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REMARKS ON THE FUNCTION $\eta(n)$

BY

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In 1980, F.SMARANDACHE introduced (see [5]) the function $\eta:\mathbb{Z}^*\to\mathbb{N}$, defined by $\eta(n) = m$, where m is the smallest natural number with the property that $m!$ is divisible by n. This function has aroused the interest of several mathematicians because of the simplicity of the definition, because of its interesting properties, important applications as well as a simple algorithm for the computation of its values. This function is now known as Smarandache's function.

First of all, the η function provides us with extremely simple answers to two fundamental problems in the number theory:

1. It helps us formulate a primeness criterion: a number $p \in \mathbb{N}\setminus\{0,1,4\}$ is prime if and only if it is a fixed point of η (i.e., $\eta(p) = p$).
2. Only one formula was known (established by W. Sierpinski in 1953) for the function $\pi : \mathbb{R}_+^*\longrightarrow\mathbb{N}$, $\pi(x)$ = the number of prime natural numbers which are less or equal to x.

A new formula has been derived from Smarandache's function, that is: $\pi(x) = \sum_{k=2}^{[x]} \frac{\eta(k)}{k} - 1$. (Here $\pi(x)+1$ means the summation function of $\left[\frac{\eta(n)}{n}\right]$ in the extended sense and $[x]$ stands for the greatest integer which is less or equal to the real number x.)

Let us remark that the η function is uniquely determined by its restriction to the set $\mathbb{N}^*$, because $\eta(-n) = \eta(n)$.

We redefine $\eta(1)=1$ (according to Smarandache's definition, $\eta(1)=0$) hence $\eta : \mathbb{N}^*\longrightarrow\mathbb{N}^*$ and η becomes invertible with respect to the Dirichlet product.

In [6] and [7] some open problems referring to the η function are presented. Here is an interesting problem:

Consider the sequence

$$\begin{array}{llllll} \eta(1)=1, & \eta(2)=2, & \eta(3)=3, & \eta(4)=4, & \eta(5)=5, & \eta(6)=3, \\ \eta(7)=7, & \eta(8)=4, & \eta(9)=6, & \eta(10)=5, & \eta(11)=11, & \eta(12)=4, \ldots \end{array}$$

and let α be the following decimal number (actually, the reasoning is valid for every basis $g \geq 2$)

$$\alpha = 0.\eta(1)\eta(2)\eta(3)...\eta(n)... \tag{1}$$

that is

$$\alpha = 0.123453746511413...$$

We ask ourselves whether the number α is irrational. Using the following result due to W. Sierpinski [4] we can solve this problem:

Theorem 1. *For every $m \in \mathbb{N}^*$ and every ciphers $c_1, c_2, ..., c_m$ in the decimal basis, $c_i \neq 0$, there exists an infinite number of primes which, written in the decimal basis, have as their first ciphers $c_1, c_2, ..., c_m$ (in this order).*

Theorem 2. *The number α given by* (1) *is irrational.*

Proof. Since $\eta(p) = p$ for every prime number p, after the point of α given by (1) there is a sequence of ciphers which represents the number p written in the decimal basis. For different prime numbers, the corresponding sequences are disjoint. Now, for every natural number n, $n \geq 1$, we consider $m = n + 2$ and take the sequence of ciphers $c_1, c_2, ..., c_n, c_{n+1}, c_{n+2}$, where $c_2 = c_3 = \cdots = c_{n+1} = 1$, $c_1 \neq 0$, $c_1 \neq 1$, $c_{n+1} \neq 1$. This sequence appears after the point of (1), even for infinitely many times (according to Theorem 1). It follows that α can be represented as a decimal fraction with an infinite number of significant ciphers which, according to the remark above, is not periodic. Therefore, the number α given by (1) cannot be rational.

Computing $\eta(n)$ for even greater values of n, we can notice a very irregular repartition of the values of η.

We remark that for every n, $\eta(n) \leq n$, and every $m \in \mathbb{N}^*$ is a value of η ($\eta(m!) = m$). Every $m \in \mathbb{N}^*$, $m \geq 3$, appears several times, but a finite numbers of times, because η is generally nondecreasing function: for every $a \in \mathbb{N}^*$ there exists $b \in \mathbb{N}^*$ so that for every $c \in \mathbb{N}^*$, $c \geq b$, we have

$\eta(c) > \eta(a)$. For every positive real number M, there exist prime numbers p satisfying $p = \eta(p) > M$. It follows that

$$\lim_{n\to\infty} \frac{\eta(n)}{n} = +\infty.$$

Now, let us formulate the following result referring to the Gauss–Dirichlet mean:

Theorem 3. $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \eta(k) = +\infty.$

The next three propositions show the nonregularity of the distribution of the values of η.

Proposition 1. *There exist subsequences* $(n_k)_{k\in\mathbb{N}^*}$ *of* $\mathbb{N}^*$ *so that*

$$\lim_{k\to\infty} \frac{\eta(n_k)}{n_k} = 0.$$

Proof. We take $n_k = k!$. Since $\eta(k!) = k$, we have

$$\lim_{k\to\infty} \frac{\eta(n_k)}{n_k} = \lim_{k\to\infty} \frac{1}{(k-1)!} = 0.$$

Proposition 2. *There exist subsequences* $(n_k)_{k\in\mathbb{N}^*}$ *of* $\mathbb{N}^*$ *so that*

$$\lim_{k\to\infty} \frac{\eta(n_k)}{n_k} = 1.$$

Proof. It suffices to consider $(n_k)_{k\in\mathbb{N}^*}$ as the sequence of prime natural numbers, because in this case $\eta(n_k) = n_k$.

Proposition 3. *For every* $m \in \mathbb{N}^*$, *there exists a subsequence* $(n_k)_{k\in\mathbb{N}^*}$ *of* $\mathbb{N}^*$ *so that*

$$\lim_{k\to\infty} \frac{\eta(n_k)}{n_k} = \frac{1}{m}\cdot$$

Proof. For $m = 1$ the result is true (see Proposition 2). Let $m > 1$ with the canonical decomposition $m = p_1^{\alpha_1} p_1^{\alpha_2} ... p_1^{\alpha_s}$. There exist infinitely many prime numbers p,

$$p > \max_{1\le i\le s} p_i^{\alpha_i} \tag{2}$$

We consider the sequence $(n_p)_p$, $n_p = m \cdot p$, where p ranges the set of prime numbers which satisfies (2). We have $\eta(n_p) = p$, hence

$$\lim_{p\to\infty} \frac{\eta(n_p)}{n_p} = \lim_{p\to\infty} \frac{p}{m\cdot p} = \frac{1}{m}\cdot$$

Although there exists a simple algorithm, based on the decomposition of natural numbers in prime factors (which allows the generation of values of η on the computer), we have no formula to give $\eta(n)$ by means of the prime factors of n. The η function is neither multiplicative nor additive. We cannot put the summation function of η in a convenient form. Also, we cannot put in a convenient form neither the function whose the summation function is η nor the inverse function of η in the Dirichlet sense.

In order to study $\eta(n)$ in an easier manner, it would be useful for us to find an invertible arithmetic function so that the Dirichlet product of η and this function should be known. This would allow the determination of the generating function and the Dirichlet series of η as well as the obtaining of some interesting identities, referring to the η function.

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