Discrete time and Kleinian structures
in duality between spacetime and particle physics

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The interplay between continuous and discrete structures results in a duality between the moduli space for black hole types and AdS_7 spacetime. The 3 and 4 Q-bit structures of quantum black holes is equivalent to the conformal completion of AdS.

Prior to the 20^{th} century physics was perfectly continuous in its formulation. Classical physics was founded with the calculus of limits of continuous functions. Max Planck proposed a statistics with discrete energy packets, which solved the black body radiation problem and paved the way for quantum mechanics. These discrete energy units lead to quantization action, where the loop integral of action for a system $\oint pdq ~=~ n\hbar$ is discrete. Products of conjugate observables with units of action are not in involution $[q,~p] ~=~ i\hbar$ and satisfy an uncertainty relationship $\Delta p\Delta q ~\ge~ \hbar/2$. Wave dynamics often imposes a spectrum of discrete eigenvalues that are physically relevant, while continuous wave functions have no direct measurable content. Hence what we observe is discrete, but these are information which evolves as potential information according to a wave function which does not conform to classical notions of reality. This loss of reality in a local sense is demonstrated with the violations of Bell's inequalities and the CHSH inequality.

It is of course the case that in our modern age we have digital computers which process all information in discrete units of binary code. The computer is employed in many lattice codes, such as with QCD. Spacetime has a minimal length scale $\ell_p ~=~ \sqrt{G\hbar/c^3}$, which is a discrete cut off . Models with lattices, grids, and related structure impose independent degrees of freedom on each vertex (a Planck unit of volume), which results in difficulties. The holographic principle by contrast works with entangled degrees of freedom which are nonlocal. Events, space and momenta are emergent properties [1][2]. The Planck scale is a discrete cut off, not necessarily a discrete lattice with independent degrees of freedom. Holographically a string longitudinal coordinate contracts on the stretched horizon to the string scale $\ell_s ~>~ \ell_p$

Discrete structure of time and quantum paths

Complementarity between conjugate observables is a foundation of quantum mechanics. However, that does not hold between time and energy. Quantum mechanics converts the Poisson brackets into commutators, $\{q,~p\} ~=~ 1, \rightarrow~ [q,~p] = i\hbar$. The Poisson brackets, $\{q,~p\} ~=~ (\partial q/\partial q)(\partial p/\partial p) - (\partial p/\partial q)(\partial q/\partial p)$, gives a unit for the bracket transformation of the units of action. The quantum commutator evaluates the action of a system per units of discrete action $\hbar$. The quantum wave function is described in configuration space, or position coordinates employed in the Lagrangian for a particular system. The configuration space has a cotangent bundle T^*M of conjugate momentum. Every cotangent bundle is a symplectic space, but not every symplectic space is a T^*M. Symplectic structure is more fundamental than coordinates or momentum, which carries over to quantum commutators.

Commutators between conjugate variables imply the development of one variable is generated by it conjugate. A quantum state $|q\rangle$ is developed into $|q'\rangle$ by $|q'\rangle = e^{-ip(q' - q)/\hbar}|q\rangle$, where the momentum is an operator. The converse is also true; development of momentum is generated by position. These exponential operators are unitary, central to the structure of quantum mechanics. Similarly the development of a state with time is generated by Hamiltonian operator with energy eigenvalues $|\psi(t')\rangle = e^{-iH(t' - t)/\hbar}|\psi(t)\rangle$, and the energy here is . The variation of this time developed state vector with respect to time gives the Schrödinger wave equation.

Energy and time as a product give units of action, and while there is an operator for energy there is none for time. A time operator would imply some energy development unitary operator that prohibits discrete energy spectra for the Hamiltonian not in general bounded below. This result of Wolfgang Pauli demonstrated there is no general time operator [3]. The result is not that surprising, for there are no Poisson brackets in classical mechanics between energy and time.

It is possible to define a discrete time operator, one which jumps the time eigenbasis along a discrete set of time jumps. This prevents the problem with a time operator generating a continuous energy development that is not properly bounded [4]. A discrete form of the operator results in the recovery of a discrete energy eigenstate, and further the discrete structure of this time operator must be tailored to the energy eigenspectrum. This matter might appear rather academic, but it illustrates one power of discrete systems, and this may be extended into a discrete structure in general relativity.

Feynman's path integral describes paths according to probability amplitudes that start at some initial point and end at a terminal point. The individual paths are given by complex valued amplitudes. The sum over histories constructs the wave mechanics, such as constructive and destructive interferences between amplitudes, which propagates observables between the initial and final points. This is equal to an additional summation over points where these histories meet. Consider intermediate points between the initial and final points, a sum over all possible intermediate points, such as an integration, the original path integral is recovered.

This may be reduced to a simple system of associative and commutative arithmetic. Philip Goyal demonstrated how the above summation over intermediate points is equivalent in a discrete setting to the concatenation of measurements, where the summation is over all possible outcomes [5]. The complex amplitudes are products of complex numbers, and in a discrete setting this is a multiplication rule which requires complex numbers. The result is that quantum mechanics is reduced to a simple system of associative and commutative mathematics of complex numbers with no reference to classical mechanics, or any notion of space or spacetime. However, the one requirement is that the points intermediate to the initial and final points be intermediate in time.

The Goyal logic is a summation of Stern-Gerlach experiments. The intermediate point corresponds to some intermediate measurement between the source of particles and the final Stern-Gerlach (SG) apparatus. If the outcome of the intermediate SG apparatus is ignored, or no measurement is performed of their outcomes, the split beams recombine as a discrete summation. So the intermediate SG apparatus represents a sum over elementary quantum events. This summation in the complex algebra corresponding to this logic recovers the quantum superposition.

These summations over SG experiments occur in a sequence. There is no ambiguity in the ordering of these events. Further, the process appears well defined in a discrete setting. The Goyal approach for discrete quantum mechanics, even if the number of elements is enormous, but not infinite, indicates some sort of quantization of time, and a discrete spacetime.

Discrete spacetime structures

Black holes are familiar to even those with a casual interest in general relativity. The canonical spacetime is the Schwarzschild metric. This spacetime is a radial gravity field where the concentration of mass is so large an event horizon is generated and the mass is folded up within a region outside the universe. For a mass M the horizon occurs at a radius $r = 2GM/c^2$, which is a region of infinite time dilation so the exterior observer never witnesses anything cross the horizon, even the material which formed it. This is the basis of the holographic principle.

Taub-NUT spacetime is similar to the black hole. The principal difference is the mass is replaced by a dual field source that shares a role the magnetic monopole holds in electromagnetism and the field dependency depends on time. The event horizon occurs at a time, a time in the past of the timelike region, and the metric coefficient is $A(t) = -1 + 2(Nt + a^2)/(t^2 + a^2)$, where N is the NUT parameter, the magnetic monopole analogue to mass, and a is the angular momentum [6]. For $a = 0$ this is similar to the Schwarzschild metric. The duality between mass and the NUT parameter is an interchange between radial and time coordinates.

The Taub-NUT spacetime has the topology $S^3 \times R$, which is generically similar to the geometry of a black hole. The metric signature for time is contained in the three-sphere instead of the real number line R. This means the hyperbolic structure of the spacetime is embedded in the 3-sphere. The spacetime is then depicted with two dimensions removed as a cylinder with spiraling geodesics. The periodicity which results from this solution means hyperbolic surfaces in the spacetime have a discrete structure with the periodic solutions of the spacetime [6]. This periodic structure exists on hyperboloids of constant proper time. This discrete structure defines a Hausdorff manifold within two adjacent patches of the Penrose conformal diagram. The discrete structure is unable to cover three patches as there are sharp geodesic separations that exist which prevent a hyperboloid of constant proper time from being extended continuously across three regions. Hawking and Ellis discuss this matter in their *Large Scale Structure of Space-Time* [6]. The manifold is also a quotient manifold, which it shares with the Anti de Sitter (AdS) spacetime.

The Taub-NUT metric may be extended to an AdS spacetime. For a range of the NUT parameter, just as with the mass in the black hole case, the metric may be transformed into the AdS spacetime. For instance, the region of spacetime around the singularity of a black hole is approximately an AdS spacetime. The symmetries of the AdS spacetime defines a conformal field theory at its boundary. The boundary of the AdS spacetime is a conformally flat spacetime, and the symmetries of the interior of the AdS spacetime are holographically projected onto a theory of open strings on the boundary [7].

The AdS spacetime on a patch is

$$ds^2 \;=\; \frac{1}{x}(dt^2 \;+\; dx^2 \;-\sum_i dz^i dz^i)$$

which in the limit $x \;\rightarrow\; 0$ defines a Minkowski metric

$$dx^2 \;=\; -dt^2 \;+\; \sum_i dz^i dz^i$$

which is a Minkowski spacetime. This means that the evolute of AdS from a spatial surface is an entire spacetime, where additional Cauchy data is a conformal completion of AdS. The Cauchy data on the AdS is defined on a conformal set of metrics. The boundary space ∂AdS_{n+1} is a Minkowski spacetime, or a spacetime E_n. The theory is a holographic unification of particle physics, existing on the AdS boundary, with the graviton in the interior of the AdS. An application of this to the QCD gauge field on the boundary associates gluon chains and the graviton. Within a discrete group setting the extension of the Taub-NUT spacetime gives a coset construction of the AdS. The quotient group on the AdS is a coset contruction with a discrete group Γ with the AdS_{n+1} spacetime. The coset model then defines the boundary of the AdS_{n+1}, which is a conformally flat Einstein spacetime E_n.

The mathematics of discrete Klein groups on AdS is contained in appendix A and further information is contained in appendix A and in [7][8]. The discrete structure on AdS defines the conformal action of AdS on a sphere of one dimension lower. This is Klein group version of the AdS/CFT correspondence. The AdS_{n+1} group of isometries $O(n,\ 2)$ contains a Möbius subgroup, or modular transformations, so that this discrete group does not necessarily act effectively on AdS_{n+1}. As a result the discrete group Γ is not necessarily convergent. Convergence means there exists a sequence $g_i \;\in\; \Gamma$ which admits a "north-south" dynamics of poles $p^{\pm}$ on a sphere, which in the hyperbolic case defines the past and future portions of a light cone. From this light cone structure emerges. Further the discrete group Γ is dense on the isomtries $O(n,\ 2)$ of AdS_{n+1}.

The Zariski topology leads to quantum structure. The discrete group action and Zariski topology constructs the quantum logic of Goyal. Consider the affine space A_n as the n-dimensional space over a closed field F. The topology is constructed from closed sets defined by the polynomial set $S \;\in\; F$ by

$$V(S) \;=\; \{x \;\in\; A_n | f(x) \;=\; 0;\ \forall f \;\in\; S\}$$

For two polynomials in the set S we have the following rules:

$$V(p_1) \cup V(p_2) \;=\; V(p_1 \times p_2),\ V(p_1) \cap V(p_2) \;=\; V(p_1 \;+\; p_2)$$

which serve as the representation map between the logic of outcomes and the algebra of quantum operations demonstrated by Goyal. This closed set topology defines the Zariski topology on the affine set A_n. So a connection to quantum mechanics exists within this system with respect to Zariski topology. This is the topology of Étale and Grothendieck, or topos theory. An overview of topos theory in physics is in [9] by Isham.

This connection to quantum structure is identified with the set of lightlike curves. This set of lightlike curves is given by the quotient of the isometries of AdS with a subgroup with Zariski topology [8]. This constructs a Borel subgroup which is a Heisenberg group of $3 \times 3 \;+\; I_n$ matrices. As a result the emergence of light cones is in a discrete model associated with a coset structure for a Heisenberg group. The two emerge from projective algebraic varieties, which are Zariski. The light cone is a projective space in a Lorentzian manifold. Quantum mechanics has a similar structure. The Hilbert space contains a projective subspace, where the fibration of the Hilbert space over the projective Hilbert space.

This connection between quantum physics and light cone structure is illustrated without the discrete group structure above. The examination of the overlap between a state $|\psi(t)\rangle$ and $|\psi(t)\ +\ \delta|\psi(t)\rangle$. This leads to the expansion

$$|\langle\psi(t)|\psi(t)\ +\ \delta\psi(t)\rangle|^2\ =\ |\langle\psi(t)|\psi(t)\rangle|^2\ -\ (\langle H^2\rangle\ -\ \langle H\rangle^2)\delta t^2,$$

with $\sqrt{\langle H^2\rangle\ -\ \langle H\rangle^2}\ =\ \delta E$. This also defines a phase

$$\phi\ =\ \int dt\sqrt{\langle H^2\rangle\ -\ \langle H\rangle^2}\ =\ \int dt\Delta E$$

which is the geometric or Berry phase. For certain systems the above overlap of states can be a measure of the entanglement of states. This is also the Fubini-Study metric for the projective space $CP^n\ \subset\ C^{n+1}$. This projective structure is analogous to light cones in relativity. The connection is made further with the observation that Heisenberg groups are nilpotent groups, which play crucial role further on.

The connection between light cones and quantum physics is drawn tighter with the discrete structure. Discrete structures are more appropriate for quantum information. In what follows the entanglement types of 3 or 4 quantum bit system is equivalent to black hole types, which is extended to the AdS spacetime as well. The Taub-NUT spacetime is essentially just a black hole with the meaning of radius and time reversed in the metric elements. Consequently the association between entanglement types and black hole types carries naturally over to Taub-NUT spacetime and to AdS.

Entanglements, quantum bits, moduli spaces and the classification of entanglements

The entanglement between two states is a nonlocal quantum effect. Most standard entangled states are bipartite entangled states. A simple case is the bipartite entanglements between two spin systems with a bases defined by a Pauli matrix σ_z the states $|+\rangle$ and $|-\rangle$ for spin up and down. The Pauli matrix acts on these states as $\sigma_z|\pm\rangle\ =\ \pm|\pm\rangle$. These states are complex numbers, which means there are 2 variables for each state and thus 4 altogether. However, there are constraints, such as the probability Born rule $1\ =\ P_+\ +\ P_-$, $P_\pm\ =\ |a_\pm|^2$ for a state $|\psi\rangle\ =\ a_+|+\rangle\ +\ a_-|-\rangle$, and irrelevance of a phase in real valued measurements. This reduces the number of variables from 4 to $4\ -\ 2\ =\ 2$.

Consider two spin systems, say two electrons. The use of electron spin state is not concrete, for these arguments hold just as well for polarization direction of photons. So we have two sets of states and operators $\{\sigma_z, |\pm\rangle\}^1\{\sigma_z, |\pm\rangle\}^2$ denoted with an additional index $i\ =\ 1, 2$ and we still have

$$\sigma_z^i|\pm\rangle^i\ =\ \pm|\pm\rangle^i.$$

We can form two independent states $|\psi\rangle^i\ =\ a_+^i|+\rangle^i\ +\ a_-^i|-\rangle^i$ for the two spin systems. For each there are 4 variables and 2 constraints. This gives 2 degrees of freedom in tota, 2 less that for two independent spinsl. These spin states are composed as

$$|\psi\rangle\ =\ (1/\sqrt{2})(|+\rangle|-\rangle\ +\ e^{i\phi}|-\rangle||+\rangle),$$

where the index is implicit, and $e^{i\phi}$ is a phase, which intertwine singlet and triplet state configurations. This is an entangled state, where access to $|\pm\rangle^1$ then you also have access to $|\pm\rangle^2$.

This basic form of entanglement is generalized for 3 and 4 qubit entanglement system in the W and Greenberger Horne Zeilinger (GHZ) state. The W state is an entanglement of 3 qubits into a quantum state of the form

$$|W\rangle ~=~ \frac{1}{\sqrt{3}}(|001\rangle ~+~ |010\rangle ~+~ |100\rangle),$$

for a set of three qubits with $\{0,~1\}$ occupation states. A density matrix $\rho_W ~=~ |W\rangle\langle W|$ under a trace of one of the entries recovers a bipartite entangled state. The GHZ state is composed of 2 dimensional state spaces for $n ~>~ 2$ qubits in the form

$$|GHZ\rangle ~=~ \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} ~+~ |1\rangle^{\otimes n})$$

where $n ~=~ 3$ for the 3-qubit system. A trace over one of the entry slots gives a density matrix $|00\rangle\langle 00| ~+~ |11\rangle\langle 11|$ which is an unentangled mixed state. Conversely a measurement of one of the three states leaves $|00\rangle$ or $|11\rangle$, which are separable pure states. The 3 qubit state then leads to highly nonclassical correlations. In particular the GHZ state results in a "wringing out" more nonlocal versions of Bell inequalities [10], where one measurement gives Bell's inequality. Quantum mechanics is not a statistical theory, but statistics are a result.

There are correspondences between these qubit entanglements and black hole types. The bipartite 2 , 3 and 4 qubit entanglements have 4, 8 and 16 complex numbers associated with them. These complex numbers are a product of elementary groups $SL(2,C)^{\otimes n}$ $n ~=~ 2,~3,~4,$ and the quantum bits are defined by a representation of a larger group of dimension 4, 8 or 16 in a quotient with this product. The product $G_{SLOCC} ~=~ SL(2,C)^{\otimes n}$ defines the Stochastic Local Operations and Classical Communication (SLOCC) group, and also maps uniquely to the group which acts on the moduli space for black hole types. Two states have equivalent entanglements if they transform according to the SLOCC group. This quotient procedure defines an invariant which maps to Noetherian numbers, conserved quantities, defined on the moduli space of certain black hole types.

The measure of entanglements is given by the determinant of these complex numbers. For the 2 qubit bipartite entanglement the entropy is a standard 2×2 determinant, which computes the entanglement entropy. A 3 qubit entanglement has a $2\times 2\times 2$ tensor determinant is different, it is not diagonalizable in the standard way, and is a hyper-determinant which exploits the $SL(2,C)^{\otimes 3}$ invariants, plus the covariants associated with each $SL(2,C)$. From these are four entanglement measures $\sigma_{ij},~i,j ~=~ a,~b,~c$ for Alice, Bob and Charlie, where these construct the W state, and σ_{abc} for the GHZ state. For the 4 qubit system things are further different. Here the SLOCC group is composed of $SL(2,C)^{\otimes 4}$, with and 4 times the 3 parameters of each $SL(2,C)$ and there are 16 complex parameters to the states ψ_{abcd}. There is a continuous coset space space $(C^2)^{\otimes 4}/SL(2,C)^{\otimes 4}$ for the remaining 4, which departs from the discrete structure for 3 qubit entanglement [11].

A moduli space is a geometric space of points which represent quantities which are fixed under the action of some group or map. These are important in gauge theory, where a gauge choice defines a moduli space of points which are fixed to that gauge. Another version are moduli spaces over algebraic curves. A moduli space then defines the set of all possible parameterizations of some element which retain the properties of the initial element. The SLOCC group would then be an example of a group on a moduli, for that group preserves the entanglement class of a state. The correspondence with a black hole is with a moduli space for the solution type. The classification of extremal black hole types is with nilpotent orbits on the moduli space, which have a Lie algebra isomorphism to the SLOCC group for entanglement types.

The G_{SLOCC} for the orbits of ψ_{abcd} is transformed into the orbits of $SO(4)\times SO(4)$, and the nilpotent orbits are classified on $SO(8,C)$. Nilpotency is defined for elements X of a semisimple algebra g if the adjoint action $ad_X : g \rightarrow g$, defined by a commutator $ad_X(Y) = [X,\ Y]$ for some integer n satisfies $(ad_X)^n = 0$. The STU model of supergravity contains an action for gravitation coupled to four vector fields. The Type IIA theory compactifies the field on $T^6 = (T^2)^{\otimes 3}$, where each T^2 corresponds to the qubit $A,\ B,\ C$. The T^6 is in the presence of Dp-branes, for p $= 0,\ 2,\ 4,\ 6$. The solutions with the Dp-branes are parameterized by the independent electric and magnetic charges associated with the four vector fields. The solution is spherically symmetric, conforming to the general geometry of black holes, which means the Killing vectors for the solution are timelike. In this setting the equations of motion for the orbits on the moduli space for these black holes are nilpotent orbits governed by real group representations, which conserve Noetherian charges. The connection is the black hole entropy is equivalent to the hyperdeterminant for the 4 qubit entanglement. The difference is that the roots or charges of the SUGRA are real valued and $SL(2,\ R)$, while the corresponding complex amplitudes for the entanglements are complex. However, the correspondence is established with the Kostant-Sekiguchi theorem [11][12]. So the occurrence of Dp-branes with the SUGRA compactification of the type IIA string is formally equivalent to the entanglement structure for a 4 qubit system.

AdS coset structure and exceptional group realizations

The AdS spacetime is a quotient manifold, $AdS_n = O(n-1,\ 2)/O(n-1,\ 1)$. The AdS spacetime may be studied in Bengtsson [13]. This quotient structure with the coset construction with discrete groups is a double coset Clifford-Klein form. The $AdS_7 = SO(6,\ 2)/SO(6,\ 1)$ is related to the G_2 group which fixes a vector basis in S^7 according to a a triality condition in the Jordan algebra $J^3(\mathcal{O})$. The triality group is $SO(8)$ where $spin(7) \sim SO(7)$ fixes a vector with the transitive action of $spin(7)$ on the S^7, $spin(7)/G_2 = S^7$. The $SO(8)$ has a **28** where the group action which fixes the a frame in the octonions is the smallest exceptional group G_2 [14].

The fixing of a vector in $spin(7)$, or framing of S^7, defines the exceptional group G_2. G_2 is defined by a three form on a 7-dimensional space. The G_2 imposes cubic constraints on $SO(7)$ which reduce the 21 generators of $SO(7)$ to 14 generators of G_2 [14]. The 7 dimensions removed are a S^7 where $SO(7) \rightarrow G_2\times S^7$. The further coset $SO(7)/SO(6)$ is equivalent to the coset $G_2/SU(3)$, where G_2 can be seen as decomposed into $SU(3)\times S^6$. The 7 and 14 dimensional representations define the "dynamics" in a cubic polynomial which defines a density $\rho^2 \sim \chi_7^3 + \dots$. The cubic nature of the density with respect to these characters results in a Z_3 center symmetry, a discrete group applicable to the AdS_7.

The framed group $GL(7,\ C)$ contains a unique orbit of 3 forms $\omega^{(3)}$. The isotropy group of $\omega^{(3)}$ is isomorphic to G_2 with 7 coefficients contained in $SO(7,C)$. $\omega^{(3)}$ is given by a contact transformation, which is invariant by a cubic determinant or Jacobian. The elements of $\omega^{(3)}$ are then equivalent to the hypermatrix G^{abcd} which gives an elliptic curve $G^{abcd}\psi_a\psi_b\psi_c = 0$ [15]. The cubic nature of the density with respect to these characters results in a Z_3 center symmetry, a discrete group applicable to the AdS. The discrete group describes a Ising-Potts model of spins, which physically correspond to the $(\mathbf{2},\ \mathbf{2},\ \mathbf{2})$ and the hyperdeterminant. These spins physically have the $\{|0\rangle,\ |1\rangle\}$ states in the $AdS_7/Z_3\ \sim\ S^6$ discrete model, and so correspond to the entanglement system governed by the hyperdeterminant. Consequently, this illustrates the potential duality between entanglements of states on a black hole horizon and with state entanglements with the AdS spacetime.

The discrete realization of AdS_n, which connects to a discrete or "quantum" time, constructs light cones and Heisenberg groups. The Heisenberg group realized is the conformal group $SL(2,\ R)$, which is the reduced $AdS_2 \times S^n$ in the near horizon environment. We then have the reductions to $AdS_2\ \sim\ CFT^1$ with the elementary Heisenberg group realization on a hyperbolic space. This is the simple $D0$-brane for the separable state. Additional correspondences exist with the Dp-branes [16]. The 3-form $\omega^{(3)}$ and the cubic polynomial or elliptic curve suggest connections with the Freudenthal determinant and the Jordan eigenvalue problem.

Discrete or continuous?

The nilpotent orbits on moduli spaces for black hole types and its parallel with a discrete structure on AdS_n preserves Noetherian charges or qubits. The decay of a black hole is simply a change in the entanglement of quantum information in a black hole to that of the embedding spacetime, or universe. The Discrete structures are based on continuous groups and manifolds. There is a relationship between elements which transform as e^α and those which transform as 2^n, where nature fundamentally employs $ln(x)\ =\ ln(2)log_2(x)$ so that the discrete binary aspect of the universe is equivalent to the continuous structure of the universe. The $SL(2,\ R)^2$ contains the braid group B_3 such that the braid group B_3 is the universal central extension of $SL(2,\ R)$ [15]. The complexification $SL(2,\ C)$ has pairs of elements given by braid groups as its central extension. While quantum mechanics describes discrete structures, as eigenvectors in Hilbert space, there are still representations of the waves or fields which have a continuous space description.

Goyal's construction involves a binary group system of Stern-Gerlach apparatuses in a discrete model[5]. The discrete elements, Q-bits, Noether charges, measurements as discrete group elements are aspects of nature which are observables. The continuous structures are less measurable, as an infinitesimal on a space is not something detected. In lines with Stenger's "Atoms and Void" metaphysics discrete aspects of physics are real, while continuous elements are not [16]. In fact quantum physics demands groups or algebras employed have complex representations. However, complex variables are not observables. Penrose assumes a dualist or Platonic view of this matter, which extends this into a triality with the inclusion of mind [17]. Penrose's M^3 metaphysics is in start contrast to Stenger's. This is a form of Platonism, similar to mathematician's idea of the objective reality of mathematical structures.

The question of whether nature is discrete or continuous is probably not demonstrable, and not provable — scientific evidence can support theories but never provide a mathematical proof. The model here indicates a curious relationship between continuity and discreteness in nature. However, there is nothing here which can be demonstrated about the reality of discrete or continuous aspects of reality. This means there should be considerable work available for philosophers long in the future.

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Appendix A

The boundary space ∂AdS_{n+1} is a Minkowski spacetime, or a spacetime E_n that is simply connected that with the AdS is such that $AdS_{n+1} \cup E_n$ is the conformal completion of AdS_{n+1} which exhibits a conformal completion under the discrete action of a Klienian group. For the Lorentzian group $SO(2,\ n)$ there exists the discrete group $SO(2, n, Z)$ which is a Mobius group. For a discrete subgroup Γ subset $SO(2,\ n,\ Z)$ that obeys certain regular properties for accumulation points in the discrete set AdS_{n+1}/Γ is a conformal action of Γ on the sphere S_n. This is then a map which constructs an AdS/CFT correspondence.

The quotient space AdS/Γ is a Kleinian structure. The group $SO(2,\ n)$ is a map from the unit ball B_{n+1}, with boundary $\partial B_{n+1}\ =\ S_n$, into R^{n+1}. The discrete group Γ acts as a conformal on the sphere S^n by the action of the Möbius transformation on S_n. The discrete set of maps on S^n has accumulation points on the limit sphere S^n_∞ are determined by the limit set $g_i\ \in\ G$, for $i\ \rightarrow\ \infty$. This is denoted by $\Lambda(G),\ G\ =\ O(2,\ n)$. The discontinuous set is then the complement of this or $\Omega(G)\ =\ S_n\ -\ \Lambda(G)$. The manifold $\Omega(G)/G$ is an orbifold. This means that the Mobius transformation on the limit sphere S^2_∞ is equivalent to the conformal transformation of N^{n+1} which is equivalent to the isometries of AdS_{n+1}. The $\Omega(\Gamma) \cap E_n/\Gamma$ is then a Lorentzian manifold ∂AdS_{n+1}, and a set of discrete points in E_n pertaining to spatial hyperbolids of equivalent data. In this way the data on any spatial surface of AdS_{n+1} is contained in its conformal completion. This is equivalent to the discrete action of Γ on S_n.

The discrete structure here is isomorphic to the discrete set for the Taub-NUT spacetime. This Taub-NUT spacetime is similar to the Schwarzschild spacetime, but where time serves the role radius does. The spacetime has a timelike region I that connects to a region which is spacelike II, which in turn is connected to a timelike region III with closed timelike curves. The manifold for TN is $S^3 \times R$, with S^3 reduce to S^1 modeled as a cylinder. The spacetime is a sort of time version of a black hole, where the time coordinate defines the horizon. The manifold $(M,\ g)$ has a discrete structure to it. An Euler angle in space wraps a geodesic around the tube in region I, and defines intervals $s^2\ =\ t^2\ -\ x^2$ that are equivalent. These discrete points define a discrete subgroup $d\ =\ SL(2,\ Z)\ \subset\ SL(2, C)$. Each of these points defines a neighborhood such that the action of the discrete group on that neighborhood determines $d(U) \cap U\ =\ $. This is a Hausdorff condition. Now the region I of M and II of $M(I,\ g)$ and $M(II,\ g)$ are cases where the $t\ >\ -x$ and then $(I\ +\ II,\ g)$ is Hausdorff. Similarly $(I,\ g)$ and $(III,\ g)$ is the case where $t\ >\ x$ and again $(I\ +\ III,\ g)$ is Hausdorff. However, this can't hold for $(I\ +\ II\ +\ III,\ g)$, which is then not Hausdorff.

Appendix B

For a 3 Q-bit system we focus on invariants. The determinant is replaced by a hyperdeterminant that transforms as a $(\mathbf{2},\ \mathbf{2},\ \mathbf{2})$, and there are elements $\sigma_i,\ \sigma_j$ and σ_k, co-invariants, that transform as $(\mathbf{3},\ \mathbf{1},\ \mathbf{1})$ $(\mathbf{1},\ \mathbf{3},\ \mathbf{1})$ and $(\mathbf{1},\ \mathbf{1},\ \mathbf{3})$ of the G_{SLOCC}. These four construct entanglement measures S_{ijk}, S_{ij}, S_{ik} and S_{jk}. S_{ABC} is a tri-partite entanglement entropy with

$$S_{ABC}\ =\ S_{A(BC)}\ -\ S_{AB}\ -\ SAC$$

The bipartite elements are pair entanglements and the tripartite involves a triplet which is entanglement. So $B\ =\ Bob$ is maximally entangled with $A\ =\ Alice$ and $C\ =\ Carl$ in the tripartite state. Where if I trace out A B or C there is a complete mixed state, classical information due to tracing, and no correlation, while with the three bipartite entanglements Bob's entanglement is with A and C, and a tracing out of his quantum state entanglement continues to have the $A - C$ entanglement. The tripartite entanglement corresponds to a large black hole, while the set of bipartite entangled states a small black hole. The tripartite state is also called the GHZ state for Greenberger, Horne, Zeilinger formulation of a three particle entanglement.

The N-partite entanglement is different from the standard bipartite entanglement. These correspond to three separable state, bipartite states plus one separated, nonseparable bipartite states and a tripartite GHZ state. These states in black hole logic correspond to small black holes 1/2 supersymmetric, small black holes with two BPS charges and 1/4 supersymmetric, two black holes with 3 charges and 1/4 supersymmetric and, the GHZ state is 4 charges and 1/8 supersymmetric.

Appendix C

The set of nilpotent orbits is a classification of $SO(8, C)$. With extremal black holes the condition is given by these nilpotent orbits on the moduli space. A nilpotent orbit is where there is a group G with algebra g, then for $a \in G$ and $b \in g$ then the adjoint action of a and b is

$$b \rightarrow b' = aba^{-1}$$

A nilpotent orbit is given by $b^n = bb...b$ (n times) and this is stationary as it is clear than $b'^n = b^n$, and are a fixed point in the moduli space. This Lie group homomorphism $SL(2, C)^2 \sim SO(4, C)$ converts ψ_{ABCD} to a $(4, 4)$ of $SO(4, C)^2$ [10], so then under $SL(2, C)^2$, ψ_{ABCD} transforms as

$$\psi_{ABCD} \rightarrow \psi'_{ABCD} = U_A \otimes U_B \otimes U_C \otimes U_D \psi_{ABCD}$$

$$= [U_A \otimes U_B(x)]\psi_{ABCD}[U_C \otimes U_D \psi_{ABCD}]^T = M_{AB}\psi_{(AB)(CD)}N^T_{CD}$$

and is a convenient transformation $M\psi N^T = \psi'$ for $U \in SL(2, C)$ and $M,\ N \in SO(4, C)$. The correspondence with $SO(4, C)$ gives this according to nilpotent orbits on $SO(4, C)^2$ on $(\mathbf{4}, \mathbf{4})$. The Kostant Sekiguchi correspondence [11] for the $SO(8, C)$ maps nilpotent orbits of $SO(4, C)^2$ on the $(\mathbf{4}, \mathbf{4})$ to the orbits of $SO(4, 4)$ on the adjoint **28** for the black hole.