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Title
Hidden Mathematical Symmetries in the 32 Crystal Point Groups?

Abstract
In a preceding paper we introduced a conjecture: the classification of the 32 crystal classes with 5 bits. In the present paper we will review our preceding result, and continue showing some further interesting issues. In the paper, it is argued that bits should be identified with five basic unknown symmetries generating these 32 groups. Probably it is not merely a coincidence that 32 means 5 bits. And probably is it not merely a coincidence that each complete subset of bits (properties) means the holohedry of a crystal system; and each new bit means a new crystal system.
The purpose of this article was of course not to draw a conclusive theory, but to suggest ideas that, we hope, will be useful for researchers in mathematics, group theory and crystallography.
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Introduction

As is well known to any mineralogist or mathematician [1], [2], there are only 32 crystal classes. These are mathematically 32 point groups, generated by symmetry operations: reflection, rotation, inversion, and the combined operation rotoinversion.
Usually an axis of rotational symmetry is represented by the capital letter A: A2, A3, A4, A6. The number of such axes present is indicated by a numeral preceding the capital A. For example, 1A2, 2A3, and 3A4 represent one 2-fold axis of rotation, two 3-fold axes, and three 4-fold axes respectively.
A center of inversion is noted by the letter “c” or “i”.
A mirror plane is denoted by “m”. A numeral preceding the m indicates how many mirror planes are present.
An axis of rotary inversion (a compound symmetry operation which is produced by performing a rotation followed by an inversion) is represented by the capital letter R.
A rotoinversion axis is sometime replaced by the equivalent rotations and reflections. For example, a 2-fold rotoinversion axis is equivalent to reflection through a mirror plane perpendicular to the rotation axis.
The reflection, rotation, inversion, and rotoinversion symmetry operations may be combined in a variety of different ways. In Table 1 there are 32 different possible combinations of these symmetry operations, or 32 point groups.
The different symmetry operations listed in Table 1 are a lot:
i, A2, m, R4, A4, A3, R6, A6.
But for a radar engineer it is inevitable to associate “32” to “5 bits”.
This means making the assumption that there are *only five* basic properties or *only five* basic symmetries generating these 32 groups.
Their presence or absence will automatically determine which class or point group of origin.
In [3] I have tentatively identified the meaning of bits but now we assume the following.

Bit c
0000c in position 00001
The bit c is not the property usually called *center*, but it’s in some unknown way related to it.

Bit m
000m0 in position 00010
The bit m is not the property usually called *planes*, but it’s in some unknown way related to it.

Bit 2
00200 in position 00100
The bit 2 is not the property usually called *axis 2*, but it’s in some unknown way related to it.

Bit 4
04000 in position 01000
The presence of bit 4 corresponds to the presence of a property that provisionally identify in a more subtle way: a symmetry that corresponds to half of a possible and / or existing symmetry.
This bit then appears to identify *axis 4*, 90° as half of 180°, or even an *axis 6*, 60° as half of 120°.

Bit 3
30000 in position 10000
The bit 3, the most significant bit, is not the property usually called *axis 3*, but it’s in some unknown way related to it.
Now we proceed with further issues.

Crystal Classes and 5 bit classification

The following table shows the 32 crystal classes and their symmetries.

TABLE 1 - Crystal classes and their Symmetries

Crystal System	**Crystal Class**	**Symmetry of Class**	**Hermann Mauguin**
	hexoctahedron	i, $3A_4$, $4A_3$, $6A_2$, 9m	m$\underline{3}$m
	gyroid	$3A_4$, $4A_3$, $6A_2$	432
Isometric	hextetrahedron	$3A_2$, $4A_3$, 6m	$\underline{4}$3m
	diploid	i, $3A_2$, $4A_3$, 3m	m3
	tetartoid	$3A_2$, $4A_3$	23
	dihexagonal dipyramid	i, $1A_6$, $6A_2$, 7m	6/mmm
	hexagonal trapezohedron	$1A_6$, $6A_2$	32
	dihexagonal pyramid	$1A_6$, 6m	6mm
Hexagonal	ditrigonal dipyramid	$1R_6$, $3A_2$, 3m	6m
	hexagonal dipyramid	i, $1A_6$, 1m	6/m
	hexagonal pyramid	$1A_6$	6
	trigonal dipyramid	$1R_6$	$\underline{6}$
	hexagonal scalenohedron	i, $1A_3$, $3A_2$, 3m	3 2/m
	trigonal trapezohedron	$1A_3$, $3A_2$	32
	ditrigonal pyramid	$1A_3$, 3m	3m
	rhombohedron	i, $1A_3$	$\underline{3}$
	trigonal pyramid	$1A_3$	3
	ditetragonal dipyramid	i, $1A_4$, $4A_2$, 5m	4/mmm
	tetragonal trapzohedron	$1A_4$, $4A_2$	422
	ditetragonal pyramid	$1A_4$, 4m	4mm
Tetragonal	tetragonal scalenohedron	$1R_4$, $2A_2$, 2m	4 2m
	tetragonal dipyramid	i, $1A_4$, 1m	4/m
	tetragonal pyramid	$1A_4$	4
	tetragonal disphenoid	$1R_4$	$\underline{4}$
Orthorhombic	rhombic dipyramid	i, $3A_2$, 3m	mmm
	rhombic disphenoid	$3A_2$	222
	rhombic pyramid	$1A_2$, 2m	mm
	prism	i, $1A_2$, 1m	2/m
Monoclinic	sphenoid	$1A_2$	2
	dome	1m	m
Triclinic	parallellohedron	i	$\underline{1}$
	monohedron	no symmetry	1

Note that here the 32 crystal classes are divided into 6 crystal systems.

TABLE 2 – Number of Classes in each System

Crystal System	Number of Classes
Triclinic	2
Monoclinic	3
Orthorombic	3
Tetragonal	7
Hexagonal	12
Isometric	5

Sometime the Hexagonal System is divided into the hexagonal and rhombohedral or trigonal divisions. But note that (Table 2):
- the whole Hexagonal System has exactly 12 Classes;
- there are exactly 8 Classes from Triclinic to Orthorombic;
- there are 7+5=12 Classes in the Tetragonal plus Isometric System.

The lowest symmetry class in the Isometric System is the Tetartoidal Class. Imagine to move that from the Isometric to the Tetragonal System in order to have:
-exactly 4 Isometric Classes;
-exactly 8 Tetragonal Classes.
This way the situation is much more suited to a 5 bit classification.

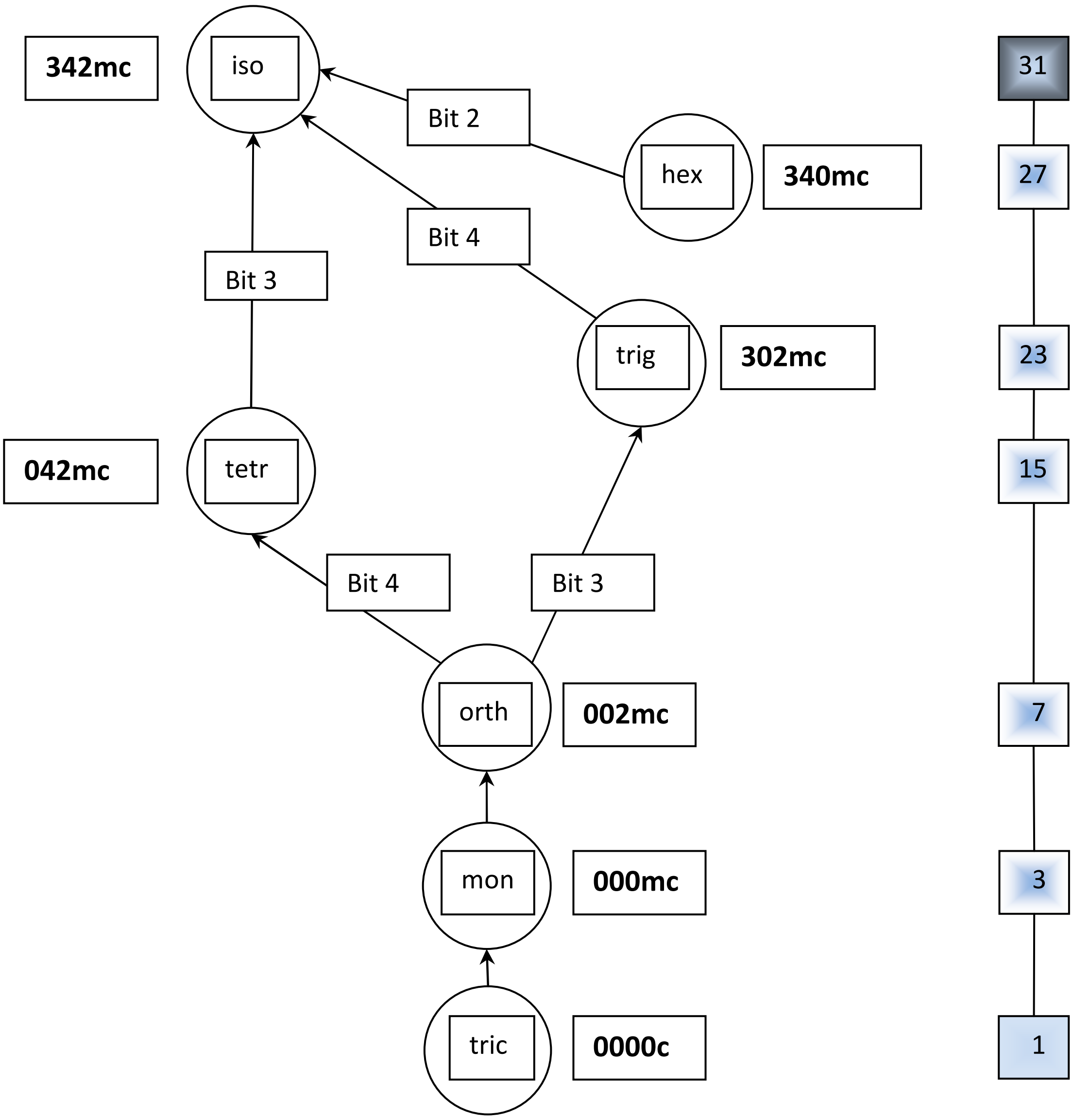
The best match I've found between 5 bits c, m, 2, 4, 3 and crystal symmetries is shown in Table 3.

TABLE 3 – 5 bit Tentative Classification

1	00000	0	triclinic
$\underline{1}$	**0000c**	**1**	**triclinic**
m	000m0	2	monoclinic
2/m	**000mc**	**3**	**monoclinic**
2	00200	4	monoclinic
222	0020c	5	orthorhombic
mm	002m0	6	orthorhombic
mmm	**002mc**	**7**	**orthorhombic**
4	04000	8	tetragonal
$\underline{4}$	0400c	9	tetragonal
23	040m0	10	isometric
4/m	040mc	11	tetragonal
422	04200	12	tetragonal
$\underline{4}$2m	0420c	13	tetragonal
4mm	042m0	14	tetragonal
4/mmm	**042mc**	**15**	**tetragonal**
3	30000	16	exagonal
$\underline{3}$	3000c	17	exagonal
$\underline{6}$	300m0	18	exagonal
3m	300mc	19	exagonal
6	30200	20	exagonal
32	3020c	21	exagonal
$\underline{6}$2m	302m0	22	exagonal
$\underline{3}$m	302mc	23	exagonal
622	34000	24	exagonal
6/m	3400c	25	exagonal
6mm	340m0	26	exagonal
6/mmm	**340mc**	**27**	**exagonal**
432	34200	28	isometric
m3	3420c	29	isometric
$\underline{4}$3m	342m0	30	isometric
m$\underline{3}$m	**342mc**	**31**	**isometric**

The class which possesses the highest possible symmetry within each crystal system is termed the holomorphic class of that system or the *holohedry* of the system (Hestenes, [1]). The holomorphic class of each crystal system is indicated in the table by bold type. For example, the m$\underline{3}$m class is the holomorphic class of the isomorphic or cubic crystal system. Note: also each complete subset of bits (properties) means the holohedry of a System

TABLE 4 – Growth of properties until the holohedry of that System



In each System we see the properties to growth (see Table 3) until that a complete subset of bits is reached, I mean for example 00011, or 00111, or 01111. In this case we reach the holohedry of that System. One more bit (see Table 4) means a new System.
The fact that so happens confirms that it's probably not a coincidence a 5 bit classification.

To further illustrate the symmetry elements, the example crystalline forms for each class were constructed as in [4] using Faces (version 3.7) by Georges Favreau.
The example crystals were constructed using the following parameters:
Crystallographic axes: a=1, b=1, c=1.
Forms: **[214]**, **[104]**, **[024]**, [100] (Color coded in all example crystals).
The following Table 5 shows the Holohedry of each System. I've added the Holohedry 3m of Trigonal System.
Table 6 and 7 show the simplified situation where the Hexagonal System is undivided.
Table 6 shows the Holohedry of each System.
Table 7 shows Classes immediately before the Holohedry (bit c disappeared).

TABLE 5 –Holohedry of each System

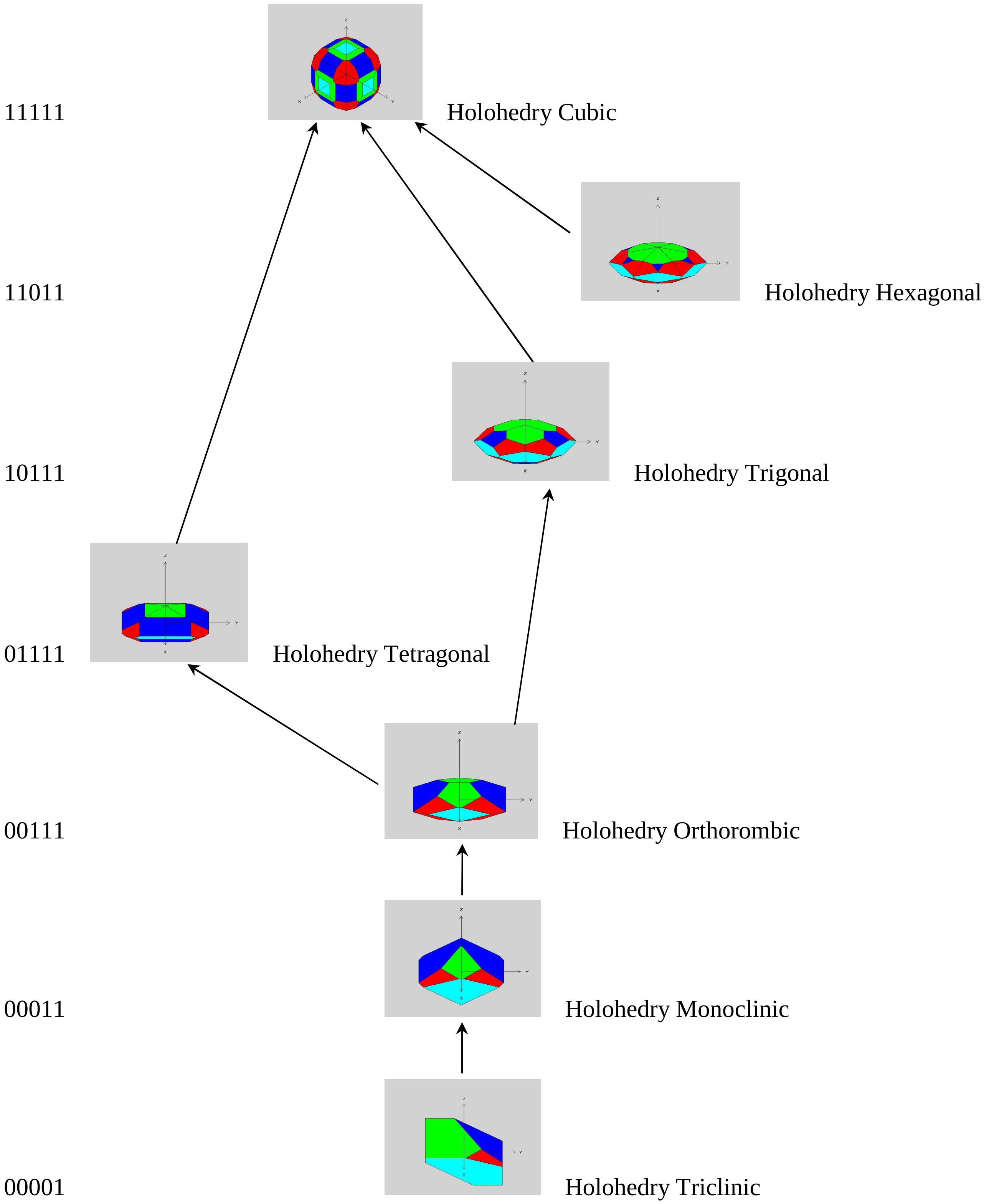


TABLE 6 – Holohedry of each System, Hexagonal System undivided

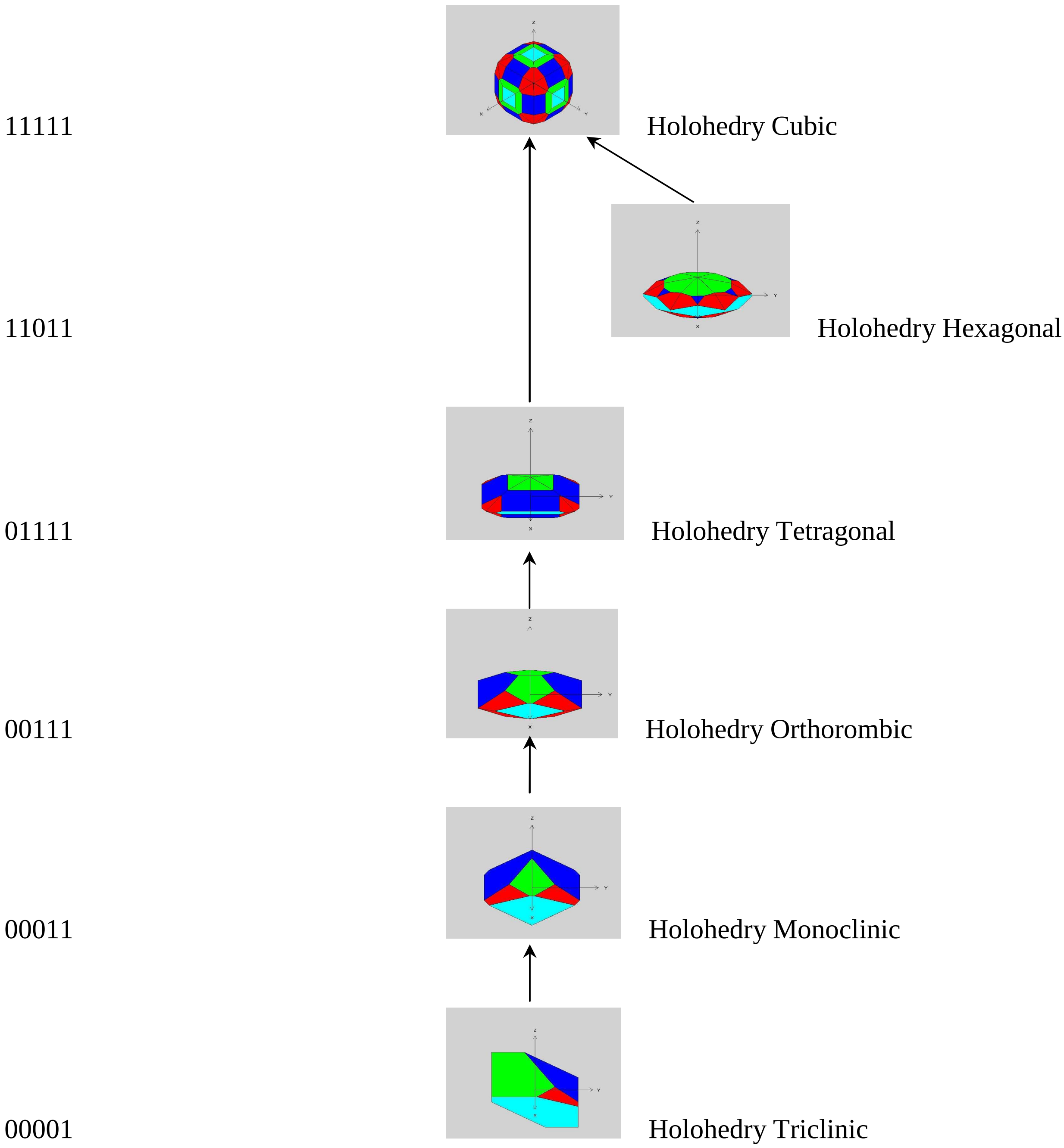
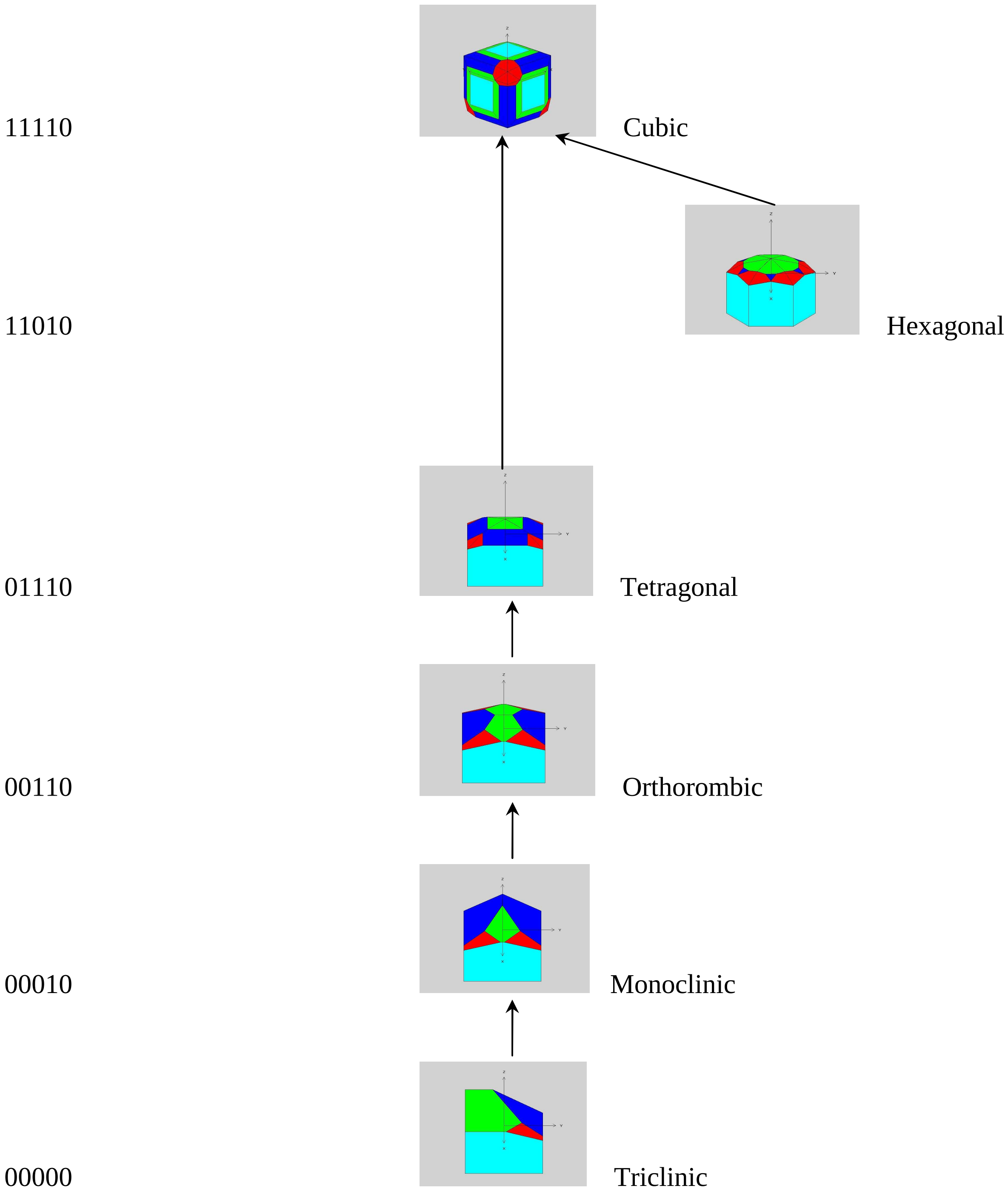


TABLE 7 – Classes immediately before the Holohedry (bit c disappeared)



Conclusions

We offer a new conjecture, which may be called *32 point groups of three dimensional crystal cells described by 5 bits*. To our knowledge this conjecture has not been discussed elsewhere, and therefore may be useful for further research, possibly in the area of mathematics, group theory and crystallography.
For the moment I do not know any theoretical justification for a possible identification with five basic symmetry properties, if not a guess, and trust that *Essentia non sunt moltiplicanda praeter necessitatem*. However, is it merely a coincidence that 32 means 5 bits? And is it merely a coincidence that each complete subset of bits (properties) means the holohedry of a system? And is it merely a coincidence that each new bit means a new crystal system?
For almost all classes the proposed 5 bit classification is entirely satisfactory but some cases seem to me still ambiguous. So the purpose of this article was of course not to draw a conclusive theory, but to suggest further study of this proposed conjecture.

References

[1] D. Hestenes, "Point Groups and Space Groups in Geometric Algebra ", in L. Dorst, C. Doran, J. Lasenby (eds.), Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Boston, 2002, pp. 3-34.

[2] E. Hitzer and C. Perwass, "Crystal Cells in Geometric Algebra", Proceedings of the International Symposium on Advanced Mechanical Engineering Between University of Fukui – Pukyong National University, November 27, 2004

[3] G. Bettini, "32 Point Groups of Three Dimensional Crystal Cells Described by 5 Bits", http://vixra.org/abs/1012.0052

[4] D. Barthelm, Mineralogy Database, "Crystallography", 2010, Internet release