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SMARANDACHE GT-ALGEBRAS

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Abstract. We introduce the notion of Smarandache GT-algebras, and the notion of Smarandache GT-filters of the Smarandache GT-algebra related to the Tarski algebra, and related some properties are investigated.

1. Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the $\{\vee, \rightarrow\}$-fragment of the propositional classical calculus. S. A. Celani ([5]) introduced Tarski algebras with a modal operator as a generalization of the concept of Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. Properties of filters in Tarski algebras were treated by S. A. Celani ([5]) and the authors ([7]). Recently, J. Kim, Y. Kim and E. H. Roh ([7]) considered decompositions and expansions of filters in Tarski algebras, and also they have shown that there is no non-trivial quadratic Tarski algebras on a field X with $|X| \geq 3$. However, we feel that the concept of Tarski algebra is relatively too strong for filters. Kim et al. ([8]) established a new algebra, called a GT-algebra, which is a generalization of Tarski algebra, and gave a method to construct a GT-algebra from a quasi-ordered set. Generally, a Smarandache Structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In this paper, we introduce the notion of $\mathcal{S}^{\mathfrak{T}}$GT-algebras and $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filters, and investigate some related properties. It's interesting to study the Smarandache Structure in GT-algebras.

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Let us review some definitions and results. By a *Tarski algebra* we mean an algebra $(X; \to, 1)$ of type $(2, 0)$ satisfying the following conditions:

(T1) $(\forall a \in X)(1 \to a = a)$.
(T2) $(\forall a \in X)(a \to a = 1)$.
(T3) $(\forall a, b, c \in X)(a \to (b \to c) = (a \to b) \to (a \to c))$.
(T4) $(\forall a, b \in X)((a \to b) \to b = (b \to a) \to a)$.

Definition 1.1. [8] By a *generalized Tarski algebra* (*GT-algebra*, for short) we mean an algebra $(X; \to, 1)$ of type $(2, 0)$ satisfying the following conditions: (T1), (T2), and (T3).

A reflexive and transitive relation $\mathfrak{R}$ on a set X is called a *quasi-ordering* of X, and the couple $(X, \mathfrak{R})$ is called a *quasi-ordered set* ([4]). Note that if X is a GT-algebra, then the relation $\leq$ by setting $x \leq y$ if and only if $x \to y = 1$ for any $a, b \in X$ is a quasi-ordering of X; with respect to this quasi-ordering 1 is the greatest element of X.

Example 1.2. Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	1
a	1	1	c	1
b	1	1	c	1
c	1	1	1	1
1	a	b	c	1

Then $(X; \to, 1)$ is a GT-algebra ([8]), and the relation

$$\mathfrak{R} := \{(a, a), (a, b), (a, 1), (b, a), (b, b), (b, 1), (c, a), (c, b), (c, c), (c, 1), (1, 1)\}$$

is a quasi-ordering of X, which is not an anti-symmetric relation of X.

Lemma 1.3. [8] *Let X be a GT-algebra. Then*

(p1) $(\forall a \in X)(a \leq 1)$.
(p2) $(\forall a, b \in X)(a \leq b \to a)$.
(p3) $(\forall a, b \in X)(a \to (a \to b) = a \to b)$.
(p4) $(\forall a, b \in X)(a \leq (a \to b) \to b)$.
(p5) $(\forall a, b, c \in X)(a \leq b \Rightarrow c \to a \leq c \to b)$.

Definition 1.4. [8] Let X be a GT-algebra. A nonempty subset F of X is called a *generalized Tarski-filter* (*GT-filter*, for short) of X if it satisfies the following conditions:

(F1) $(\forall a, b \in X)(b \in F \Rightarrow a \to b \in F)$.
(F2) $(\forall a, b \in X)(a \to b \in F, a \in F \Rightarrow b \in F)$.

Note that every GT-filter contains the element 1 by (T2) and (F1).

Theorem 1.5. [8] *Let F be a nonempty subset of a GT-algebra X. Then F is a GT-filter of X if and only if it satisfies $1 \in F$ and* (F2).

2. Main Theorem

Lemma 2.1. *Let $(X; \to, 1)$ be a nontrivial GT-algebra. For every $a(\neq 1) \in X$, the set $\{a, 1\}$ is a Tarski algebra under the operation on X.*

Proof. Straightforward. □

Lemma 2.1 shows that every nontrivial GT-algebra $(X; \to, 1)$ has a Tarski algebra of order 2. The following example shows that there is a GT-algebra in which there are no proper Tarski algebra of order more than equal to 3.

Example 2.2. Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

It is routine to check that $(X; \to, 1)$ is a GT-algebra which is not a Tarski algebra, and the sets $\{a, b, 1\}, \{a, c, 1\}, \{b, c, 1\}$ are not Tarski algebras.

Definition 2.3. A *Smarandache GT-algebra* (briefly, *$\mathcal{S}^{\mathfrak{T}}GT$-algebra*) is defined to be a GT-algebra X in which there exists a proper subset Ω of X such that

(i) $1 \in \Omega$ and $|\Omega| \geq 3$,
(ii) Ω is a Tarski algebra with respect to the same operation on X.

Note that any GT-algebra of order 3 cannot be an $\mathcal{S}^{\mathfrak{T}}$GT-algebra. Hence, if X is an $\mathcal{S}^{\mathfrak{T}}$GT-algebra, then $|X| \geq 4$. Notice that the GT-algebra X in Example 2.2 is not an $\mathcal{S}^{\mathfrak{T}}$GT-algebra.

Example 2.4. Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	1
a	1	b	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

It is easy to check that $(X; \to, 1)$ is an $\mathcal{S}^{\mathfrak{T}}$GT-algebra since $\Omega := \{a, b, 1\}$ is a Tarski algebra which is properly contained in X.

In what follows, let X and Ω denote an $\mathcal{S}^{\mathfrak{T}}$GT-algebra and a nontrivial proper Tarski algebra of order more than 2, respectively, unless specified.

DEFINITION 2.5. A nonempty subset F of X is called a *Smarandache GT-filter of X related to* Ω (briefly, $\mathcal{S}^{\mathfrak{T}}_{\Omega}$*GT-filter* of X) if it satisfies the following conditions:

(SF1) $1 \in F$,
(SF2) $(\forall x \in \Omega)(\forall a \in F)(a \to x \in F \Rightarrow x \in F)$.

EXAMPLE 2.6. Let $X := \{a, b, c, 1\}$ be the $\mathcal{S}^{\mathfrak{T}}$GT-algebra with $\Omega := \{a, b, 1\}$ in Example 2.4. Then the sets $F_1 := \{a, 1\}, F_2 := \{c, 1\}, F_3 := \{a, c, 1\}, F_4 := \{b, c, 1\}$ are $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filters of X.

EXAMPLE 2.7. Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	d	1
a	1	1	1	d	1
b	1	1	1	d	1
c	1	1	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

It can be readily check that $(X; \to, 1)$ is an $\mathcal{S}^{\mathfrak{T}}$GT-algebra with $\Omega := \{a, d, 1\}$. Then the set $F_1 := \{b, d, 1\}$ is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filters of X. But $F_2 := \{c, 1\}$ is not an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X since $c \to a = 1 \in F_2$ and $a \notin F_2$.

THEOREM 2.8. *Let Ω_1 and Ω_2 be Tarski algebras contained in a Smarandache GT-algebra X and $\Omega_1 \subset \Omega_2$. Then every $\mathcal{S}^{\mathfrak{T}}_{\Omega_2}$GT-filter of X is an $\mathcal{S}^{\mathfrak{T}}_{\Omega_1}$GT-filter of X, but the converse is not true.*

Proof. Straightforward. □

EXAMPLE 2.9. Let $X := \{a, b, c, d, e, 1\}$ be a set with the following Cayley table:

$\rightarrow$	a	b	c	d	e	1
a	1	1	1	d	1	1
b	a	1	c	d	1	1
c	a	b	1	d	1	1
d	a	b	c	1	1	1
e	a	b	c	d	1	1
1	a	b	c	d	e	1

Then $(X; \rightarrow, 1)$ is a GT-algebra, $\Omega_1 := \{a, d, 1\}$ and $\Omega_2 : \{a, b, c, d, 1\}$ are Tarski algebras. Hence we know that X is a Smarandache GT-algebra, and the subset $F := \{a, c, 1\}$ is an $\mathcal{S}^{\mathfrak{T}}_{\Omega_1}$GT-filter of X, but not an $\mathcal{S}^{\mathfrak{T}}_{\Omega_2}$GT-filter of X since $a \rightarrow b = 1 \in F$ and $a \in F$ but $b \notin F$.

Example 2.9 shows that there exists a Tarski algebra Ω contained in a Smarandache GT-algebra X such that an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X is not a GT-filter of X.

THEOREM 2.10. *For any $a \in X$, the set $[a) := \{x \in X | a \leq x\}$ is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X.*

Proof. Obviously, $1 \in [a)$. Let $z \in \Omega$ and $x \in [a)$ and $x \rightarrow z \in [a)$. Then we have

$$a \rightarrow z = 1 \rightarrow (a \rightarrow z) = a \rightarrow (x \rightarrow z) = 1.$$

Hence $z \in [a)$. Therefore, $[a)$ is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X. □

LEMMA 2.11. *Every $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter F of X satisfies the following inclusion:*

$$\Omega \rightarrow F \subseteq F,$$

where $\Omega \rightarrow F := \{x \rightarrow a | x \in \Omega, a \in F\}$.

Proof. Let $z \in \Omega \rightarrow F$. Then $z = x \rightarrow a$ for some $x \in \Omega$ and $a \in F$. Thus we have $z \in F$ since $a \rightarrow z = a \rightarrow (x \rightarrow a) = 1 \in F$. □

Lemma 2.11 shows that every $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter F of X satisfies the conditions

$$\Omega \rightarrow F \subseteq F, \text{ and (SF2)}.$$

The following example shows that the converse is not true in general.

EXAMPLE 2.12. Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

$\rightarrow$	a	b	c	1
a	1	b	c	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

It is ready to check that $(X; \rightarrow, 1)$ is an $\mathcal{S}^{\mathfrak{T}}$GT-algebra with $\Omega := \{a, c, 1\}$. Let $F := \{b\}$. Then F satisfies the conditions $\Omega \rightarrow F \subseteq F$ and (SF2). But $1 \notin F$.

If F is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X satisfies $\Omega \cap F \neq \emptyset$ and $\Omega \rightarrow F \subseteq F$, then there exists $a \in \Omega \cap F$, and so we have $1 = a \rightarrow a \in F$. Hence we obtain the following theorem.

THEOREM 2.13. *Let F be a nonempty subset of X that satisfies $\Omega \cap F \neq \emptyset$. Then F is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X if and only if $\Omega \rightarrow F \subseteq F$ and* (SF2).

For any GT-algebra X and $x, y \in X$, we denote

$$A(x, y) := \{z \in X | x \leq y \rightarrow z\}.$$

THEOREM 2.14. *For any $x, y \in X$, the set $A(x, y)$ is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X.*

Proof. Straightforward. □

Now, we give a characterization of $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filters.

THEOREM 2.15. *Let F be a nonempty subset of X. Then F is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X if and only if for any $x, y \in F$, either $A(x, y) \subseteq F$ or $A(y, x) \subseteq F$.*

Proof. The necessity is straightforward. Suppose that either $A(x, y) \subseteq F$ or $A(y, x) \subseteq F$ for every $x, y \in F$. Then we have $1 \in A(x, x) \subseteq F$. Let $x \in \Omega$ and $y \in F$ satisfy $y \rightarrow x \in F$. Then we have $x \in A(y \rightarrow x, y) \subseteq F$. Hence F is an $\mathcal{S}^{\mathfrak{T}}_{\Omega}$GT-filter of X. □

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References

[1] M. Abad, J. P. Dias Varela, and M. Zander, *Varieties and Quasivarieties of Monadic Tarski algebras,* Sientiae Mathematicae Japonicae **56** (2002), no. 3, 599-612.

[2] J. C. Abbott, *Semi-boolean algebras,* Mathematicki Vesnik **4** (1967), no. 19, 177-198.

[3] J. C. Abbott, *Implicational algebras,* Bull. Math, R. S. Roumaine **11** (1967), 3-23.

[4] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Colloq. Publ. **25**, second edition 1948; third edition, 1967, Providence.

[5] S. A. Celani, *Modal Tarski algebras* Reports on Mathematical Logic **39** (2005), 113-126.

[6] Y. B. Jun, *Smarandache BCI-algebras,* Sientiae Mathematicae Japonicae Online **e-2005**, 271-276.

[7] J. Kim, Y. Kim, and E. H. Roh, *Decompositions and expansions of filters in Tarski algebras,* Journal of the Chungcheong Mathematical Society **20** (2007), no. 4, 457-463.

[8] J. Kim, Y. Kim, and E. H. Roh, *A note on GT-algebras,* J. Korean Soc. Math. Educ. Ser. B: Pure. Appl. Math. **16** (2009), no. 1, 59-68.

[9] W. B. Vasantha Kandasamy, *Smarandache groupoids,* http://www.gallup.unm.edu/ smarandache/Groupoids.pdf.

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