Counter to Zhou's Criticism of Jones' Proof of the Irrationality of π and π^2

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Abstract

A geometric proof of the irrationality of π is given. It uses an evaluation of the area given by the product of two symmetric functions, together with bounds on the integral. The symmetric functions embed the assumption of rational π; one function is dependent on n; as the evaluation of the integral exceeds the upper bound for large n for any given rational π, a contradiction is obtained. This proof has been criticized, but here some counters to the criticism are offered.

1 The proof in brief

Assuming $\pi = p/q$ we note that $\sin x$ and the quadratic $x(p-qx)$ will have the same shape, roots, and symmetry in the interval $[0, \pi]$. The function $f(x)\sin x$ with $f(x) = x^n(p-qx)^n$ will share these properties. The maximum of this function occurs at $p/2q$. We have

$$0 < \int_0^{\frac{p}{q}} f(x) \sin x \ \mathrm{dx} \ \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^n, \tag{1}$$

Where the upper and lower bounds follow from the symmetry of the curve. Evaluating this integral using integration by parts (or tabular integration) gives

$$\int_0^{\frac{p}{q}} f(x) \sin x \ \mathrm{dx} \ = \sum_{k=0}^{n} (-1)^k f^{(2k)}(\pi, 0), \tag{2}$$

where $f^{(2k)}(\pi, 0) = f^{(2k)}(\pi) + f^{(2k)}(0)$. Using the symmetry of $f(x)$ we have $f^{(2k)}(0) = f^{(2k)}(\pi)$. As $f(x)$ is a polynomial whose least power of x is n, the first n derivatives evaluate to 0 at 0. After this all derivatives have an $n!$ in all their coefficients. We conclude that $n!$ divides the sum in the last equation. As factorial growth exceeds polynomial growth we have a contradiction.

2 Establishing a domain of discourse

There are ways to criticize mathematics. Certainly the most obvious and least disputed way is to find an error in the reasoning. These can be typos, things that can be fixed easily, and errors that can not be fixed: mistakes. If there are mistakes in the article then certainly the article should be withdrawn or somehow removed from consideration. *It doesn't work*, if proven by Zhou, can not be disputed and there can really be no discussion. Other types of criticism are much more difficult to agree upon. One such avenue is to argue that the reasoning is arbitrary in nature. The article is open to this type of criticism: its title suggests that discovering a result is one of its main themes. Discovery generally involves heuristics or plausible reasoning that helps in the formulation of precise mathematical proofs. Any article that is seeking to discover a result is susceptible to such criticism, but the criticism may not be appropriate or in a sense valid. We will explore this below.

Another avenue of criticism, related to heuristics, is the potential to lay artificially simple reasoning on a complex proof. This is more subtle and may touch on the main drift of Zhou's criticism. The question posed by the above proof is then does its simplicity distort or camouflage or somehow detract from some form of mathematical truth. This seems to be a unique problem. Is there another set of competing proofs that potentially has this issue? Perhaps. Euler's initial proof that

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

is generally thought to not be above reproach. Is the above proof of this type? Does it wait on more precise analysis and formal considerations? Is it, rather, a valid discovery tool that is, however, an historical anachronism in the present world? Is it a could warm-up to earlier proofs?

3 Responding to these avenues of criticism

3.1 Errors

There are typos in the article. All typos are fixable.

There are no mistakes in the article that I have been made aware of.

3.2 Arbitrary in nature

The criticism in the section "Hay and Needles that are not Golden" seems to say that the proof is arbitrary in nature. The initial statement, however, that the article does not show how symmetry is used seems inaccurate. Symmetry is used to establish the upper and lower bounds of the integral and to shorten the proof of the result. It is true that the upper and lower bounds can be established without symmetry, but they can also be established succinctly with symmetry. The arbitrary nature of the proof is then suggested by Zhou using counter-examples of symmetric candidate functions. The idea seems to be that if symmetry is at the heart of this proof then it should work for any symmetric functions one might devise. The counter-examples show that some symmetric function work and some don't and that a function that is not symmetric works for a proof of the irrationality of e.

The counter I offer is that the quadratic $x(p-qx)$ is more natural (or easier to conjure) than all the others he mentions. It looks like the sin function, afterall, and it can be within a heuristic avenue the one to try. This is a matter of opinion and maybe taste. The fact that the proof of a different number's irrationality (e's) does not use a polynomial symmetric with e^x is irrelevant.

The section entitled "More hay" gives another criticism that seems placeable under the rubric "arbitrary in nature." This section, however, does not address the proof part of the article, but rather the heuristic part. The heuristic argument is given that we can eliminate certain rational candidate values of π using the upper and lower bounds given in (1). Zhou states accurately that irrational values are also eliminated and makes the point that this heuristic is eliminating values per the upper and lower bound and not per rationality qua rationality. This seems a valid criticism, but it as at the typo level:

the words are not precise enough. A better written article would state that we have shown π is not 1, 2, or 22/7 and left off "rational candidate values." It is another matter of opinion as to whether or not showing that 22/7 can't be the value of π serves a purpose.

3.3 Artificial simplification

The main criticisms of the article reside in Zhou's statements that suggest an oversimplification and what I think could be called a related irrelevance of its statements. I first point out some inaccuracies related to this criticism. His statement that the article's purpose is to motivate I_n is inaccurate: I_n the symbol and its definition do not occur anywhere in the article. It is not true that the purpose of the article is to motivate any extant proof of the irrationality of π. It is stated in the article that its purpose is to provide a simplifying concept for Niven's article. This aside, Zhou's statements that it is more significant than other things mentioned in the article that $(\sin x)'' = -\sin x$ and that certain integrals are polynomials points to the criticism of irrelevance.

I will strengthen his argument and then consider it. Jones' article is bad because it artificially lays irrelevant and easy facts on a complicated phenonemon. He uses a geometric argument about area to show π is irrational using an integral. The integral he uses resides within a more accurate article that does not use a geometric argument. He's laying down true statement's about this integral knowing the conclusion in advance: the integral shows π is irrational.

Looking at the proof given above this seems to be in a sense valid. It seems unfair; it's too easy. This to me points to the article's philosophical fallacy: the human fallacy. An initial referee remarked that the article was a gloss on Niven. That, I confess, pinched. But over the month's I have evolved in my psychological sphere a counter: Niven and Hermite and others are a confused Jones. The area under the curve is not well computed with an integral under an assumption; the assumption is a rational value of π. All of the history of the irrationality of π is irrelevant to this fact. I'm genuinely sorry for all the hard work that Hermite, Niven, and Zhou and others have done, but all of it, apart from my human empathy, is irrelevant.

The test of the proof resides in a poll not in prose: can new readers to these irrationality proofs believe Jones's proof without ever having read or considered Niven, et. al.? The human fallacy also resides in the fact that humans want the easiest version of something. It also resides in the fact that humans prefer geometric arguments, being a visual creature, to recursion formulas and other algebraic fair. My main counter is "we will see" which proofs go on and which die. There is another counter.

4 The second proof

Unmentioned by Zhou is the second proof: that for π^2. Without this proof the claims of irrelevance and oversimplification would have some credence, albeit within human perception. With this second proof, however, there is real damage to the claim of laying artificial logic on somebody else's hard work – if that is a fair assessment of Zhou's main criticism. Frankly, I, the author feel guilty of such. That's the nature of simplification, though.

This proof builds directly on the area argument and the symmetry used previously. We modify sin to make it symmetric with $x(a - bx)$ when a/b is assumed to be π^2. We arrive then at

$$0 < \int_0^{a/b} x^n(a - bx)^n \sin x/\sqrt{a/b}\ \mathrm{dx} \leq \frac{a}{b}\left(\frac{a^2}{4b}\right)^n .$$

This integrand does not occur in other proofs of the irrationality of π. It is constructed per the concept used for the earlier proof. Once again the area is not computed correctly under the assumption. One can once again appeal to golden threads and the like, but a refutation of the argument, apart from metaphors, is still wanting.

5 The Temple Should be Destroyed

Zhou has been very candid in his feelings about Jones' article and I do thank him for this candidness. My candid feelings are that Niven's proof is obsolete and many results on π's irrationality have historic interest, but for a quick and clear proof of the irrationality of π there is a new and better proof. The simpler the better: the simplest is the best.

I also wish to thank the MAA for publishing the article. I guess the high priests have been shaken and if given their way, I guess, would rather it not exist.