Units of a metric tensor and physical interpretation of the gravitational constant.

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It is shown that writing the metric tensor in dimensionfull form is mathematically more appropriate and allows a simple interpretation of the gravitational constant as an emergent parameter. It is also shown that the value of the gravitational constant is due to the contribution of all the particles in the Universe. Newton's law of gravitation is derived from atomic considerations only. The Dirac's large number is related to the number of particles in the Universe.

The question why gravitational force is so small in comparison to nuclear forces has been a major puzzling question of the last centuries. If one accepts that gravitation is a fundamental force one must conclude that there must be "particles" (or non trivial physics) with a size of Plank's length $10^{-35}m$, which is 20 orders of magnitude smaller than the expected size of a proton (nuclei) - which is about the same ratio as the size of a proton to the size of the Earth. This disparity in sizes leads to a logical question whether gravitation (and space in itself) is an independent (fourth) interaction or an emergent phenomenon [1].

In this paper we will try to show that gravitation could be understood as collective effect of all the particles of the Universe governed only by their atomic description.

It is a standard practice to write a metric tensor in the dimensionless form [2]. For the flat Minkowski space, for example, the metric is a diagonal matrix (1, -1, -1, -1), and the Schwarzchild metric in spherical coordinates has this form:

$$g_{ij} = diag[(1 - \frac{r_0}{r}), (1 - \frac{r_0}{r})^{-1}, r^2, r^2 sin^2\theta] \tag{1}$$

Even though this form is sufficient to perform all needed calculations, mathematically - from the tensor analysis point of view - it is incorrect and because of that it lacks physical clarity.

To explain this let us consider a flat 3D space. Let us assume that in a system of coordinates where 'x' is measured in meters (m), the metric tensor has the form:

$$g_{ij} = diag[1, 1, 1] \tag{2}$$

Let us now make a coordinate transformation $x \to \bar{x}$, such that $\bar{x}$ is a distance measured in centimeters (cm). According to the rules of tensor analysis we will have:

$$\begin{aligned} &x = \bar{x}/100 \\ &\bar{g}_{ij} = diag[\frac{1}{10^4}, \frac{1}{10^4}, \frac{1}{10^4}] \end{aligned} \tag{3}$$

In other words, if we say that the metric is a metric of a flat space, we must write it as:

$$g_{ij} = diag[\frac{1}{L_G^2}, \frac{1}{L_G^2}, \frac{1}{L_G^2}] \tag{4}$$

where L_G is a constant with units of length, which scales depending on a system of coordinates (or the units of measurements).

The physical meaning of the constant L_G and its value (say in meters) can be understood if we consider the static point-mass problem. In conformal Euclidean coordinates (t, x, y, z) the metric tensor has this form:

$$\begin{aligned} &g_{ij} = g_t(r)(dt)^2 - g_E(r)(dl)^2; \\ &(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 \quad r = \sqrt{x^2 + y^2 + z^2} \end{aligned} \tag{5}$$

At large distances ($r \to \infty$) in linear approximation the functions g_t and g_E can be written as:

$$g_t \approx \frac{1}{L_G^2} - \frac{2C}{r}; \qquad g_E \approx \frac{1}{L_G^2} + \frac{2C}{r}; \quad C = const \tag{6}$$

(the factor 2 is taken for convenience only). The fact that g_t and g_E depend only on one constant C (with different signs) is required in order to explain the light bending experiment.

The constant C in the first order approximation (no particle-to-particle interactions) must be:

a) proportional to the number of "elementary" particles (N) and

b) inversely proportional - to give the dimension $\frac{1}{length^2}$ - to the particle's (atomic) length parameter L_a:

$$C = \frac{N}{L_a}; \quad where\ L_a = \frac{\hbar}{m_p c}; \quad m_p = proton\ mass \tag{7}$$

The geodesic curves (motion of test particle) is described by Γ-s which depend only on the ratio of the functions g_t and g_E thus is a function of the $\frac{L_G^2}{l_a}$:

$$\begin{aligned} g_t &= \frac{1}{L_G^2}(1 - \frac{L_G^2 c}{\hbar}\ \frac{2Nm_p}{r}); \quad g_E = \frac{1}{L_G^2}(1 - \frac{L_G^2 c}{\hbar}\ \frac{2Nm_p}{r}) \\ \frac{\phi}{c^2} &\approx -\frac{L_G^2 c}{\hbar}\ \frac{Nm_p}{r} \end{aligned} \tag{8}$$

Thus we get Newton's law of gravitation (gravitation potential proportional to the mass of the body and inversely proportional to the distance) derived from most general considerations and atomic length parameter only.

One of the major (and puzzling one) differences between gravitation and particle physics lies in the fact that the characteristic length of gravitation (GM/c^2) is proportional to mass while the characteristic length of the particle theories ($\hbar/mc$) is inverse to the particle mass. Writing the metric tensor in dimensionfull form - see eq.(6 thru 8) explains that.

From eq. (8) also follows that L_G is Plank's length:

$$\begin{aligned} L_G &= \sqrt{\frac{G\hbar}{c^3}} = 1.6 \cdot 10^{-35} m \\ g_{ij}|_{r=\infty} &= 3.9 \cdot 10^{69}\ [\frac{1}{m^2}]\ diag(1, -1, -1, -1) \end{aligned} \tag{9}$$

Since the metric tensor g_{ij} depends, per equation (5), on two functions g_t and g_E, there are two possible explanations for the parameter L_G - the value of g_{ij} at infinity - depending which of the two functions, g_t or g_E, we think has more physical meaning.

In the first scenario, we assume - as it is suggested in all text books on gravitation - that $g_t(r)$ has a physical significance of the gravitational potential. Because of the minus sign (gravitational attraction) in front of 1/r term, we must conclude that there is a background value $1/L_G^2$ of the metric tensor and matter (proton, neutron, etc) replaces that "background" metric, so the value of g_t decreases. This background value of the metric could be due to the λ-term in Einstein's equation. The difficulty here is in fact the existence of a curved space without any matter (vacuum), which of course could be remedied by introducing "dark energy" with its energy-momentum tensor T_{ij} proportional to the metric tensor.

In the second scenario, we assume that it is g_E that has more physical importance -gravitational potential - and g_t is related to g_E through equations of motion (for example, Einstein's equation) - in the first order of magnitude $g_t \approx \frac{1}{L_G^4 g_E}$. In this case, because of the plus sign in front of the 1/r term, we must conclude that adding one more particle to the existing metric increases the value of the gravitational potential g_E (at least at large distances). This makes us wonder if the background value of the metric ($\frac{1}{L_G^2}$) is nothing more than a collection of all metric terms for all particles that surround us.

It is obvious first to check the value produced by all particles of our Milky Way galaxy:

$$Mass: \quad M_{gal} = 6\,10^{11} M_{sun} = 1.2 \cdot 10^{42} kg$$
$$Number\,of\,particles\ (barions): \quad N_{gal} = M_{gal}/M_{proton} = 7 \cdot 10^{68}$$
$$Distance\,gal - sun: \quad L_{gal-sun} = 2700ly = 2.6 \cdot 10^{20} m$$
$$g_E \equiv \frac{1}{L_G^2} = \frac{N_{gal}}{L_{gal-sun} L_a} = 3 \cdot 10^{63} \tag{10}$$

This value is about 6 orders of magnitude less than the excepted value of $3.9 \cdot 10^{69}$ per eq. (9). In our calculations we did not take into account the mass of the Milky-Way galaxy core. According to some estimations it's a black hole with 10^6 the mass of our Sun, which is a mere fraction of the overall mass of the Milky Way galaxy.

If we calculate a similar value for the Universe we get:

$$Mass\ of\ Universe: \quad M_U = 10^{11} M_{gal} = 1.2 \cdot 10^{53} kg$$
$$Number\ of\ particles (barions): \quad N_U = M_U / M_{proton} = 7 \cdot 10^{79}$$
$$Radius\ of\ Universe: \quad L_U = 10^{11} ly = 9 \cdot 10^{26} m$$
$$g_E = \frac{1}{L_G^2} = \frac{N_U}{L_U L_a} = 7.5 \cdot 10^{67} \tag{11}$$

This value is only 50 times less than the excepted value $(3.9 \cdot 10^{69})$.

It appears that the estimate above suggests that the value of the gravitational constant (Plank length) is defined by all the particles of the Universe.

This of course is an estimate and the factor 50 when dealing with numbers 10^{80} - the numbers of particles - should not be taken with rejection.

What is important in this second scenario is that it explains the value of gravitational constant as a composition (summation) of all gravitational terms of all particles in the Universe.

$$\frac{1}{L_G^2} = \sum_i \frac{1}{L_i L_a} = \frac{1}{L_a} \sum_i \frac{1}{L_i} \approx \frac{N_U}{L_U L_a} \tag{12}$$

where L_i is the distance to the i-th particle.

We would like to emphasize that writing the metric tensor in dimensionfull form is equivalent to the statement that the gravitational constant is not a universal (given a priori) constant, but rather a value of the dimensionfull metric at a given point. And as such it is defined by the position of the particles in the Universe and their atomic lengths.

Hence, it can very with time. Cosmologically the Universe is increasing in size - the value of L_U increases. Because of that the value of $g_E \equiv \frac{1}{L_G^2}$ per formula (12) decreases, which is effectively equivalent to increasing of the gravitational constant.

This (the second scenario) also fits into the "Large Numbers" theory, associated with the names of Weyl, Addington, and Dirac [3]. This theory suggests that the big numbers - such as the ratio between electrical and gravitation interaction of two protons ($\approx 4 \cdot 10^{39}$) - in some ways are related to the number of particles (protons, electrons, etc) in the Universe. As we can see from the formula (12), the total particles in the Universe can be calculated as:

$$N_U = (\frac{L_a}{L_G})(\frac{L_U}{L_G}) \tag{13}$$

where L_a atomic size of a proton, L_G - Plank's length, and L_U is the radius of the Universe.

From the atomic (quantum mechanics) point of view, the main question would be whether or not the "elementary" particle within its size creates (modifies) the metric comparable to the Plank metric($4\,10^{69}\frac{1}{m^2}$).

If it does not, that is to say that the add-on metric of a particle (say proton) is about $\frac{1}{L_a^2} = 10^{30}\frac{1}{m^2} << \frac{1}{L_G^2} = 4 \cdot 10^{69}\frac{1}{m^2}$, then for all intents and purposes one can consider the space as a flat Minkowski space. And as a consequence there is no physics at the Plank's length. The gravitational interactions are only important on a macro level.

On the other hand, if the answer is yes, and the particle metric (within the particle) changes with a singularity to the level of the Plank's length ($1.6 \cdot 10^{-35} m$), the metric must be included in the equations of quantum mechanics as an essential dynamic variable. For example, the metric is proportional to the fourth power of the particle's spinor. With this scenario, from the point of view a standard quantum mechanics (Minkowski space), the particle has singularity at the point $r = 0$, which is removed at the Plank distance due to the space-matter interaction.

If we consider Einstein's equations for gravitation and electromagnetism we can show that the gravitational constant can be absorbed into the energy-momentum tensor expressed in atomic ($\hbar c$) units.

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{G}{c^4} T_{ij} \tag{14}$$
$$with \quad T_{ij} = -E_{im} E_{jn} g^{mn} + \frac{1}{2} g_{ij} (E_{km} E_{ln} g^{kl} g^{mn})$$

where E_{ij} is the electromagnetic tensor and g_{ij} is the dimensionless - that is $g_{ij}|_\infty = diag(1, -1, -1, -1)$ - metric tensor.

This can be written as:

$$\hat{R}_{ij} - \frac{1}{2} \hat{R} \hat{g}_{ij} = \frac{1}{\hbar c} \hat{T}_{ij} \tag{15}$$

where symbol 'hat' ($\hat{}$) indicates that the metric is dimensionfull - that is $\hat{g}_{ij}|_\infty = \frac{1}{L_G^2}\, diag(1, -1, -1, -1)$ and L_G is the Plank's length.

Indeed, the Ricci tensor R_{ij} depends only on Christoffel symbols and thus does not change, if the metric tensor is multiplied by a constant.

$$R(g_{kl})_{ij} = R(\hat{g}_{kl})_{ij} \equiv \hat{R}_{ij}; \quad and \quad R g_{ij} = \hat{R} \hat{g}_{ij} \tag{16}$$

The RHS of the Einstein equation (14) can be writen as:

$$\begin{aligned}
\frac{G}{c^4} T_{ij} &= \frac{1}{\hbar c} \{ -E_{im} E_{jn} (\frac{G\hbar}{c^3} g^{mn}) \\
&\quad + \frac{1}{2} (\frac{c^3}{G\hbar})\, g_{ij} [E_{km} E_{ln} (\frac{G\hbar}{c^3})\, g^{kl}\ (\frac{G\hbar}{c^3})\, g^{mn}] \} \\
&= \frac{1}{\hbar c} \{ -E_{im} E_{jn} \hat{g}^{mn} + \frac{1}{2} \hat{g}_{ij} [E_{km} E_{ln} \hat{g}^{kl} \hat{g}^{mn}] \} \equiv \frac{1}{\hbar c} \hat{T}_{ij} \\
where \quad \hat{g}_{ij} &= (\frac{G\hbar}{c^3})\, g_{ij} = \frac{1}{L_G^2} g_{ij}
\end{aligned} \tag{17}$$

As we can see, when the dimensionfull form of metric is used, the Einstein equations take a more universal form without the gravitational constant, which becomes a part of the dimensionfull metric tensor.

The dimensionfull form of Einstein equations - eq. (15) - holds true for any type of matter, if the corresponding Lagrangian density ($L_m \sqrt{g}$) is an invariant (does not change) with respect to a transformation when metric is multiplied by a constant - $g_{ij} \to C g_{ij}$ - as it is for the Maxwell (or any other vector field) theory.

Conclusion

What we showed is that gravitation (as a curves space) in its root could be understood on an atomic level. It is due to nuclear interactions the space becomes warped with long reach ($1/L_a r$) asymptotic. The addition of such warpages over all the particles of the Universe makes the combined metric (g_E) reach an enormous value of $10^{69}\,\frac{1}{m^2}$. On that level, the additional effect of one particle - or to that matter of our Sun - is extremely small, which explains the unusually small value of gravitation as compared to the nuclear or electromagnetic forces.

This explanation supports the idea that the large numbers - such as a ratio of the electrical and gravitational forces - are related to the number of "elementary" particle in the Universe.

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