_n}\right\}=\frac{p}{\varepsilon},\ p\in\mathbb{N}^*$ but such that $0\le\frac{p}{\delta}<1;\ \delta,k\in\mathbb{Z}$.

These two cases can be written as a single case:

$\left\{x_{j_n}\right\}=\frac{p}{\delta}, p\in\mathbb{N}$ and $0\le\frac{p}{\delta}<1$

Let s_{j_n} be the number of solutions for $\left\{x_{j_n}\right\}$. Now one will follow the inverse procedure until the determination of all the $\left\{x_{j_n}\right\}$. One replaces the value(s) of $\left\{x_j\right\},\ j=\overline{1,n}$ in the previous system of 2 equations with 2 unknowns. One obtains the values of $\left\{x_{j_{n-1}}\right\}$.

Let $s_{j_{n-1}}$ be the number of these.

The inverse procedure continues until when one determines $\left\{x_{j_1}\right\}$ which has s_{j_1} solutions: $\left(\left\{j_1,...,j_1\right\}=\left\{1,2,..n\right\}\right)$

One remarks that $\left\{x_i\right\}\in\mathbb{Q},\ i=\overline{1,n}$. If the system has solutions it results that these are in $\mathbb{Q}$.

Until now one has obtained $\prod_{i=1}^{n}s_i$ solutions.

(3) Reporting all the values of $\left\{x_1\right\},...,\left\{x_n\right\}$ in (2) on obtains a linear system of n equations with n unknowns: $[x_1],...,[x_n]$ which will be solved in integer numbers. One normally solves in $\mathbb{R}^n$, and if the solution belongs to $\mathbb{Z}^n$ then this solution is correct (one then performs the relation (1) to obtain $x_1,...,x_n$); otherwise, it is not convenient.

One will execute the paragraph (3) for all the values of $\left\{x_1\right\},...,\left\{x_n\right\}$.

Thus, the system is well solved. The number of solutions of this is ≥ 0 and $\le\prod_{i=1}^{n}s_i$.

7.104.

Solve in $\mathbb{N}$ the equation: $3x-7y+2z=-18$

Solution:

The general solution in $\mathbb{Z}$ is the following: $x=k_1,\ y=k_1+2k_2,\ z=2k_1+7k_2-9$ with: $k_1,k_2\in\mathbb{Z}$

Because $x \geq 0,\ y \geq 0,\ z \geq 0$ it results that $k_1 \geq 0$ and also that $k_2 \geq \left[-\frac{k_1}{2}\right]+1$ and $k_2 \geq \left[\frac{9-2k_1}{7}\right]+1$, that is one has $k_2 \geq \left[\frac{2-2k_1}{7}\right]+2$. From which, the general solution in $\mathbb{N}$ will be: $x = k_1,\ y = k_1 + 2k_2,\ z = 2k_1 + 7k_2 - 9$ with $k_1,\ k_2$ in $\mathbb{N}$ and $k_2 \geq \left[\frac{2-2k_1}{7}\right]+2$

7.105.

Solve in $\mathbb{N}$ the equation: $2x + 15y + 9z = 44$

Solution:

One has $0 \leq y \leq \left[\frac{44}{15}\right] = 2$.

A) $y = 0 \Rightarrow 2x + 9z = 44 \Rightarrow 0 \leq z \leq \left[\frac{44}{9}\right] = 4$

a) $z = 0 \Rightarrow x = 22$

b) $z = 1 \Rightarrow x = \frac{35}{2} \notin \mathbb{N}$

c) $z = 2 \Rightarrow x = 13$

d) $z = 3 \Rightarrow x \notin \mathbb{N}$

e) $z = 4 \Rightarrow x = 4$

B) $y = 1 \Rightarrow 2x + 9z = 29 \Rightarrow 0 \leq z \leq 3$

a) $z = 1 \Rightarrow x = 10$

b) $z = 3 \Rightarrow x = 1$

c) $z \in \{0, 2\} \Rightarrow x \notin \mathbb{N}$

C) $y = 2 \Rightarrow 2x + 9z = 14 \Rightarrow 0 \leq z \leq 1$

a) $z = 0 \Rightarrow x = 7$

b) $z = 1 \Rightarrow x \notin \mathbb{N}$

All the solutions are:

$(22,0,0);\ (10,1,1);\ (7,2,0)$

$(13,0,2);\ (1,1,3)$

$(4,0,4)$

Therefore there is a limited number of solutions.

7.106.

Solve in $\mathbb{Z}$ the equation:

$17x_1 + 20x_2 - 18x_3 = -34$

Solution:

One writes the equation thus:

$20x_2 - 18x_3 + 17(x_1 + 2) = 0$

One notes $x_1 + 2 = t \in \mathbb{Z}$. It results: $20x_2 - 17(x_3 - t) - x_3 = 0$

That is: $20x_2 - 17h - x_3 = 0$

One notes $x_3 - t = h \in \mathbb{Z}$

It results:

$$\begin{cases} x_1 + 2 = t \\ x_3 - t = h \end{cases} \Rightarrow x_1 = t - 2 = (x_3 - h) - 2 = -h - 2 + 20x_2 - 17h = 20x_2 - 18h - 2$$

The general solution is:

$$\begin{cases} x_1 = 20k_1 - 18k_2 - 2 \\ x_2 = k_1 \\ x_3 = 20k_1 - 17k_2, \ (k_1, k_2) \in \mathbb{Z}^2 \end{cases}$$

7.107.

Solve in the set of integer numbers the equation:

$15x - 17y + 9z = \alpha$

Where α is an integer parameter.

Solution:

The equation can be written:

$15x + 9(z - 2y) + y = \alpha$

Or again:

$15x + 9t + y = \alpha$ (1)

where $t = z - 2y$ (2)

It results from (1) that: $y = -15x - 9t + \alpha$ (3)

It results from (2) that: $z = t + 2y$

It results from (3) that: $z = -30x - 17t + 2\alpha$

The integer general solution is:

$$\begin{cases} x = k_1 \\ y = -15k_1 - 9k_2 + \alpha \qquad (k_1, k_2) \in \mathbb{Z}^2 \\ z = -30k_1 - 17k_2 + 2\alpha \end{cases}$$

(one has noted $x = k_1$ and $t = k_2$)

7.108.

Solve in $\mathbb{Z}$ the equation $3x + 70y - 35z + 6 = 76$.

Solution:

The equation can be written:

$$3\left(\underbrace{x + 23y - 12z + 2w - 25}_{t_1}\right) + \left(\underbrace{y + z}_{t_2}\right) = 1$$

With these notation one has: $3t_1 + t_2 = 1$, equation which admits the general solution:

$$\begin{cases} t_1 = k \\ t_2 = -3k + 1 \end{cases} \text{with } k \in \mathbb{Z}$$

Therefore $y + z = t_2 = -3k + 1$ from where $y = -z - 3k + 1$ with $z \in \mathbb{Z}$ (1)

In a similar way $x + 23y - 12z + 2w - 25 = t_1 = k$, where

$x - 23z - 69k + 23 - 12z + 2w - 25 = k$ (one has used (1)).

Thus $x = 35z - 2w + 70k + 2$

The general solution of the equation in $\mathbb{Z}^4$ will be:

$$\begin{cases} x = 35z - 2w + 70k + 2 \\ y = -z - 3k \end{cases} \quad z, w, k \in \mathbb{Z}$$

7.109.

Solve the equation $xy + 5z - 2 = 0$ in the set of integer numbers.

Solution:

$\forall x \in \mathbb{Z}$ one can write: $x = 5k_1 + r_1$ with $k_1 \in \mathbb{Z}$ and $r_1 \in \{0,1,2,3,4\}$ (1)

$\forall y \in \mathbb{Z}$, one can also write: $y = 5k_2 + r_2$, with $k_2 \in \mathbb{Z}$ and $r_2 \in \{0,1,2,3,4\}$ (2)

Using (1) and (2) in the initial equation, one has

$5(5k_2k_2 + k_1r_2 + k_2r_1 + z) + r_1r_2 - 2 = 0$

It results that 5 divides $(r_1r_2 - 2)$. Therefore $(r_1, r_2) \in \{(1,2),(2,1),(3,4),(4,3)\}$ (3)

From where, the general solution will be:

$$\begin{cases} x = 5k_1 + r_1 \\ y = 5k_2 + r_2 \\ z = -5k_1k_2 - k_1r_2 - k_2r_1 + \dfrac{2 - r_1r_2}{5} \end{cases}$$

With $(k_1, k_2) \in \mathbb{Z}^2$ (arbitrary parameters), and $(r_1, r_2) \in \{(1,2),(2,1),(3,4),(4,3)\}$.

The unknown z has been obtained by starting from the initial equation because there were known the values of x and y.

7.110.

It is given the equation $x^2 + 3k_1 + 2 = (3k_2 + 1)^y$. Show that the equation does not admit a natural solution, for any $(k_1, k_2) \in \mathbb{Z}^2$. Generalize.

Solution:

One has: $3k_1 + 2 \equiv 2 (\bmod 3)$ and $3k_2 + 1 \equiv 1 (\bmod 3)$ therefore $(3k_2 + 1)^y \equiv 1 (\bmod 3)$ where $x^2 \equiv 1 - 2 \equiv 2 (\bmod 3)$

a) if $x = M_3 + 1 \Rightarrow x^2 \equiv 1 \not\equiv 2 (\bmod 3)$.

b) if $x = M_3 + 2 \Rightarrow x^2 \equiv 1 \not\equiv 2 (\bmod 3)$

c) if $x = M_3 + 2 \Rightarrow x^2 \equiv 0 \not\equiv 2 (\bmod 3)$

From which $\forall x \in \mathbb{N},\ x^2 \not\equiv 2 (\bmod 3)$, Thus the equation does not admit a natural solution.

Generalization:

The equation $x^2 + 3y + 2 = (3z + 1)^w$ does not admit an integer solution.

The proof is the same. First of all one shows that if $k_2 \neq 0$ then $y \geq 0$, because if $y < 0$ it would result that an integer number (the left member of the equation) is equal to a non-integer number (the right member).

7.111.

Solve the equation $xy + 4t - 7w + 14 = 0$ in the set of integer numbers.

Solution:

One writes: $xy + 4t - 8w + 12 + w + 2 = 0$ we note $t - 2w + 3 = v$ which will be a new unknown.

The equation becomes: $xy+4v-w+2=0$ where $w=-xy-4v-2$. And $t=v+2w-3=v+2(-xy-4v-2)-3-2xy-7v-7$. If one changes the notations (for the sake of mathematical esthetic) one has the integer general solution:

$$\begin{cases} x=k_1 \\ y=k_2 \\ t=-2k_1k_2-7k_3-7 \\ w=-k_1k_2-4k_3-2 \end{cases}$$

with $(k_1,k_2,k_3)\in\mathbb{Z}^3$ (parameters).

7.112.

Show that the equation:

$$2x^2+5xy-12y^2-2x+3y-1=0$$

does not have a solution in the set of integer numbers.

Solution:

The equation can be written:

$$2x^2+8xy-3xy-12y^2-2x+3y=1$$

Or again $x(2x-3y)+4y(2x-3y)-1(2x-3y)=1$

Therefore $(2x-3y)(x+4y-1)=1$

As $x,y\in\mathbb{Z}$ it results that one has the following possibilities:

a) either

$$(2x-3y)=1 \quad (1)$$

$$x+4y-1=1 \quad (2)$$

(2) implies $x=-4y+2$, and substituting in (1) it comes

$$-11y=-3\Rightarrow y=\frac{3}{11}\notin\mathbb{Z}$$

b) or

$$2x-3y=-1 \quad (3)$$

$$x+4y-1=-1 \quad (4)$$

(4) $\Rightarrow x=-4y$ and substituting in (3) it comes

$$-8y-3y=-1\Rightarrow y=\frac{1}{11}\notin\mathbb{Z}$$

Therefore the equation does not have a solution in $\mathbb{Z}$.

7.113.

Prove that the equation: $5x^2 + 50y^2 - 26xy - 8x - 46y + 15 = 0$
does not admit a solution in the set of natural numbers.

Solution I:

The equation can be written:

$$\left(4x^2 + 49y^2 + 9 - 28xy + 12x - 42y\right) + \left(x^2 + y^2 + 4 + 2xy - 4x - 4y\right) + 2 = 0$$

Or

$$(2x - 7y + 3)^2 + (x + y - 2)^2 + 2 = 0$$

But this equation does not admit a solution in $\mathbb{R}$, because
$(2x - 7y + 3)^2 + (x + y - 2)^2 + 2 > 0$. Therefore it does not admit even more a solution in $\mathbb{N}$.

Solution II:

The equation can be written:

$$5x^2 + 2(-13y + 4)x + \left(50y^2 - 46y + 15\right) = 0$$

$$\Delta = b'^2 - ac = 169y^2 + 16 - 104y - 250y^2 + 230y - 75 = -\left[(9y - 7)^2 + 10\right] < 0$$

It results that the equation does not admit a solution in $\mathbb{R}$. Therefore it does not admit even more a solution in $\mathbb{N}$.

7.114.

Solve in the set of integer numbers the equation:

$$x^3 - 3y = 2$$

Solution:

The equation can be written: $x^3 - 2 = 3y$.

Therefore $x^3 - 2$ is divisible by 3, that is

$$x^3 = M_3 + 2$$

$$x = 3k + r,\ \ r = 0,1,2,\ k \in \mathbb{Z}$$

$$x = 3k \Rightarrow x^3 = M_3 \neq M_3 + 2$$

$$x = 3k + 1 \Rightarrow x^3 = M_3 + 1 \neq M_3 + 2$$

$$x = 3k + 2 \Rightarrow x^3 = M_3 + 8 = M_3 + 2$$

Let's consider $x = 3k + 2,\ k \in \mathbb{Z}$

$$y = \frac{x^3 - 2}{3} = \frac{(3k + 2)^3 - 2}{3} = 9k^3 + 18k^2 + 12k + 2$$

The solution of the equation is:

$$\begin{cases} x = 3k+2 \\ y = 9k^3 + 18k^2 + 12k + 2 \end{cases}$$
$k \in \mathbb{Z}$

7.115.

Prove that the equation $x^4 - 7y_1 ... y_n = 14z + 10,\ n \geq 1$ does not admit any integer solution.

Solution:

One can write: $x^4 - 7(y_1 ... y_n - 2z - 1) = 3$, or $x^4 - 3 = 7(y_1 ... y_n - 2z - 1)$

From which 7 divides $x^4 - 3$, that is $x^4 = M_7 + 3$.

Let's consider $x = M_7 + r$ with $r \in \{0, \pm 1, \pm 2, \pm 3\}$. Then $x^4 = M_7 + r^4$; but $r^4 \in \{0^4, 1^4, 2^4, 3^4\}$

But one can see that $r^4 \neq 3 (\bmod 7)$, therefore $x^4 \neq M_7 + 3$. It results that the equation does not admit any integer solution.

7.116.

Solve in the set of integer numbers the equation:

$5x^4 - 6y = 20$

Solution:

$5x^4 - 6y = 20 \Leftrightarrow 6y = 5(x^4 - 4)$

Therefore y is divisible by 5.

Let's consider $y = 5z$, $z \in \mathbb{Z}$

The equation becomes:

$$5x^4 - 30z = 20 \Leftrightarrow x^4 - 6z = 4 \Leftrightarrow z = \frac{x^4 - 4}{6} \in \mathbb{Z}$$

Therefore $x^4 \equiv 4 (\bmod 6)$

It results:

$x = 6k + 2$, $x = 6k + 4$, $k \in \mathbb{Z}$

The integer solutions of the equation are:

$$\begin{cases} x = 6k+2 \\ y = 5\dfrac{(6k+2)^4}{6} \\ k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} x = 6k + 4 \\ y = 5\dfrac{(6k+2)^4 - 4}{6} \\ k \in \mathbb{Z} \end{cases}$$

7.117.

Solve in the set of integer numbers the equation:

$$4x^y - 7z = 5$$

Solution:

$4x^y = 7z + 5$; since $z \in \mathbb{Z}$ one has $4x^y \in \mathbb{Z}$.

Therefore $x^y = \alpha \in \mathbb{Z}$ where $x^y \in \left\{\pm\frac{1}{2}, \pm\frac{1}{4}\right\}$ but $z = \frac{4x^y - 5}{7}$, from where $x^y = \frac{1}{2}$ or $x^y = \alpha \in \mathbb{Z}$ A solution is: $x = -2,\ y = -1,\ z = -1$, since $x^y = \frac{1}{2}$ admits the solution in integer numbers $x = -2, y = -1$.

If $x^y = \alpha \in \mathbb{Z}$, the initial equation becomes $4\alpha - 7z - 5 = 0$ which admits the integer general solution.

$$\begin{cases} \alpha = 7k + 3 \\ z = 4k + 1 \end{cases}$$

with $k \in \mathbb{Z}$ (parameter).

Therefore $x^y - 7k - 3 = 0$, with x, y, z of $\mathbb{Z}$. It results $k = \frac{x^y - 3}{7}$. But $x = M_7 + r$, with $r \in \{0,1,2,..,6\}$.

One writes $x = 7s + r,\ s \in \mathbb{Z}$, and then $\frac{(7s+r)^y - 3}{7} \in \mathbb{Z}$ if and only if $r = 3$ or 5

$$x = 7s + 3 \Rightarrow y = 6t + 1,\quad t \in \mathbb{N}$$
$$x = 7s + 5 \Rightarrow y = 6t + 5,\quad t \in \mathbb{N}$$

Therefore the integer solutions of the equation are:

$$\begin{cases} x = -2 \\ y = -1 \\ z = -1 \end{cases}$$

$$\begin{cases} x = 7s + 3, \quad s \in \mathbb{Z} \\ y = 6t + 1, \quad t \in \mathbb{N} \\ z = \dfrac{4(7s+3)^{6t+1} - 5}{7} \end{cases}$$

$$\begin{cases} x = 7s + 5, \quad s \in \mathbb{Z} \\ y = 6t + 5, \quad t \in \mathbb{N} \\ z = \dfrac{4(7s+5)^{6t+1} - 5}{7} \end{cases}$$

7.118.

Solve in $\mathbb{Z}$ the equation:

$$x^y + 5z - 2 = 0$$

Solution:

The equation can be written: $x^y + 5z - 2 = 0$

x has the form $5k_1 + r_1,\ k_1 \in \mathbb{Z},\ r_1 \in \{0,1,2,3,4\}$ and it must have $x^y \equiv 2 \pmod 5$ or $(5k_1 + r_1)^y \equiv r_1^y \equiv 2 \pmod 5$

Thus $r_1 \neq 0,\ r_1 \neq 1,\ r_1 \neq 4$

For $r_1 = 2$ one has $2^{M_4+1} \equiv 2 (\text{mod}\,5)$ and

For $r_1 = 3$ one has $3^{M_4+3} \equiv 2 \pmod 5$

Therefore

$$\begin{cases} x = 5k_1 + 2 : \ k_1 \in \mathbb{Z} \\ y = 4k_2 + 1 : \ k_2 \in \mathbb{N} \\ z = \dfrac{2 - (5k_1 + 2)^{4k_2+1}}{5} \end{cases} \qquad (1)$$

and

x = $5k_1 + 3$, $k_1 \in$ Z;

y = $4k_2+3$, $k_2 \in$ Z; (2)

z = $\{2 - (5k_1 + 3)^{4k2+3}\} / 5$.

One can observe that $z \in \mathbb{Z}$ in (1) and also in (2). It results that:

$2^{4k_2+1} \equiv 2 \pmod 5$ and $3^{4k_2+3} \equiv 2 \pmod 5$

The integer general solution is obtained by bringing together (1) and (2).

7.119.

Solve in $\mathbb{N}$ the equation: $x! = y^z$

Solution:

$$\begin{cases} x = 0 \Rightarrow \begin{cases} y = 1 \text{ and } z \in \mathbb{N} \\ y \in \mathbb{N}^* \text{ and z=0} \end{cases} \\ x = 1 \Rightarrow \begin{cases} y = 1 \text{ and } z \in \mathbb{N} \\ y \in \mathbb{N}^* \text{ and z=0} \end{cases} \\ x \geq a \Rightarrow y = x! \text{ and } z = 1 \\ \end{cases}$$

We prove the last affirmation, because the first two are banal. One supposes (by absurd) that $z = k \geq 2$. One has $x! = y^k$. One excludes the case $x = 2$ which implies $y = 2$ and $z = 1$. Therefore $x \geq 3$. From which $x!$ contains at least two prime divisors (2 and 3). Therefore y^k also admits at least two prime divisors (2 and 3), from which y admits at least two prime divisors (2 and 3).

Let's consider $x_1,...,x_p$ to be all the prime numbers inferior or equal to x (one has proved that $p \geq 2$) $x! = x_1^{\alpha_1}...x_p^{\alpha_p}$, $\alpha_p \in \mathbb{N}^*$, $1 \leq i \leq p$. Thus $x_1^{\alpha_1}...x_p^{\alpha_p} = y^z$. From which it results that $\alpha_1 : z, \alpha_2 : z....\alpha_p : z$

One considers x_p, the greatest prime number inferior or equal to x. In accordance to the theorem of Chebyshev, for $\frac{x}{2} \leq x_p \leq x$ there exists at least a prime number. Therefore $\frac{x}{2} < x_p \leq x$. Then $\alpha_p = \left[\frac{x}{x_p}\right] + \left[\frac{x}{x_p^2}\right] + ... = 1$. It results that $z = 1$. Thus (s) represents all the solutions of the equation.

7.120.

Determine the general form of the solution in the set of integer numbers of the equation:

$$\left|\frac{\sum_{i=1}^{n} x_i^{p_i}}{\sum_{j=1}^{m} x_j^{r_j}}\right| = 1$$

where $m, n, p_i, r_j \in \mathbb{N}^*$, $m \le n, r_j < p_i, j \in \{1,...,m\}$, where all the p_i are even numbers and all the r_s are odd numbers.

Solution:

One has

$$\left|\sum_{i=1}^{n} x_i^{p_i}\right| = \left|\sum_{j=1}^{m} x_j^{r_j}\right| \le \sum_{j=1}^{m}\left|x_j^{r_j}\right| = \sum_{j=1}^{m}\left|x_j\right|^{r_j}$$

and

$$\sum_{i=1}^{n}\left|x_i\right|^{p_i} = \sum_{j=1}^{m}\left|x_j\right|^{r_j} \ge \sum_{j=1}^{m}\left|x_j\right|^{r_j}$$

From which $\sum_{i=1}^{m}\left|x_i\right|^{p_i} = \sum_{j=1}^{m}\left|x_j\right|^{r_j}$, it results:

$$\sum_{j=1}^{m}\left|x_j\right|^{p_j} - \left|x_j\right|^{r_j} + \sum_{j=m+1}^{n}\left|x_j\right|^{p_j} = 0$$

Since $\left|x_j\right|^{p_j}\left|x_j\right|^{r_j} \ge 0$ and $\left|x_j\right|^{p_j} \ge 0$, $\forall j \in \{1,...,n\}$ one has $x_j = 0$ for $j \in \{m+1,...,n\}$ and $\left|x_j\right|^{p_j} - \left|x_j\right|^{r_j} = 0$ for $j \in \{1,...,m\}$. Therefore $x_j \in \{0,1,-1\}$ for $j \in \{1,...,m\}$.

The general form of the solution in integer numbers of the equation is:

$$\begin{cases} x_1 = \varepsilon_1, ..., x_m = \varepsilon_m, x_{m+1} = ... = x_n = 0; \text{ with } \varepsilon_j = 0 \text{ or } 1 \\ \sum_{1}^{m} \varepsilon_j \ne 0 \\ x_1 = \xi_1, ..., x_m = \xi_m, x_{m+1} = ... = x_n = 0; \text{ with } \xi_j = 0 \text{ or } 1 \\ \sum_{1}^{m} \xi_j \ne 0 \end{cases}$$

The number of solutions is $2\left(C_m^1 + C_m^2 + ... + C_m^m\right) = 2\left(2^m - C_m^0\right) = 2^{m+1} - 2$

7.121.

Determine the linear equation which admits the following solution in the set of integer numbers:

$$\begin{cases} x_1 = 3k_1 - 7k_2 + 5 \\ x_2 = k_1 + 2k_2 \\ x_3 = 4k_1 + 13k_2 - 71 \end{cases}$$

where k_1 and k_2 are parameters in $\mathbb{Z}$.

Solution:

The equation has three unknowns x_1, x_2, x_3.

Its general form is: $a_1^{'}x_1 + a_2^{'}x_2 + a_3^{'}x_3 = b$, with $a_i^{'}, b_i^{'} \in \mathbb{Q},\ i = 1,2,3$. Or $x_1 + \frac{a_2^{'}}{a_1^{'}}x_2 + \frac{a_3^{'}}{a_1^{'}}x_3 = \frac{b'}{a_1^{'}}$

By differently noting the coefficients, we obtain: $x_1 + a_2x_2 + a_3x_3 = b$ with $a_2, a_3, b \in \mathbb{Q}$.

One can write for k_1, k_2 arbitraries in $\mathbb{Z}$:

$$3k_1 - 7k_2 + 5 + a_2k_1 + 2a_2k_2 + 4a_3k_1 + 13a_3k_2 - 71a_3 = b$$

$$\left.\begin{array}{l} \text{For } k_1 = k_2 = 0 \Rightarrow 5 - 71a_3 = b \quad (1) \\ \text{For } k_1 = 0, k_2 = 1 \Rightarrow -7 + 2a_2 + 13a_3 = 0 \quad (2) \\ \text{For } k_1 = 0, k_2 = 1 \Rightarrow -7 + 2a_2 + 13a_3 = 0 \quad (3) \end{array}\right\} \qquad (4)$$

(One has used (1) to obtain (2) and (3).)

But (4) is a system of three equations with three unknowns which will be normally solved.

From (3) it results $a_2 = -4a_3 - 3$. From (2) it results now $-7 + 13a_3 - 8a_3 - 6 = 0$. Or $a_3 = \frac{13}{5}$, therefore $a_2 = \frac{67}{5}$.

From (1) it results $b = \frac{25 - 923}{5} = -\frac{898}{5}$.

Therefore the equation is:

$$5x_1 - 67x_2 + 13x_3 = -898$$

7.122.

One considers a natural $n \geq 3$ and $a \in \mathbb{R}$. Solve the inequality:

$\left[\frac{x+a}{n}\right] + \left[\frac{x-a}{n}\right] \geq [x]$; Discussion.

Solution:

Let's consider $x = ng + r,\ 0 \leq r < n,\ r \in \mathbb{R},\ q \in \mathbb{Z}$

I) If $a = nq_a,\ q_a \in \mathbb{Z}$, then the inequality of the problem becomes: $2\left[\frac{x}{n}\right] \geq [x]$ (1) which is equivalent to $(2-n)q \geq 0$.

Therefore $x \in M_1 = \{y \mid y = nq + r:\ 0 \leq r < n,\ r \in \mathbb{R},\ q \leq 0,\ q \in \mathbb{Z}\}$

II) If $a \neq nq_a:\ q_a \in \mathbb{Z}$, then a can be written

$$a \neq nq_a + r_a:\ 0 < r_a < n,\ r_a \in \mathbb{R},\ q_a \in \mathbb{Z} \qquad (2)$$

One can suppose $0 < a < n$, since, with (2), the inequality of the problem becomes equivalent to

$$q_a + \left[\frac{x + r_a}{n}\right] - r_a + \left[\frac{x - r_a}{n}\right] \geq [x]$$

One has $\left[\frac{x+a}{n}\right] + \left[\frac{x-a}{n}\right] \geq [x]$ (3)

which is equivalent to $(2-n)q + E(r) \geq [r]$ (4)

where $E(r) = \left[\frac{r+a}{n}\right] + \left[\frac{r-a}{n}\right] = \begin{cases} -1 \text{ if } r < \min\{a, n-a\} \\ 0 \quad \text{if } \min\{a, n-a\} \leq r < \max\{a, n-a\} \\ +1 \quad \text{if } r \geq \max\{a, n-a\} \end{cases}$

1) $0 \leq r < \min\{a, n-a\}$ then (4) $\Leftrightarrow (2-n)q - 1 \geq [r]$.

where $q \leq -1$.

$\alpha)(q = -1) \Rightarrow (4) \Leftrightarrow (n - 3 \leq [r]) \Rightarrow x \in M_2 = \{y \mid y = qn + r, 0 \leq r < \min\{a, n-a\}, [r] \leq n-3\}$

$\beta)(q = -2) \Rightarrow (4) \Leftrightarrow (2n - 5 \geq [r]) \Rightarrow x \in M_3 = \{y \mid y = qn + r, 0 \leq r < \min\{a, n-a\}, [r] \leq 2n-5\}$

$\gamma)(q \leq -3) \Rightarrow (4)$ is true

$\Rightarrow x \in M_4 = \{y \mid y = qn + r, 0 \leq r < \min\{a, n-a\}, q \leq -3, q \in \mathbb{Z}\}$

2)$\min\{a, n-a\} \leq r < \max\{a, n-a\}$.Then (4) $\Leftrightarrow (2-n)q \geq [r]$

$\alpha)(q = -1) \Rightarrow x \in M_5 = \{y \mid y = -n + r : \min\{a, n-a\} \leq r < \max\{a, n-a\}, [r] \leq n-2\}$

$\beta)(-q \leq -2) \Rightarrow x \in M_6 = \{y \mid y = qn + r, \min\{a, n-a\} \leq r <, \max\{a, n-a\}, q \leq -2, q \in \mathbb{Z}\}$

3)$\max\{a, n-a\} \leq r < n \Rightarrow (4) \Leftrightarrow ((2-n)q\_1 \geq [r]) \Rightarrow x \in M_7 =$

$= \{y \mid y = qn + r, \max\{a, n-a\} \leq r < n, q \leq -1, q \in \mathbb{Z}\}$

And we analyzed all the cases.

7.123.

Find a method for solving in natural numbers the equation:

$$\sum_{i=1}^{m} x_i + a = b \cdot \sqrt[p]{c \cdot \prod_{j=1}^{n} y_j}$$

with $a,b,c\in\mathbb{N},\ p\in\mathbb{N}^*$.

Solution:

The equation will have an infinity of natural solutions when $b\cdot c\neq 0$.

$\forall h\in\{1,...,n-1\}$ one takes $y_h\in\mathbb{N}$ arbitrary. We construct y_n such that $cy_1...y_n=k^p$ with $k\in\mathbb{N}$. (1)

Let's consider $y_n^{'}$ the smallest natural number which has the property (1). (There exists a such $y_n^{'}$ because, if one writes each $y_h=\prod_i p_i^{\alpha_{ih}}$ with $\alpha_{ih}\in\mathbb{N}$ and p_i being the i-th prime (positive) number $h\in\{1,...,n-1\}$, one takes $y_n^{'}=\prod_i p_i^{\beta_{in}}$, with $\beta_{in}\in\mathbb{N}$ and the β_{in} are selected such that $\gamma_i+\sum_{h=1}^{n-1}\alpha_{ih}+\beta_{im}=M_p$, where one has written $c=\prod_i p_i^{\gamma_i}$ with $\gamma_i\in\mathbb{N}$, and β_{in} (for each i) is the smallest natural number which verifies this property.)

We construct $y_{n=}y_n^{'}\cdot t^p$, with $t\in\mathbb{N}^*$, t being a parameter. The equation becomes $\sum_{i=1}^{m}x_i+a=bkt$ where the unknowns are $x_1,...x_n,\ t$. One has: $\sum_{i=1}^{m}x_i+a=\alpha t$, where we noted $bk=\alpha\in\mathbb{N}$.

A) If $bc=0$, then the equation $\sum_{i=1}^{m}x_i+\varepsilon=0$ does not admit a solution in natural numbers.

B) If $bc\neq 0$, then $\alpha\neq 0$. The equation admits an infinity of natural solutions: $\forall s\in\{1,...,m-1\}$, $x_s=\alpha w_s+r_s$ where $bc\neq 0\le r_s\le\alpha-1$, a $r_s\in\mathbb{N}$ and w_s is a natural parameter, and $x_m=\alpha w_m+r_m$, where $0\le r_m\le\alpha-1$ but r_m is chosen such that $\sum_{i=1}^{m}r_i+a=M_\alpha$ (one has noted M_α a multiple of α) and also $r_m\in\mathbb{N}$, w_m = natural parameter.

7.124.

It is given the equation $P(x_1,..,x_n)=0$ with $P(x_1,..,x_n)$ a second degree polynomial in $x_1,..,x_n$ with real coefficients. Show that Δ_{x_i} is perfect square if and only if Δ_{x_j} is a perfect square. (By Δ_{x_h} one has noted the determinant of the initial second degree equation relative to the unknown x_h.)

Solution:

Necessity.

(The reciprocal proof will be similar.)

The equation can be written: $A_i x_i^2 + B_i x_i + C_{x_i} = 0$ (1)

Where A is a constant and B_i a first degree linear function in $x_1,..,x_{i-1},x_{i+1},...,x_n$. (A is a constant, because otherwise it would result that P has a degree strictly superior to 2.)

Δ_{x_i} being a perfect square, it implies that $\Delta_{x_i} = k_i^2\left(x_1,..,x_{i-1},x_{i+1},...,x_n\right)$ with k_i a first degree linear function in $x_1,..,x_{i-1},x_{i+1},...,x_n$.

(1) becomes $A\left(x_i - \frac{B_i + k_i}{2A}\right)\left(x_i - \frac{B_i - k_i}{2A}\right) = 0$, where $A\frac{2Ax_i + B_i - k_i}{2A}\frac{2Ax_i + B_i + k_i}{2A} = 0$ (2)

Since $2Ax_i + B_i + k_i$ are first degree functions in $x_1,..,x_n$ one can compute x_j in function of $x_1,..,x_{j-1},x_{j+1},...,x_n$.

(2) becomes $A\left(\frac{x_j - f_1\left(x_1,..,x_{j-1},x_{j+1},...,x_n\right)}{2A}\right)\cdot\left(\frac{x_j - f_2\left(x_1,..,x_{j-1},x_{j+1},...,x_n\right)}{2A}\right)$ by notation.

By notation $B\left(x_j - g_1\left(x_1,..,x_{j-1},x_{j+1},...,x_n\right)\right)\cdot\left(x_j - g_2\left(x_1,..,x_{j-1},x_{j+1},...,x_n\right)\right) = 0$

where g_1, g_2 are also linear functions in $x_1,..,x_{j-1},x_{j+1},...,x_n$. Therefore Δ_{x_j} is a perfect square.

7.125.

Having $a_{\sigma(n)}x^n + ... + a_{\sigma(1)}x + a = 0$ all the equations obtained by circular permutations of coefficients, on the set $\{a_n,....,a_1,a_0\}$, $a_i \in \mathbb{R}^*$, $0 \le i \le n$, n even.

a) Show that these equations admit a real common root if and only if $a_n + ... + a_1 + a_0 = 0$

b) Let x_0 be the real common root, S the sum of all the roots of the equations, P the product of all the roots of the equations. Then:

$$S - x_0 + (n+1)\frac{P}{x_0} = -\left(\sum_{i=0}^{n-2}\frac{a_{n-i}}{a_{n-i-2}} + \frac{a_0}{a_{n-1}} + \frac{a_1}{a_n}\right)$$

Solution:

In total one has n+1 equations.

a) One has on $a_{\sigma_k(n)}x_0^n + ... + a_{\sigma_k(1)}x_0 + a_{\sigma_k(0)} = 0,\ \ 0 \le k \le n$.

Therefore let x_0 be the real common root. One does the sum of all the relations (1), and it comes:

$$S_1\left(x_0^n + x_0^{n-1} + ... + x_0^1 + 1\right) = 0$$

It results $S_1 = 0$ or $x_0^n + x_0^{n-1} + ... + x_0^1 + 1 = 0$, but it does not exists a $x_0 \in \mathbb{R}$ which annulets the equation $x^n + ... + x^1 + 1 = 0$, with n even. From which $S_1 = 0$; but $S_1 = a_n + ... + a_1 + a_0$

Reciprocal: $S_1 = 0$. This implies that $a_{\sigma_k(n)}1^n + ... + a_{\sigma_k(1)}1 + a_{\sigma_k(0)} = 0,\ \ 0 \le k \le n$.

Therefore all the equations admit the real common root $x_0 = 1$.

b) $S-x_0=\left(-\frac{a_1}{a_n}-1\right)+\left(-\frac{a_0}{a_{n-1}}-1\right)+\left(-\frac{a_n}{a_{n-2}}-1\right)+...+\left(-\frac{a_2}{a_0}-1\right)$

$P=\frac{a_0}{a_n}\frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_1}{a_0}=1$

Therefore $S-x_0+(n+1)\frac{P}{x_0}=-\left(\frac{a_0}{a_{n-1}}+\frac{a_1}{a_n}+\sum_{i=0}^{n-2}\frac{a_{n-i}}{a_{n-i-2}}\right)$.

7.126.

Solve in $\mathbb{R}$ the equation $(x+1)^x+(x+2)^x=(x+3)^x$

(Amer. Math. Monthly, 1985)

Solution:

Obviously $x>-1$, since the bases of the powers must be non-negative, and that for $x=-1$ the operation 0^{-1} does not have sense.

If one has $(x+3)^x\neq 0$, we divide the equation by this. It results that

$$\left(\frac{x+1}{x+3}\right)^x+\left(\frac{x+2}{x+3}\right)^x=1$$

Let's consider $g_1(x)=\left(\frac{x+1}{x+3}\right)^x$ and $g_2(x)=\left(\frac{x+2}{x+3}\right)^x$ and $f(x)=g_1(x)+g_2(x)$ which have the same domain of definition $]-1,+\infty[$

We show that g_1 and g_2 are strictly declining, from which it results that f is also strictly decreasing.

On constructs the graphic representations of g_1 and g_2.

For

The line with the equation is a horizontal asymptote when x tends toward

The line with the equation is a vertical asymptote when x tends toward

The graph of is found in the figure (1).

For g_1

$$\lim_{x\to\infty} g_1(x)=\lim_{x\to\infty}\left[\left(1+\frac{-2}{x+3}\right)^{\frac{x+3}{-2}}\right]^{\frac{-2}{x+3}x}=x^{-2}=\frac{1}{x^2}<\frac{1}{4}$$

$x=0\Rightarrow g_1(0)=1$

From which the line with the equation $y=\frac{1}{e^2}$ is a horizontal asymptote when x tends toward $+\infty$.

The graph of g_1 is found in the figure (2).

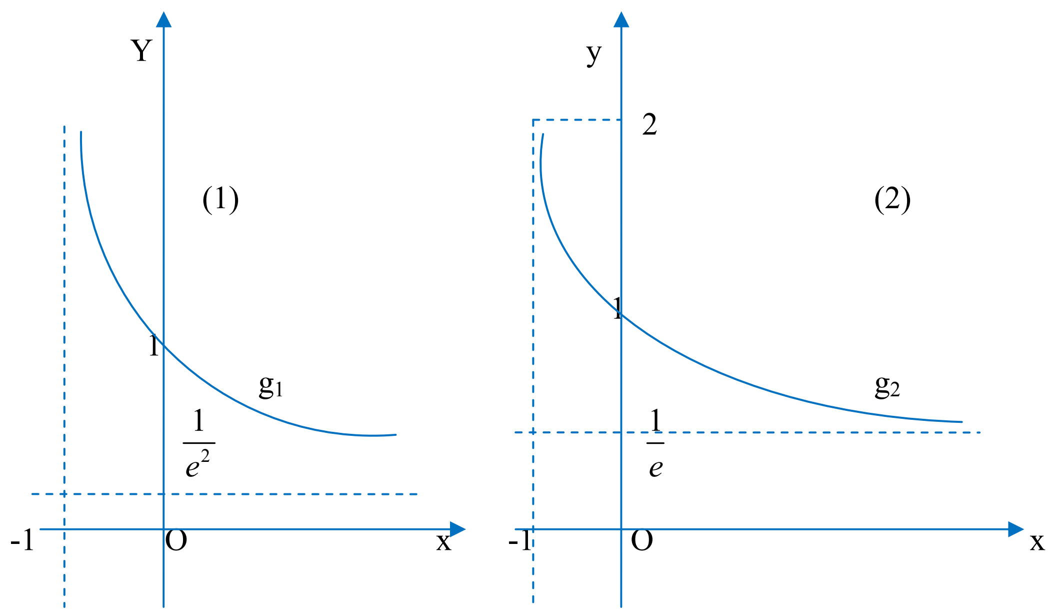
For $g_2 : x = 0 \Rightarrow g_2(0) = 1;\ x = -1 \Rightarrow g_2(-1) = 2$

$$\lim_{x\to\infty} g_2(x) = \lim_{x\to\infty}\left[\left(1+\frac{-1}{x+3}\right)^{\frac{x+3}{-1}}\right]^{1+\frac{-1}{x+3}\cdot x} = e^{-1} = \frac{1}{e} < \frac{1}{2}$$

From where the line from equation $y = \frac{1}{e^2}$ is a horizontal asymptote when x tends toward $+\infty$

The graphic of g_2 can be seen in figure (2)

From (1) and (2) it results that g_1 and g_2 are strictly declining on $]-1,+\infty[$, therefore one has the same property for f. Because $sf(2) = 1$ it results that $x = 2$ is the only real solution of the equation.



7.127.

Knowing that $b^2 - 4ac$ is a perfect square, find a method for solving in the set of integer numbers the equation $ax^2 + bxy + cy^2 + dx + fy + e = 0$, with a,b,c,d,f,e integers.

Solution:

We try to write the equation using the form

$$ax^2 + bxy + cy^2 + dx + fy + e = (\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2) + \delta$$

where $\alpha_1, \beta_1, \gamma_1, \delta \in \mathbb{Q},\ i \in \{1,2\}$. One has

$$\alpha_1\alpha_2 x^2 + \alpha_1\beta_2 xy + \alpha_1\gamma_2 x + \beta_1\alpha_2 xy + \beta_1\beta_2 y^2 + \beta_1\gamma_2 y + \gamma_1\alpha_2 x + \gamma_1\beta_2 y + \gamma_1\gamma_2 + \delta =$$

$ax^2+bxy+cy^2+dx+fy+e$.
By identification it results:

$$(1) \qquad \begin{cases} \alpha_1\alpha_2 = a \\ \beta_1\beta_2 = c \\ \alpha_1\beta_2 + \beta_1\alpha_2 = b \end{cases}$$

and

$$(2) \qquad \begin{cases} \gamma_1\alpha_2 + \alpha_1\gamma_2 = d \\ \beta_1\gamma_2 + = e\gamma_1\beta_2 = f \\ \gamma_1\gamma_2 = e \end{cases}$$

which is a second degree system of 6 equations with 7 unknowns $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta$. From (1) one obtains $\alpha_2 = \frac{a}{\alpha_1}, \beta_2 = \frac{c}{\beta_1}$ and $\alpha_1\frac{c}{\beta_1} + \frac{a}{\alpha_1}\beta_1 = b$

It results that $cz + \frac{a}{z} = b$ where $z = \frac{\alpha_1}{\beta_1} \in \mathbb{Q}$. From which $cz^2 - bz + a = 0$; but it is necessary that Δ_z to be a perfect square, that is $b^2 - 4ac = k^2, k \in \mathbb{Z}$, what it is satisfied by hypothesis.
Therefore $z = \frac{\alpha_1}{\beta_1} = \frac{b \pm k}{2c}$. Then $\beta_1 = \alpha_1 \cdot \frac{2c}{b \pm k}$ and $\beta_2 = \frac{b \pm k}{2\alpha_1}$
One replaces in (2), and one obtains:
$\gamma_1\frac{a}{\alpha_1} + \alpha_1\gamma_2 - d$; $\frac{2c}{b \pm k}\alpha_1\gamma_2 + \frac{b \pm k}{2}\alpha_1\frac{1}{\alpha_1} = f$ from which one finds $\frac{\gamma_1}{\alpha_1}$ and $\alpha_1\gamma_2$ as rationales. (3) since one has a linear system of 2 equations with two unknowns $\left(w = \frac{\gamma_1}{\alpha_1},\ t = \alpha_1\gamma_2\right)$
From (3) it results that one can express γ_1 and γ_2 in function of α_1. From the equation $\gamma_1\gamma_2 + \delta = e$ one can take δ in function of α_1. One gives a convenient value to α_1 and one thus determinates all the unknowns. One has: $(\alpha'_1 x + \beta'_1 y + \gamma'_1)(\alpha'_2 x + \beta'_2 y + \gamma'_2) = \delta'$. One puts the coefficients to have the same denominator and one eliminates this. Then one finds $\alpha'_1,\alpha'_2,\beta'_1,\beta'_2,\gamma'_1,\gamma'_2,\delta' \in \mathbb{Z}$.
Now one decomposes δ' an integer factors and one tries all the possibilities, which will give a system of Diophantine equations: $\alpha'_i x + \beta'_i y + \gamma'_i = d_i$ with $d_1 d_2 = -\delta'$ and $i \in \{1,2\}$

7.128.

One considers the equation $\sum_{i=0}^{n} a_i x^i = 0$ with all the real coefficients $a_n \neq 0$ .and $n \geq 0$ natural. Show that if $(n-1)a_{n-1}^2 - 2na_n a_{n-2} < 0$ then the equation does not have all its roots in $\mathbb{R}$.

Solution:

$$S=(x_1-x_2)^2+(x_1-x_3)^2+...+(x_1-x_n)^2+(x_2-x_3)^2+...+(x_2-x_n)^2+...+(x_{n-1}-x_n)^2=$$
$$=(n-1)\left[x_1^2+...+x_n^2\right]-2\left[x_1x_2+x_1x_3+...+x_1x_n+x_2x_3+...+x_2x_n+...+x_{n-1}x_n\right]=$$
$$=(n-1)\left[x_1+...+x_2\right]^2+\left[-2-2(n-1)\right]\left[x_1x_2+...+x_1x_n+x_2x_3+...+x_2x_n+...+x_{n-1}x_n\right]=$$
$$=(n-1)\frac{a_{n-1}}{a_n^2}+\frac{a_{n-2}}{a_n}(-2n)=\frac{1}{a_n^2}\left[(n-1)a_{n-1}^2-2na_na_{n-2}\right]<0$$

(One has noted $x_1, x_2, ..., x_n$ the roots of the equation.)

It results that the given equation does not have all its roots in $\mathbb{R}$, since otherwise it would result that $S \geq 0$.

Remark: for n=2 one obtains the well-known result that if the determinant of a second degree equation, $\Delta < 0$, then the equation has complex roots.

7.129.

Solve the following system:

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} x_i = \alpha_i,\ \ 1 \leq j \leq n \ \text{ with } n \geq 2$$

Solution:

One explicitly writes the system:

$$\begin{cases} \quad\ \ x_2+x_3+...+x_{n-1}+x_n=\alpha_1 \\ x_1 \quad\ \ +x_3+....+x_{n-1}+x_n=\alpha_2 \\ .... \\ x_1+x_2+x_3+....+x_{n-1} \quad\ \ =\alpha_n \end{cases}$$

One does the subtraction between the first equation and each other equation. One has:

$$-x_1+x_k=\alpha_1-\alpha_k,\ \ 2\leq k\leq n$$

One replaces $x_k=\alpha_1-\alpha_k$, $2\leq k\leq n$ in the first equation and one obtains:

$$(n-1)x_1+(n-1)\alpha_1=\alpha_1+\alpha_2+...+\alpha_n$$

From which $x_1=\frac{1}{n-1}\left[-(n-2)\alpha_1+\alpha_2+...+\alpha_n\right]$

One determines in a similar way the unknowns $x_2, ..., x_n$. The solution of the system is:

$$x_i=\frac{1}{n-1}\left[\alpha_1+...+\alpha_{i-1}-(n-2)\alpha_i+\alpha_{i+1}+...+\alpha_n\right],\ \ 1\leq i\leq n.$$

7.130.

Solve in the set of integer numbers the system:

$$\begin{cases} -17x+52y=130 \\ 35x-27y+26z=84 \end{cases}$$

Solution:

We solve in integer numbers the first equation of the system, which is a Diophantine equation, and its general solution will be:

$$(1)\begin{cases} x=52t-26 \\ y=17t-6 \end{cases}, \text{ with } t\in\mathbb{Z}.$$

By replacing the values of x and y in the second equation, one has:

(2) $1361t+26z=832$

with $(t,z)\in\mathbb{Z}^2$

One solves this equation in integer numbers; its general solution will be

$$\begin{cases} t=26k \\ z=-1361k+32 \end{cases}, \textit{ with } k\in\mathbb{Z}$$

This is used in (1):

$$\begin{cases} x=52\cdot 26k-26 \\ y=17\cdot 26k-6 \end{cases}, \text{ with } k\in\mathbb{Z}$$

Therefore, the general solution of the initial system is:

$$\begin{cases} x=1352k-26 \\ y=442k-6 \\ z=-1361k+32 \end{cases}, \text{ with } k\in\mathbb{Z}$$

<u>Observation</u>: The method which was used is the normal substitution, which is also utilized to solve in real numbers.

7.131.

Let's consider a linear homogeneous system having as associated matrix $A\in M(m,n,\mathbb{Q})$, which admits the rank $r(A)<n$. (The rank of a matrix is the order of the greatest non-null determinant which can be extracted from this matrix.) Show that the system admits non-banal integer solutions

Solution:

One considers the initial system $\sum_{j=1}^{n} a_{ij}x_j = 0$, $1 \le i \le m$, with all the $a_{ij} \in \mathbb{Q}$. One brings the coefficients to the same denominator and eliminates it. One obtains a system which has all the coefficients integers. One notes $r(\Delta) = r < n$ (according to the hypothesis). If one eliminates the secondary equations then r principal equations remain. We normally solve in $\mathbb{R}^n$, applying the method of Cramer. Without diminishing the generality one supposes that $x_1, ..., x_r$ are the principal variables.

Thus $x_{r+1}, ..., x_n$ will be the secondary variables. Because $r < n$, there exists at least a secondary variable. The real solutions of the system are:

$x_h = \frac{1}{\Delta}\sum_{t=r+1}^{n} b_{ht}x_t$, $1 \le h \le r$, with all the b_{ht}, Δ integers, where Δ is the determinant which contains the columns $1, ..., r$ and the lines $1, ..., r$.

If one notes $x_t = \Delta k_t$, $r+1 \le t < n$, with $k_t \in \mathbb{Z}$ (parameters) from which

$$x_h = \sum_{t=r+1}^{n} b_{ht}k_t, \ 1 \le h \le r$$

It results an integer solution r undetermined for our system. If we give non null values to parameters $k_{r+1}, ..., k_n$ we obtain a particular integer solution non trivial .

7.132.

Determine the matrices A and B of order n such that:

$\left(1\ x\ x^2 ...\ x^{n-1}\right) \cdot A = \left(1\ 1{+}x\ 1{+}x^2 ...\ 1{+}x^{n-1}\right)$ and

$\left(1\ 1{+}x\ 1{+}x^2 ...\ 1{+}x^{n-1}\right) \cdot B = \left(1\ x\ x^2 ...\ x^{n-1}\right)$

for any x real.

a) Compute A^m and B^m, for $m \in N^*$.

b) Show that $A^k B^e = B^e A^k$ and $(AB)^p = (BA)^p \quad \forall p, e, k \in N^*$.

c) Show that if $\sum_{i=1}^{s} k_i = \sum_{i=1}^{s} e_i$ then $\prod_{i=1}^{s} A^{k_i} B^{e_i} = I_n$, where I_n is the unitary matrix of order n; and $k_i, e_i \in N^*$.

Solution:

Let's consider the matrix

$$A = \begin{pmatrix} a_{11} ... a_{1j} ... a_{1n} \\ a_{21} ... a_{2j} ... a_{2n} \\ .... \\ a_{n1} ... a_{nj} .... a_{nn} \end{pmatrix}$$

We note $v=\left(1\ x\ x^2 ...\ x^{n-1}\right)$ and $u=\left(1\ 1{+}x\ 1{+}x^2 ...\ 1{+}x^{n-1}\right)$.

Multiplying v by the first column of A we obtain:
$a_{11}+a_{21}x+...+a_{n1}x^{n-1}=1$, $\forall x\in\mathbb{R}$. One does $x=0$ and it results $a_{11}=1$, therefore $a_{21}x+...+a_{n1}x^{n-1}=0$, $\forall x\in\mathbb{R}$; that is this is the null polynomial because it has more than $n-1$ roots; from which $a_{21}=...=a_{n1}=0$.

One multiplies v by the column j of A ($2\le j\le n$) and one obtains:
$a_{1j}+a_{2j}+...+a_{j-1j}x^{j-2}+a_{jj}x^{j-1}+...+a_{nj}x^{n-1}=1+x^{j-1}+...+a_{nj}x^{n-1}=0 \quad \forall x\in\mathbb{R}$.
For $x=0$ one finds $a_{1j}=1$. Therefore
$a_{2j}+a_{2j}+...+a_{j-1j}x^{j-1}+\left(a_{jj}-1\right)x^{j-1}+...+a_{nj}x^{n-1}=0\ \forall x\in\mathbb{R}$. This polynomial, also, is null, thus $a_{2j}=a_{j-1,j}=...=a_{nj}=0$ and $a_{jj}-1=0$, or $a_{jj}=1$, with $2\le j\le n$. It results that

$$A=\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 0 & ... & 0 \\ ....0 \\ 0\ ....0 & 1 \end{pmatrix}$$

Let's consider $B=\begin{pmatrix} b_{11}...b_{1j}...b_{1n} \\ b_{21}...b_{2j}...b_{2n} \\ .... \\ b_{n1}...b_{nj}....b_{nn} \end{pmatrix}$. One multiplies u by the first column of B, and one finds:

$b_{11}+b_{21}+...+b_{n1}+b_{21}x+...+b_{n1}x^{n-1}=1$, $\forall x\in\mathbb{R}$.
For $x=0$, that implies $b_{11}+b_{21}+...+b_{n1}=1$. One has also $b_{21}x+...+b_{n1}x^{n-1}=0$, $\forall x\in\mathbb{R}$
From which $b_{21}=....=b_{n1}=0$

Multiplying u by the column j of B, we obtain:

$b_{1j}+b_{2j}+...+b_{nj}+b_{2j}x+...+b_{jj}x^{j}+...+b_{nj}x^{n-1}=x^{j-1}$, $\forall x\in\mathbb{R}$, $2\le j\le n$. If $x=0$, one finds $b_{1j}+b_{2j}+...+b_{nj}=0$. Therefore,
$b_{2j}x+...+b_{jj}x^{j}+...+b_{nj}x^{n-1}=x^{j-1}\Leftrightarrow b_{2j}x+...+b_{j-1j}x^{j-2}+\left(b_{jj}-1\right)x^{j-1}+...+b_{nj}x^{n-1}=0$
$\forall x\in\mathbb{R}$.
The same thing: $b_{2j}x=...=b_{j-1j}=b_{j+1j}=...=b_{nj}=0$ and $b_{jj}-1=0$, or $b_{jj}=1$. Because $b_{1j}+b_{2j}+...+b_{nj}=b_{1j}+1=0\Rightarrow b_{1j}=-1$. From which

$$B=\begin{pmatrix} 1-1... & -1 \\ 0\ 1\ 0\ ... & 0 \\ ....0 \\ ....0 \\ 0............01 \end{pmatrix}$$

a) One shows by recurrence that

$$A^m = \begin{pmatrix} 1m & \ldots\ldots & m \\ 0\ 1\ 0\ \ldots & & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & & 0 \\ 0\ldots\ldots\ldots\ldots & & 01 \end{pmatrix}$$

The case $m = 1$ is obvious.
One supposes the property true for m, it must be shown for $m+1$:

$$` A^{m+1} = A^m A \begin{pmatrix} 1 & m+1\ \ldots\ldots & m+1 \\ 0 & 1\quad 0\ \ldots\ldots & 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & 0 \\ 0\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots 0. & 1 \end{pmatrix}$$

In a similar mode one proves that

$$B^m = \begin{pmatrix} 1 & -m & \ldots\ldots -m \\ 0 & 1 & \ldots.0.\quad 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots.0.\quad 1 \end{pmatrix}$$

b) One sees that $AB = BA - I_n$.

$$A^k B^e = \underbrace{A\ldots A}_{k}\underbrace{B\ldots B}_{e} = \underbrace{A\ldots A}_{k-1} BA \underbrace{B\ldots B}_{e-1}\ldots = BA^k B^{e-1} = \ldots =$$

$$= B^{2e}A^k \Rightarrow AB = BA \Rightarrow (AB)^p = (BA)^p .$$

c) $A^{k_1}B^{e_1}A^{k_2}B^{e_2}\ldots.A^{k_s}B^{e_s} = A^{k_1}A^{k_2}B^{e_1}B^{e_2}\ldots A^{k_s}B^{e_s} = A^t B^t = (AB)^t = I_n^t = I_n$,

where one has noted $t = \sum_{i=1}^{s} k_i = \sum_{i=1}^{s} e_i$

7.133.

Let's consider the matrix: $A = \begin{pmatrix} ab \\ ba \end{pmatrix}$ with $a, b \in \mathbb{R}$.

1) Compute the matrix A^n, $n \in \mathbb{N}^*$.
2) Discuss the limit:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \det\left(A^k\right)}{\det\left(\sum_{k=1}^{n} A^k\right)}$$

Solution

I) One will prove by the recurrence method that

$$A^n = \begin{pmatrix} \alpha_n \beta_n \\ \beta_n \alpha_n \end{pmatrix}, \text{ with } \alpha_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} C_n^{2i} a^{n-2i} b^{2i} \tag{1}$$

and

$$\beta_n = \sum_{i=0}^{\left[\frac{n+1}{2}\right]} C_n^{2i-1} a^{n-2i+1} b^{2i-1} \tag{2}$$

where $[x]$ represents the integer part of x.

The case $n=1$ is evident. We suppose the property true for n, it must be proved that the property is also true for $n+1$

$$A^{n+1} = A^n \cdot A = \begin{pmatrix} \alpha_n \beta_n \\ \beta_n \alpha_n \end{pmatrix} \begin{pmatrix} ab \\ ba \end{pmatrix} = \begin{pmatrix} a\alpha_n + b\beta_n & b\alpha_n + a\beta_n \\ b\alpha_n + a\beta_n & a\alpha_n + b\beta_n \end{pmatrix} = \begin{pmatrix} \alpha_{n+1} \ldots\beta_{n+1} \\ \beta_{n+1} \ldots \alpha_{n+1} \end{pmatrix},$$

it must be proved that $\alpha_{n+1} = a\alpha_n + b\beta_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} C_n^{2i} a^{n+1-2i} b^{2i} + \sum_{i=1}^{\left[\frac{n+1}{2}\right]} C_n^{2i-1} a^{n-2i+1} b^{2i} =$

$$= a^{n+1} + \sum_{i=1}^{\left[\frac{n}{2}\right]} \left(C_n^{2i} + C_n^{2i-1}\right) a^{n+1-2i} b^{2i} + \sum_{i=\frac{n+1}{2}} C_n^{2i-1} a^{n+1-2i} b^{2i} \tag{3}$$

if $\frac{n+1}{2} \in \mathbb{Z}$.

i) If $n = 2k$, then $\frac{n+1}{2} \notin \mathbb{Z}$, then the second sum from (3) is equal to zero, and $\left[\frac{n}{2}\right] = \left[\frac{n+1}{2}\right]$ it results that the expression of α_{n+1} from (1).

ii) If $n = 2k+1$, then $\frac{n+1}{2} \in \mathbb{Z}$ and the second sum of (3) is equal to $C_n^n a^0 b^{n+1} = b^{n+1}$, from where it results the expression of α_{n+1} from (1) because the first sum from (3) will have the same terms of $i = 0$ until $i = \left[\frac{n}{2}\right]$.

Also, it must be shown that

$$\beta_{n+1} = b\alpha_n + a\beta_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} C_n^{2i} a^{n-2i} b^{2i+1} + \sum_{i=1}^{\left[\frac{n+1}{2}\right]} C_n^{2i-1} a^{n-2i+2} b^{2i-1} =$$

$$= \sum_{i=1}^{\left[\frac{n+2}{2}\right]} C_n^{2i-2} a^{n-2i+2} b^{2i-1} + \sum_{i=1}^{\left[\frac{n+1}{2}\right]} C_n^{2i-1} a^{n-2i+2} b^{2i-1} =$$

$$\sum_{i=1}^{\left[\frac{n+1}{2}\right]} \left(C_n^{2i-2} + C_n^{2i-1}\right) a^{n+1-2i+1} b^{2i-1} + \sum_{\substack{i=\frac{n+2}{2} \\ if\, \frac{n+2}{2} \in \mathbb{Z}}} \left(C_n^{2i-2} a^{n-2i+2} b^{2i-1}\right); \tag{4}$$

if $\frac{n+2}{2} \in \mathbb{Z}$

i) If $n = 2k+1$, then $\frac{n+2}{2} \notin \mathbb{Z}$ and therefore the second sum from (4) is null. Also $\left[\frac{n+1}{2}\right] = \left[\frac{n+2}{2}\right]$, from which it results the expression of β_{n+1} from (2).

ii) If $n = 2k$, then $\frac{n+2}{2} \in \mathbb{Z}$, from which the second sum of (4) is equal to

$$C_n^{\left[\frac{n}{2}\right]+2-2} a^{n-2\left[\frac{n}{2}\right]+2-2} b^{2\left[\frac{n}{2}\right]+2-1} = b^{n+1}.$$

Similar like in (4), if one does the addition between the two sums, it results that for the first sum i takes the values from 1 until $\frac{n+2}{2} = \left[\frac{n+2}{2}\right]$, from which it results the expression of β_{n+1} from (2)/

2) $\det A^k = \alpha_k^2 - \beta_k^2 \Rightarrow \sum_{k=1}^{n} \det\left(A^k\right) = \sum_{k=1}^{n}\left(\alpha_k^2 - \beta_k^2\right) = \sum_{1}^{n}\left(\alpha_k - \beta_k\right)\left(\alpha_k + \beta_k\right) =$

$$= \sum_{1}^{n}(a-b)^k (a+b)^k = \sum_{1}^{n}\left(a^2-b^2\right)^k$$

C1) if $a^2 - b^2 \neq 1$ then $\sum_{1}^{n}\left(a^2-b^2\right)^k = \left(a^2-b^2\right)\frac{\left(a^2-b^2\right)^n - 1}{a^2-b^2-1}$;

$$\sum_{1}^{n} A^k = \begin{pmatrix} \sum_{1}^{n}\alpha^k & \sum_{1}^{n}\beta^k \\ \sum_{1}^{n}\beta^k & \sum_{1}^{n}\alpha^k \end{pmatrix} \Leftrightarrow \det\left(\sum_{1}^{n} A^k\right) = \left(\sum_{1}^{n}\alpha^k\right)^2 - \left(\sum_{1}^{n}\beta^k\right)^2 =$$

$$= \left(\sum_{1}^{n}\alpha^k - \sum_{1}^{n}\beta^k\right)\left(\sum_{1}^{n}\alpha^k + \sum_{1}^{n}\beta^k\right) = \left(\sum_{1}^{n}\left(\alpha^k - \beta^k\right)\right)\left(\sum_{1}^{n}\left(\alpha^k + \beta^k\right)\right) =$$

$$= \left(\sum_{1}^{n}(a-b)^k\right)\left(\sum_{1}^{n}(a+b)^k\right)$$

C2) If $a - b \neq 1$, then $\sum_{1}^{n}(a-b)^k = (a-b)\frac{(a-b)^n - 1}{a-b-1}$

C3) If $a + b \neq 1$, then $\sum_{1}^{n}(a+b)^k = (a+b)\frac{(a+b)^n - 1}{a+b-1}$;

<u>Discussion</u>

The cases:

A) The conditions C1, C2, C3 are satisfied.

B) It exists at least a condition from these which is not satisfied.

A) $\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{1}^{n}\det\left(A^k\right)}{\det\left(\sum\limits_{1}^{n}A^k\right)}\right)=\dfrac{(a-b-1)(a+b-1)}{a^2-b^2-1}\cdot\lim\limits_{n\to\infty}\dfrac{\left(a^2-b^2\right)^n-1}{\left(a^2-b^2\right)^n-(a-b)^n-(a+b)^n+1}=L$

One has the sub-cases:

A.I. $\left|a^2-b^2\right|<1$

A.II. $\left|a^2-b^2\right|>1$

A.III. $\left|a^2-b^2\right|=1\Rightarrow a^2-b^2=-1$ because one has C1.

A.I. It admits the situations:

A.I.1. $|a-b|<1$ and $|a+b|<1\Rightarrow L=-\dfrac{(a-b-1)(a+b-1)}{a^2-b^2-1}$

A.I.2. $|a-b|<1$ and $|a+b|>1\Rightarrow L=0$

A.I.3. $|a-b|<1$ and $|a+b|=1\Leftrightarrow|a-b|<1$ and $a+b=-1$, because one has C3. It results that the limit does not exist.

A.I.4. $|a-b|>1$ and $|a+b|<1\Rightarrow L=0$

A.I.5. $|a-b|>1$ and $|a+b|>1$ (this case is not possible, because it would result $\left|a^2-b^2\right|>1$, which is in contradiction with A.I.)

A.I.6. $|a-b|>1$ and $|a+b|=1$ (the same this case is not possible).

A.I.7. $|a-b|=1$ and $|a+b|<1\Leftrightarrow a-b=-1$ and $|a+b|<1$, because one has C2. It results that the limit does not exist.

A.I.8. $|a-b|=1$ and $|a+b|=1$

A.I.9. $|a-b|=1$ and $|a+b|>1$ these two cases don't exist because of A.I.

A.II. admits the situations:

A.II.1. $|a-b|>1$ and

$$|a-b|>1\Rightarrow L=\frac{(a-b-1)(a+b-1)}{a^2-b^2-1}\cdot\lim_{n\to\infty}\frac{1-\dfrac{1}{\left(a^2-b^2\right)^n}}{1-\dfrac{1}{(a+b)^n}-\dfrac{1}{(a-b)^n}+\dfrac{1}{\left(a^2-b^2\right)^n}}=$$

$$=\frac{(a-b-1)(a+b-1)}{a^2-b^2-1}$$

A.II.2. $|a-b|>1$ and $|a+b|<1\Rightarrow L=0$

A.II.3. $|a-b|>1$ and $|a+b|=1\Leftrightarrow|a-b|>1$ and $a+b=-1$, because one has C3. It results that the limit does not exist.

A.II.4. $|a-b|=1$ and $|a+b|>1\Leftrightarrow a-b=-1$ and $|a+b|>1$, because one has C2. It results that the limit does not exist.

A.II.5. $|a-b|=1$ and $|a+b|=1$

A.II.6. $|a-b|=1$ and $a+b<1$

A.II.7. $|a-b|<1$ and $|a+b|=1$

A.II.8. $|a-b|<1$ and $|a+b|<1$

These 4 cases don't exist because of A.I.

A.II.9. $|a-b|<1$ and $|a+b|>1 \Rightarrow L=0$

A.III. It results: $L=-\dfrac{(a-b-1)(a+b-1)}{a^2-b^2-1}\lim\limits_{n\to\infty}\dfrac{(-1)^n-1}{(-1)^n+1-(a-b)^n-(a^2+b^2)^n}$

But $a-b\neq 0$ and $a+b=\dfrac{a^2-b^2}{a-b}-\dfrac{1}{a+b}$, therefore

$$L=\frac{(a-b-1)(a+b+1)}{a^2-b^2-1}\lim_{n\to\infty}\frac{(-1)^n-1}{(-1)^n+1-(a-b)^n-\left(\dfrac{1}{a-b}\right)^n}$$

$a-b\notin\{-1,1\}$ because of C2 and because if $a-b=-1$ it would result $-1=a^2-b^2=(a-b)(a+b)=-(a+b)$, that is $a+b=1$, contradiction with C3.

A.III. admits the situations:

A.III.1. $|a-b|<1 \Rightarrow \left|-\dfrac{1}{a+b}\right|>1 \Rightarrow L=0$

A.III.2. $|a-b|>1 \Rightarrow \left|-\dfrac{1}{a-b}\right|<1 \Rightarrow L=0$

A.III.3. $|a-b|=1 \Leftrightarrow a-b=1$ or $a-b=-1$, which is not possible.

B) It exists at least a condition between C1, C2 or C3 which is not satisfied.

- if the condition C1 is not satisfied, then $a^2-b^2=1 \Rightarrow \sum\limits_{1}^{n}\det\left(A^k\right)=n$

- if the condition C2 is not satisfied, then $a-b=1 \Rightarrow \sum\limits_{1}^{n}(a-b)^k=n$

- if the condition C3 is not satisfied, then $a+b=1 \Rightarrow \sum\limits_{1}^{n}(a+b)^k=n$

We analyze all the possibilities for this case.

B.I. $a-b=1$ and $a+b\neq 1$. It results $a^2-b^2\neq 1$, $a=b+1$ and

$$L=\lim_{n\to\infty}\frac{(a^2-b^2)(a^2-b^2)^n-1}{a^2-b^2-1}\cdot\frac{1}{n}\cdot\frac{a+b-1}{(a+b)\left[(a+b)^n-1\right]}=$$

$$=\frac{(a^2-b^2)(a+b-1)}{(a+b)(a^2-b^2-1)}\lim_{n\to\infty}\frac{1}{n}\frac{(2b+1)^n-1}{(2b+1)^n-1}=0$$

B.II. $a-b=1$ and $a+b=1 \Rightarrow a^2-b^2=1 \Rightarrow a=1$ and $b=0 \Rightarrow L=\lim\limits_{n\to\infty}\dfrac{n}{n:n}=0$

B.III. $a-b\neq 1$ and $a+b=1 \Rightarrow a^2-b^2\neq 1$,

$$a=1-b \Rightarrow L=\lim_{n\to\infty}\frac{\left(a^2-b^2\right)\left[\left(a^2-b^2\right)^n-1\right]}{a^2-b^2-1}\cdot\frac{1}{n}\cdot\frac{a-b-1}{(a-b)\left[(a-b)^n-1\right]}=$$

$$=\frac{\left(a^2-b^2\right)(a-b-1)}{\left(a^2-b^2-1\right)(a-b)}\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\frac{(1-2b)^n-1}{(1-2b)^n-1}=0$$

B.IV. $a-b\neq 1$ and $a+b\neq 1 \Rightarrow a^2-b^2=1$ because of B).

$$\Rightarrow L=\lim_{n\to\infty}\frac{a-b-1}{(a-b)\left[(a-b)^n-1\right]}\cdot\frac{a+b-1}{(a+b)\left[(a+b)^n-1\right]}\cdot n=$$

$$=\frac{(a-b-1)(a+b-1)}{a^2-b^2}\lim_{n\to\infty}\frac{n}{\left(a^2-b^2\right)^n-(a-b)^n-(a+b)^n+1}=$$

$$=(a-b-1)(a+b-1)\lim_{n\to\infty}\frac{n}{2-(a-b)^n-(a+b)^n}=2(1-a)\lim_{n\to\infty}\frac{n}{2-(a-b)^n-(a+b)^n}$$

B.IV.1. $|a-b|<1$ and $|a+b|>1$. Applying the theorem of Stolz-Césaro, one obtains:

$$L=2(1-a)\lim_{n\to\infty}\frac{n}{(a+b)^n\left[\frac{2}{(a+b)^n}-\left(\frac{a-b}{a+b}\right)^n-1\right]}=0$$

B.IV.2. $|a-b|<1$ and $|a+b|=1 \Leftrightarrow |a-b|<1$ and $a+b=-1$.

This case does not exist because it would result $\left|a^2-b^2\right|<1$, therefore $a^2-b^2\neq 1$. Contradiction with B.IV.

$$\left.\begin{array}{l}
\text{B.IV.3. } |a-b|<1 \text{ and } |a+b|<1\\
\text{B.IV.4. } |a-b|=1 \text{ and } |a+b|<1\\
\text{B.IV.5. } |a-b|=1 \text{ and } |a+b|>1\\
\text{B.IV.6. } |a-b|>1 \text{ and } |a+b|>1\\
\text{B.IV.7. } |a-b|>1 \text{ and } |a+b|=1
\end{array}\right\}\quad \text{these 5 cases don't exist because of B).}$$

B.IV.8. $|a-b|>1$ and $|a-b|<1 \Rightarrow L=0$

B.IV.9. $|a-b|=1$ and $|a+b|=1 \Leftrightarrow a-b=-1$ and $a+b=-1 \Rightarrow a=-1$ and $b=0$, implies that the limit does not exist.

And, look, all the cases are analyzed.

The discussion about limit is very long, but it necessitate a good arrangement of cases (which depend of the real parameters a and b).

7.134.

Let $a_0,a_1,\ldots,a_n$ be real numbers, $a_n>0,\ n\in\mathbb{N}$

One considers the polynomial

$$P(x) = a_n x^n + ... + a_1 x + a_0$$

such that: if there exist $a_i < 0,\ \ 1 \le i \le n-1$ then the first non null coefficient $a_{i+k}:\ \ 1 \le k \le n-1$, before a_i verifies $|a_i| \le a_{i+k}$. If $x_1, ..., x_m \in \mathbb{R}_+$ determine the minimum of the expression:

$$E(x_1, ..., x_m) = \sum_{j=1}^{m} \left(P(x_j) + P\left(\frac{1}{x_j} \right) \right)$$

and the point $(x_1, ..., x_n)$ where this minimum is obtained.

Solution:

If $x > 0$ and $i > j$, then $x^i + \frac{1}{x^i} \ge x^j + \frac{1}{x^j} \ge 2$, the two equalities taking place when $x = 1$.

$$E(x_1, ..., x_m) = \sum_{j=1}^{m} \left(a_n \left(x_j^n + \frac{1}{x_j^n} \right) + ... + a_1 \left(x_j + \frac{1}{x_j} \right) + 2a_0 \right)$$

Let's consider

$$F(x_j) = P(x_j) + P\left(\frac{1}{x_j} \right) = \sum_{j=1}^{m} a_n \left(x_j^n + \frac{1}{x_j^n} \right) + ... a_1 \left(x_j + \frac{1}{x_j} \right) + 2a_0$$

If $a_i \ge 0,\ \ 1 \le i \le n-1$, then

$$\min_{x_j \in \mathbb{R}_+^*} F(x_j) = 2(a_0 + a_1 + ... + a_n)$$

and this is only realized for $x_j = 1$

If there exist $a_i < 0,\ \ 1 \le i \le n-1$, then the hypothesis of the problem will give $|a_i| \le a_{i+k},\ \ 1 \le k \le n-1$, and for the other coefficients one has $a_{i+1} = ... = a_{i+k-1} = 0$

Therefore:

$$a_{i+k} \left(x_j^{i+k} + \frac{1}{x_j^{i+k}} \right) + a_i \left(x_j^i + \frac{1}{x_j^i} \right) = (a_{i+k} |a_i|) \left(x_j^{i+k} + \frac{1}{x_j^{i+k}} \right) + |a_i| \left(x_j^{i+k} + \frac{1}{x_j^{i+k}} - x_j^i - \frac{1}{x_j^i} \right) \ge$$

$$\ge (a_{i+k} |a_i|) \cdot 2 + 0 = 2(a_i + a_{i+k})$$

the equality taking place only for $x_j = 1$, and the same it results that

$$\min_{x_j \in \mathbb{R}_+^*} F(x_j) = 2(a_0 + a_1 + ... + a_n)$$

which it is only realized for $x_j = 1$

One finds that $\min\limits_{\substack{x_j \in \mathbb{R}_+^* \\ j \in \{1, ..., m\}}} E(x_1, ..., x_m) = 2m(a_0 + a_1 + ... + a_n)$

and this is only realized for $(x_1, ..., x_m) = \underbrace{(1, ..., 1)}_{m}$

7.135.

Show that

a) The sum of the power of order $2p+1$ of $2k+1$ natural consecutive numbers is divisible by $2k+1$.

b) The sum of the power of order $2p+1$ of $2k$ natural consecutive numbers is divisible by $2k$ if and only if $p \geq 1$ and k is divisible by 2.

Solution

a) Let S_1 the sum of the powers of order $2p+1$ of $2k+1$ natural consecutive numbers. The $2k+1$ natural consecutive numbers constitute a complete system of residues modulo $2k+1$

That means:

$S_1 = 0^{2p+1} + 1^{2p+1} + 2^{2p+1} + ... + k^{2p+1} + (k+1)^{2p+1} + ... + (2k-1)^{2p+1} + (2k)^{2p+1} \pmod{2k+1}$

$2k - i \equiv -(i+1) \pmod{2k+1}$ for $0 \leq i \leq k-1$

$(2k-i)^{2p+1} \equiv -(i+1)^{2p+1} \pmod{2k+1}$

$0 \leq i \leq k-1$

Therefore:

$(2k)^{2p+1} \equiv -1^{2p+1} \pmod{2k+1}$

$(2k-1)^{2p+1} \equiv -k^{2p+1} \pmod{2k+1}$

....

$(k+1)^{2p+1} \equiv -k^{2p+1} \pmod{2k+1}$

$\Rightarrow$

$S_1 = 0^{2p+1} + 1^{2p+1} + 2^{2p+1} + ... + (k-1)^{2p+1} + k^{2p+1} + (k+1)^{2p+1} + ... + (2k-2)^{2p+1} + (2k-1)^{2p+1} \pmod{2k}$

Therefore

$S_1 : 2k+1$

b) Similarly, let S_2 the sum of the powers of order $2p+1$ of $2k$ consecutive natural numbers. The $2k$ natural consecutive numbers constitute a complete system of residues modulo $2k$. That is:

$S_2 = 0^{2p+1} + 1^{2p+1} + 2^{2p+1} + ... + (k-1)^{2p+1} + k^{2p+1} + (k+1)^{2p+1} + .$

$.. + (2k-2)^{2p+1} + (2k-1)^{2p+1} \pmod{2k}$. But $2k - i \equiv -i \pmod{2k}$ for $1 \leq i \leq k-1 \Rightarrow$

$(2k-i)^{2p+1} \equiv -i^{2p+1} \pmod{2k}$:

for $1 \leq i \leq k-1$

Then:

$(2k-1)^{2p+1} \equiv -1^{2p+1} \pmod{2k}$

$$\left(2k-2\right)^{2p+1} \equiv -2^{2p+1} (\bmod 2k)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(k+1\right)^{2p+1} \equiv -\left(k-1\right)^{2p+1} \left(\bmod 2k\right)$$

$$S_2 = 0 + 1^{2p+1} + 2^{2p+1} + \ldots + \left(k-1\right)^{2p+1} + k^{2p+1} - \left(k-1\right)^{2p+1} - \ldots - 2^{2p+1} - 1^{2p+1} \equiv k^{2p+1} \left(\bmod 2k\right)$$

Then

$$\left(S_2 : 2k\right) \Leftrightarrow \left(k^{2p+1} : 2k\right) \Leftrightarrow \left(k^{2p} : 2\right) \Leftrightarrow \left(p \geq 1 \text{ and } k : 2\right)$$

i.e. k is divisible by *2*.

7.136.

Prove that if a and m are integers, $m \neq 0$, then $\left(a^{|m|} - a\right)\left(|m|-1\right)$ is divisible by m.

Solution

I) m is prime.

a) $a = M_m$ (multiple of m) then $a^{|m|} - a = M_m$, and we find the conclusion.

b) $a \neq M_m$ we have, using the Fermat theorem $a^{|m|} - a = M_m$

II) m is not prime, $m \neq 0$

a) |m| = 4. Then

$$E = \left(a^{|m|} - a\right)\left(m-1\right)! \equiv 2a\left(a^{|m|-1} - 1\right) \equiv 2a\left(a^3 - 1\right)\left(\bmod 4\right)$$

If $a = M_2$ (multiple of 2), it results that $E \equiv 0\left(\bmod 4\right)$

If $a = M_2 + 1$, it results that $a^3 - 1 = M_2$, from where $E \equiv 0\left(\bmod 4\right)$

b) $|m| \neq 4$. Then $\exists a, b \in \mathbb{Z} - \{0, -1, +1\}$ such that $|m| = |a| \cdot |b|$

If $|a| \neq |b|$, because $|a| < |m| - 1$, $|b| < |m| - 1$, it is clear that $|a|$ and $|b|$ are among the factors of $\left(|m|-1\right)!$, then $\left(|m|-1\right)! \equiv 0 (\bmod m)$, from where we have the conclusion.

If $|a| = |b|$, because $|m| \neq 4$ and $|a| < \left(|m|-1\right)$, $|b| < |m| - 1$, we have $2|b| < |m| - 1$, therefore $|a|$ and $2|b|$ are found among the factors of $\left(|m|-1\right)!$, then $\left(|m|-1\right)! \equiv 0 (\bmod m)$, then $E \equiv 0\left(\bmod m\right)$.

Remark:

In II we proved the following assertion: if $m \in \mathbb{Z} - \{0, \pm 2\}$, then $\left(|m|-1\right)! \equiv 0 (\bmod m)$.

REFERENCES

[1] *Gazeta Matematică*, Gazette de la Société de Mathématique de Roumanie;

[2] *Revista Matematică a Elevilor din Timişoara;*

[3] *Ştiinţă şi Tehnică*, Bucharest;

[4] *Licăriri*, Craiova;

[5] *Matematikai Lapok*, Cluj-Napoca;

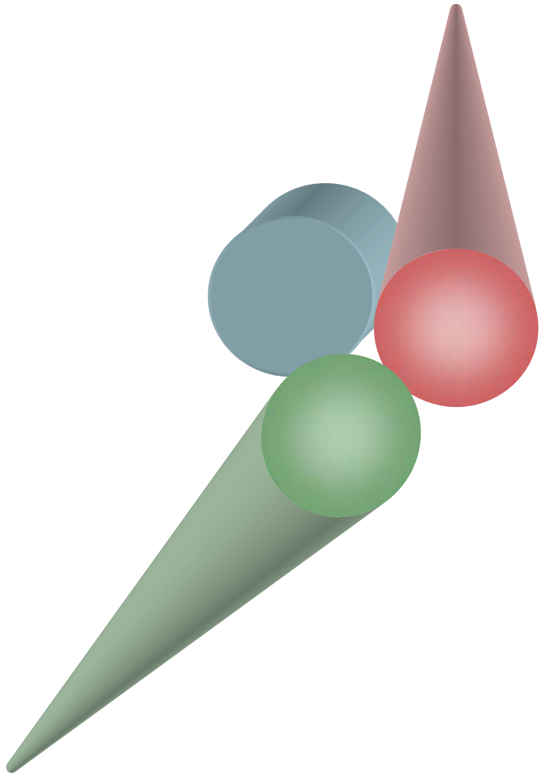
[6] *Les Annales de l'Université de Timişoara;*

[7] *Buletinul Ştiinţific şi Tehnic al Institultului Politehnic "Traian Vuia"*, Timişoara

[8] *Les Annals de l'Université de Craiova;*

[9] W. Sierpinski – *250 problèmes de théorie élémentaire des nombres* – Classiques Hachette, Paris, 1972;

[10] D. Gorll, G. Girard – *Les olympides internationals de mathématique* - Classiques Hachette, Paris, 1976.



This book is addressed to College honor students, researchers, and professors.

It contains 136 original problems published by the author in various scientific journals around the world.

The problems could be used to preparing for courses, exams, and Olympiads of Mathematics.

Many of these have a generalized form.

For each problem we provide a detailed solution.

I was a *professeur coopérant* between 1982-1984, teaching mathematics in French language, at *Lycée Sidi EL Hassan Lyoussi* in Sefrou, Province de Fès, Morocco.

I used many of these problems for selecting and training, together with other Moroccan professors, in Rabat city, of the Moroccan student team for the International Olympiad of Mathematics in Paris, France, 1983.

ISBN 9781599731421