# Random motion, harmonic oscillator and dark energy


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**ABSTRACT**

We explore the random motion of a quantum free particle and re-introduce an additional term to the textbook solution for the variance. Prior experiments could have missed this term yet it should be possible to test the hypothesis. A quadratic potential energy derived from the self gravitational potential of the particle is hypothesized resulting in the well known quantum harmonic oscillator in the special condition that the particle rests exactly in the ground state, i.e. $\hbar\omega_0/2 = k_B T$. Radiation is found trapped in this gravitational potential and when the particle carries the reduced Planck mass the density of radiation is exactly that of the black body. We argue this "dark" particle is responsible for the open question of dark energy and has a relic density of only 17% more than the commonly accepted value.



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## I. BACKGROUD

Cosmological observations early last century indicate the Universe is expanding. These observations came by measuring the speed at which objects are moving away from Earth and noticing the strong correlation with their distance, known as Hubble's Law. However it was not until the end of the last century when we had observations of Type Ia supernova that indicated the Universe is also accelerating [1].

The most popular explanation for these findings is an elusive energy density with an equation of state, $w < -1/3$ [2] coined "Dark Energy" making up ~73% [3] of the energy of the Universe. Despite many attempts to explain dark energy's origin [4], those attempts have fallen short [2,5]. Yet there are still many theories under review [6]; with the most accepted being the Lambda Cold Dark Matter model, ΛCDM where Lambda, the Cosmological Constant, provides a negative pressure $w = -1$ [1,2].

We argue that hypothesizing a "dark particle" begins to answer the question of missing dark energy density.

## II. RANDOM MOTION

The quantum diffusion of a free particle is not a new discovery, yet its reference is scanty. The quantum free particle's Hamiltonian can be solved vis a vie the Fourier Transform [7].

$$H = \frac{p^2}{2m}$$

$$\frac{d}{dt}\psi = \frac{i\hbar}{2m}\frac{d^2}{dx^2}\psi$$

The magnitude squared of the solution is a Gaussian distribution with a variance that is quadratic in time [8].

$$\left(\Delta x_{quadratic}\right)^2 = \frac{\hbar^2}{4mk_BT} + \frac{k_BT}{m}t^2$$

However if one makes a Minkowski transformation [8,9] to imaginary time the quantum free particle's Hamiltionian becomes the regular diffusion equation with diffusion constant $D = \hbar/2m$

$$\frac{d}{dt}f = \frac{\hbar}{2m}\frac{d^2}{dx^2}f$$

This diffusion constant has been hypothesized before [10,11] and the solution to the equation is a Gaussian with variance linear in time

$$(\Delta x_{linear})^2 = \frac{\hbar}{m}t$$

More evidence in favor of a linear diffusion term embedded within the quantum particle is that it can be solved directly from Heisenberg's Uncertainty Principle if looked at as a differential equation.

$$\Delta p \Delta x = m\big(\Delta \dot{x}(t)\big)\big(\Delta x(t)\big) = \frac{\hbar}{2}$$

We solve for an important time constant, $\tau$, by equating this diffusion constant with the one from Einstein's kinetic theory [12].

$$D = \mu k_B T = \frac{\tau}{m}k_B T$$

$$\tau = \frac{\hbar}{2k_BT}$$

## III. POSSIBLE EXPERIMENTAL VALIDATION

The text book solution gives $\left(\Delta x_{quadratic}\right)^2$ [8], however this paper hypothesized that the actual variance, if measured, will be

$$\left(\Delta x_{hypothesized}\right)^2 = \left(\Delta x_{quadratic}\right)^2 + (\Delta x_{linear})^2$$

It is quite possible that this linear diffusion term slipped past prior measurements of a free particle's variance. Only in very cold temperatures and small time scales is the linear term noticeable.

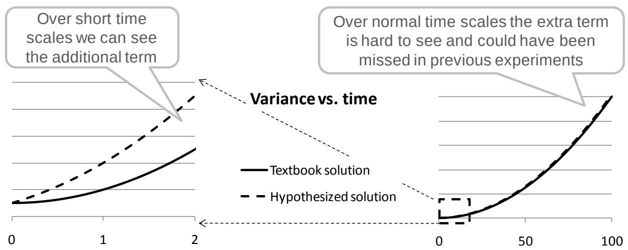


**Figure 1** – Divergence from known solution is only noticeable in small time scales and cold temperatures.

## IV. RESISTIVE SPRING FORCE

Now we can show how a resistive spring force places the particle in the ground state of the harmonic oscillator.

First from kinematic arguments, assuming the particle is initally at rest relative to a reference frame, the particle will begin to move due to its random motion a total displacement, $x$ in time $\tau$. From Netwon's second Law [13] the force on the particle will be

$$F_{particle} = ma = \frac{mx}{(\tau)^2}$$

With the particle being at rest on average we can look at Newton's first Law [13] backward, i.e. given a particle at rest, the sum of the forces acting on the particle is zero, we have

$$\sum F = F_{spring} + F_{particle} = 0$$

$$F_{spring} = \frac{-mx}{(\tau)^2}$$

Here we see Hook's Law which has an associated potential energy $\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$ leading to the harmonic osscilator with 1-D groud state energy, $E_{x0}$ [14].

$$E_{x0} = \frac{\hbar\omega_0}{2} = \frac{\hbar}{2}\sqrt{\frac{m/\tau^2}{m}} = k_B T$$

The quantum mechanical harmonic oscillator has just enough thermal energy to populate the ground state, an interesting finding if this is interpreted as the vacuum state.

We can see this potential energy is derived from the curvature of space due to the gravitational mass of the particle by making use of the derivation of Freidmann's equation and equating the average gravitational potential to $k_B T/2$, or for 3 dimensions, $3k_B T/2$ [1].

$$\overline{PE_{gravity}} = \frac{\overline{-GMm}}{r} = \frac{3k_B T}{2}$$

When $M = 4\pi r^3 \rho/3$

$$\frac{\overline{-GMm}}{r} = \frac{-4\pi G\rho m\overline{r^2}}{3}$$

Due to symmetry we can re-write $\overline{r^2}$ as $3(\Delta x)^2 = 3\hbar^2/4mk_B T$ [14] to arrive at,

$$\frac{-8\pi G\rho}{3} = \frac{(2k_B T)^2}{\hbar^2} = \frac{1}{\tau^2}$$

Plugging this back into the relationship between potential energy and force [14] we have

$$F_r = \frac{-d}{dr}PE = \frac{-d}{dr}\left(\frac{-GMm}{r}\right) = \frac{-d}{dr}\left(\frac{-4\pi G\rho m r^2}{3}\right)$$
$$= \frac{8\pi G\rho m r}{3} = \frac{-m}{\tau^2}r$$

When the particle moves away from the initial center of mass, the gravitational field curves the space such that any deviation has a resistive spring force; keeping the particle in the ground state of the harmonic oscillator.

Yet why do we see non-stationary diffusion of a free particle? An obvious answer is that this effect is only noticeable within a Schwarzschild radius of the center of the particle (i.e. where the space is curved). Since all particles we have experimental observation on, have a quantum wavelength much greater than its Schwarzschild radius, this force does not have a noticeable effect.

## V. DARK PARTICLE

However if we look at a hypothetical particle, coined "dark particle," which carries the reduced Planck mass, $m = \sqrt{\hbar c/8\pi G}$, we find this resistive force is in play. Also as we expect the particle to be coupled to a radiation field [15] we find black body radiation bound within the gravitational potential.

The energy in the 3-D oscillator will be

$$\hbar\omega = \frac{m}{2\tau^2}(x^2 + y^2 + z^2) + \frac{1}{2m}\left(p_x^2 + p_y^2 + p_y^2\right)$$

As the harmonic oscillator is in the ground state, we start with dual Gaussian wavefunctions as solution for both position and momentum and assume Heisenberg's Uncertainty relation and the Equipartition theorem hold [14].

$$x \sim p(x)dx = \frac{1}{\sqrt{2\pi\Delta x^2}} \cdot e^{-\frac{x^2}{2\Delta x^2}}dx$$

$$p_x \sim p(p_x)dp_x = \frac{1}{\sqrt{2\pi\Delta p_x{}^2}} \cdot e^{-\frac{p_x{}^2}{2\Delta p_x{}^2}}dp_x$$

We can solve for the probability distribution on $\omega$ [15].

$$\omega \sim p(\omega)d\omega = \frac{\hbar^3\omega^2}{2(k_B T)^3}e^{\frac{-\hbar\omega}{k_B T}}d\omega$$

The average energy of this distribution is the three dimensional ground state energy of the harmonic oscillator, $3k_BT$. However before we can associate this probability with the radiation we must account for the fact that multiple photons can occupy the same state [14]. Thus we have,

$$M \cdot \overline{\hbar\omega} = \frac{6 \cdot k_B T}{2}$$

$$M \cdot \overline{\frac{m}{2}\left(\frac{x}{\tau}\right)^2} = M \cdot \overline{\frac{1}{2m}(p_x)^2} = \frac{k_B T}{2}$$

$$T \rightarrow \frac{T}{M}$$

And

$$p_{radiation}(\omega)d\omega = \frac{M^3\hbar^3\omega^2}{2(k_BT)^3} e^{\frac{-M\hbar\omega}{k_BT}} d\omega$$

From here we find the density of bound radiation within the gravitational potential by using Poynting's theorem [17]. Assume the particle radiates energy $\omega$ every $\tau$ seconds originating at a distance $dx = cdt = \hbar/2mc$. As the radiation moves away from the particle it's energy will red-shift and $\tau$ will dilate due to the photon moving through the high gravitational field defined by the Schwarzschild metric [16]. At the distance $c\tau$ away from the particle we can assume the relativistic effects on the radiation field are negligible and the power is reduced by $(1 - r_s/dx)$.

$$P_1 = S_1 \cdot A_1 = \frac{\hbar\omega}{\tau}$$

$$P_2 = S_2 \cdot A_2 = \frac{\hbar\omega}{\tau}(1 - r_s/dx)$$

Here $P_1$ and $P_2$ is the power respectively flowing through the spherical areas surrounding the particle, $A_1 = 4\pi(dx)^2$ and $A_2 = 4\pi(c\tau)^2$, and $r_s = 2Gm/c^2$ is the Schwarzschild radius.

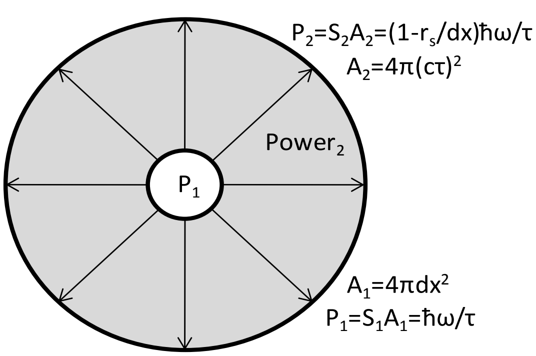


**Figure 3** – Diagram of Poynting's vector – Difference in power flux between inner and outer circles represents trapped energy density.

One can interpret $(P_1 - P_2)/A_2$ divided by the speed of light and multiplied by the probability distribution on the radiated energy as the average energy density of the radiation field gravitationally trapped by the mass of the particle.

$$\rho_{radiation-M}(\omega)d\omega = \frac{(P_1 - P_2)}{c \cdot A_2} p_{radiation}(\omega)d\omega$$

$$= \frac{\hbar\omega^3}{2\pi^2c^3} e^{\frac{-M\hbar\omega}{k_BT}} d\omega$$

Lastly sum over all states M from 1 to ∞, and both degrees of polarization [18] since all are possible.

$$\rho_{radiation}(\omega)d\omega = \sum_{pol=1,-1} \sum_{M=1}^{\infty} \rho_{radiation-M}(\omega)d\omega$$

$$= \frac{\hbar\omega^3 d\omega}{\pi^2c^3} \sum_{M=1}^{\infty} e^{\frac{-M\hbar\omega}{k_BT}} = \frac{\hbar\omega^3}{\pi^2c^3} \frac{1}{\left(e^{\hbar\omega/_{kT}} - 1\right)} d\omega$$

One will recognize the energy density of Black Body radiation, [18]. In normal black body radiation, a macroscopic cavity provides the confinement of the radiation [18]. Here it is the gravitational effects from the mass of the particle that confines the radiation as illustrated in figure 2 below.

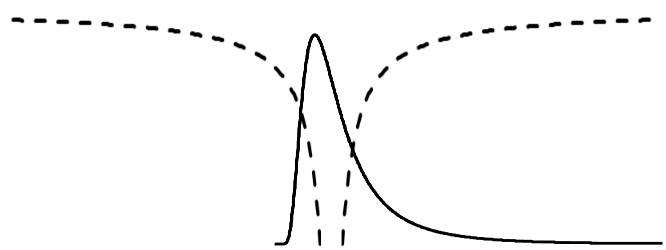

**Figure 2** – Black Body Radiation trapped within the gravitational potential defined by the Schwarzschild metric.

## VI. DARK ENERGY

*Now hypothesize that a local group of dark particle's are able to exchange heat with the local surroundings when neutral hydrogen atoms or other sinks are nearby to capture the radiation from its gravitational binding but that they become frozen when neutral hydrogen atoms are not nearby*.

During the dark ages, the time between decoupling and re-ionization [19], the Universe was filled with hydrogen atoms that provided the coupling mechanism between the dark particles and regular energy. In these conditions the dark particles were coupled to the Cosmic Microwave Background (CMB). However after re-ionization, the hydrogen was ionized and the dark particles and its associated radiation energy density became frozen. The red-shift of re-ionization and the current temperature of the CMB provide an estimate of the temperature of the dark particle before they became frozen at re-ionization.

$$T_{\mathrm{DP}} = (1 + z_{Re-ionization})T_{today} = constant$$

If we know the temperature of the dark particles at re-ionization, then we should have an idea for the total energy density that contributes to the Cosmological constant.

$$\rho_{\text{Dark Particles}-\text{BBR}} = \frac{\pi^2}{15}\frac{(k_B T_{DP})^4}{\hbar^3 c^5}$$

Because we have estimates of today's z value of re-ionization and today's temperature of the CMB we can estimate the density, $\rho_{DP-BBR}$.

The Lambda Cold Dark Matter model, ΛCDM, provides a completely independent estimate of the density of dark energy, $\rho_{\Lambda\text{CDM}}$ [1], which we can estimate using the parameter, $\Omega_\Lambda$, and today's Hubble constant.

$$\rho_{\Lambda CDM} = \Omega_\Lambda \cdot \rho_{\text{critical}} = \Omega_\Lambda \cdot \frac{3H^2}{8\pi G}$$

Using the 7 year Wilkinson Microwave Anisotropy Probe [3] as a source for our estimates, Table 1 shows how our model's estimate of dark energy is high by 17% but well within the confidence range defined by $z_{re-ionization}$

| | Low | Average | High |
|---|---|---|---|
| $\rho_{\text{DP-BBR}}$ $(kg/m^3)$ | 5.22E-27 | 8.12E-27 | 1.21E-26 |
| $T_{today}$ $(degrees)$ | 2.725 | 2.725 | 2.725 |
| $z_{re-ionization}$ | 9.3 | 10.5 | 11.7 |
| $\rho_{\Lambda CDM}$ $(kg/m^3)$ | 6.21E-27 | 6.95E-27 | 7.74E-27 |
| $H$ $\left(\frac{km}{sec \cdot Mpc}\right)$ | 68.5 | 71.0 | 73.5 |
| $\Omega_\Lambda$ | 0.705 | 0.734 | 0.763 |

**Table 1** Estimate and confidence rages of Dark Energy from the Dark Particles BBR model and the Lambda Cold Dark Matter model using 7 year WMAP data

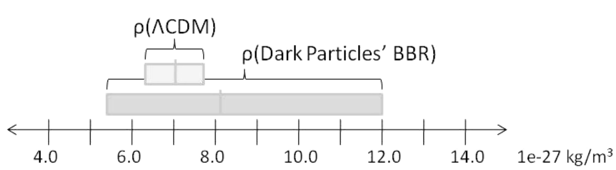


**Figure 4** Visualization of estimates of dark energy from two models

## VII. DISCUSSION

With a simple fundamental analysis we show how a hypothetical particle with the reduced Planck mass could be responsible for the missing Dark Energy. We propose an experimental environment where we can validate the kinetic forces acting on the particle. While it is not possible to artificially create a particle with the reduced Planck mass, it would explain why prior experiments have been unable to locate the missing energy.

Much more analysis is needed for dark particles to fully answer the questions of dark energy [2,5] or for that matter other open questions like dark matter[20]. However the philosophical guidance provided by Occam's Razor [21] suggests more investigation is justified.

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