A TREATY OF SYMMETRIC FUNCTION

An Approach in Deriving General Formulation for Alternating Sums of Power for an Arbitrary Arithmetic Progression.

Paper Part III

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In remembrance of my beloved father who passed away on the 23rd of June 2009 and my special thanks to my brother Mohd Yunus Abd Shukor for introducing me Fermat's Last Theorem when I was a teenager. Although I didn't get the proof for this theorem, it enhanced my understanding towards developing the generalized equations for Symmetric Function for Sums of Powers and expressing Riemann Zeta Function using Sum of Power. The finding also contributes to a formulation of a new conjecture of Prime Number of a Power Sum Origin. Lastly to my sister Nazirah Abd Shukor, thanks for all the supports and patience.

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Abstract. An extension of Sum of Power formulation into alternating system. The general formulation is given as follows:

For odd p

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^p=\sum_{j=0}^{m}\left[\frac{1}{2^{2j}}O_j\binom{p}{2j}s^{2j}\frac{\left[\sum_{i=1}^{n}x_i\right]^{p-2j}}{n^{p-2j}}\frac{\left(1-(-1)^n\right)}{2}-\frac{1}{2^{(2j+1)}}Q_j\binom{p}{2j+1}s^{2j+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{p-(2j+1)}}{n^{p-(2j+2)}}\frac{\left(1+(-1)^n\right)}{2}\right] \quad [1]$$

For even p

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^p=\sum_{j=0}^{m}\left[\frac{1}{2^{2j}}O_j\binom{p}{2j}s^{2j}\frac{\left[\sum_{i=1}^{n}x_i\right]^{p-2j}}{n^{p-2j}}\frac{\left(1-(-1)^n\right)}{2}\right]-\sum_{j=0}^{m-1}\left[\frac{1}{2^{(2j+1)}}Q_j\binom{p}{2j+1}s^{2j+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{p-(2j+1)}}{n^{p-(2j+2)}}\frac{\left(1+(-1)^n\right)}{2}\right] \quad [2]$$

Where: $p-(2j+1)\geq -1$ if p is even, $p-(2j+1)\geq 0$ if p is odd, $s=x_{i+1}-x_i$, O_j and Q_j are coefficients, where $p\geq(2j+1)$ and $O_0=1$, $Q_0=1$ and $m=\begin{cases}\frac{p-1}{2}\ for\_odd\_p\\ \frac{p}{2}\ for\_even\_p\end{cases}$ [3]

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1 Introduction.

Alternating sum of powers has been studied by few researcher and the studies dealt mainly on the integers. One of the researches was done by T. Kim [1]. He proposed the formula for alternating sums of the n-th power for positive integers up to $k-1$, the formula is given as follows:

$$T_n(k)=\frac{(-1)^{k+1}}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l k^{n-l}+\frac{E_n}{2}\left(1+(-1)^{k+1}\right) \quad [4]$$

Where E_l is Euler's number.

Other alternating sums of power can be seen on the website "Series Math Study" [2]. The formula deals mainly on the odd alternating sums of power. Some of the formulae are given as follows:

$$1-3+5-7+9-11+\cdots-+(2n-1)=-n(-1)^n \quad [5]$$

$$1^2-3^2+5^2-7^2+9^2-11^2+\cdots-+(2n-1)^2=-2n^2(-1)^n+\frac{(-1)^n-1}{2} \quad [6]$$

$$1^3-3^3+5^3-7^3+9^3-11^3+\cdots-+(2n-1)^3=(3n-4n^3)(-1)^n \quad [7]$$

This paper is about finding new generalized formulation which can be used to generate alternative sum of power without using complicated methodology. This formulation works on all arithmetic progressions for both odd and even *n*. The method how this formula is derived can be seen as follows:

Expanding equation [1] for *p* up to 10, yields these equations:

For *p*=1

$$\sum_{i=1}^{n}(-1)^{i+1}x_i=\sum_{j=0}^{0}\left[\frac{1}{2^{2j}}O_j\binom{1}{2j}s^{2j}\frac{\left[\sum_{i=1}^{n}x_i\right]^{1-2j}}{n^{1-2j}}\frac{\left(1-(-1)^n\right)}{2}-\frac{1}{2^{(2j+1)}}Q_j\binom{1}{2j+1}s^{2j+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{1-(2j+1)}}{n^{1-(2j+2)}}\frac{\left(1+(-1)^n\right)}{2}\right]$$

$$=\frac{1}{2^{2(0)}}O_0\binom{1}{2(0)}s^{2(0)}\frac{\left[\sum_{i=1}^{n}x_i\right]^{1-2(0)}}{n^{1-2(0)}}\frac{\left(1-(-1)^n\right)}{2}-\frac{1}{2^{(2(0)+1)}}Q_0\binom{1}{2(0)+1}s^{2(0)+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{1-(2(0)+1)}}{n^{1-(2(0)+2)}}\frac{\left(1+(-1)^n\right)}{2}$$

$$=\frac{\left[\sum_{i=1}^{n}x_i\right]}{n}\frac{\left(1-(-1)^n\right)}{2}-\frac{ns}{2}\frac{\left(1+(-1)^n\right)}{2} \quad [8]$$

For p=2

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2=\sum_{j=0}^{1}\left[\frac{1}{2^{2j}}O_j\binom{2}{2j}s^{2j}\frac{\left[\sum_{i=1}^{n}x_i\right]^{2-2j}}{n^{2-2j}}\frac{\left(1-(-1)^n\right)}{2}\right]-\sum_{j=0}^{0}\left[\frac{1}{2^{(2j+1)}}Q_j\binom{2}{2j+1}s^{2j+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{2-(2j+1)}}{n^{2-(2j+2)}}\frac{\left(1+(-1)^n\right)}{2}\right]$$

$$=\left[\frac{1}{2^{2(0)}}O_0\binom{2}{2(0)}s^{2(0)}\frac{\left[\sum_{i=1}^{n}x_i\right]^{2-2(0)}}{n^{2-2(0)}}+\frac{1}{2^{2(1)}}O_1\binom{2}{2(1)}s^{2(1)}\frac{\left[\sum_{i=1}^{n}x_i\right]^{2-2(1)}}{n^{2-2(1)}}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{1}{2^{(2(0)+1)}}Q_0\binom{2}{2(0)+1}s^{2(0)+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{2-(2(0)+1)}}{n^{2-(2(0)+2)}}\right]\frac{\left(1+(-1)^n\right)}{2}$$

$$=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^2}{n^2}+\frac{O_1s^2}{4}\right]\frac{\left(1-(-1)^n\right)}{2}-s\left[\sum_{i=1}^{n}x_i\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [9]$$

For p=3

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^3=\sum_{j=0}^{1}\left[\frac{1}{2^{2j}}O_j\binom{3}{2j}s^{2j}\frac{\left[\sum_{i=1}^{n}x_i\right]^{3-2j}}{n^{3-2j}}\frac{\left(1-(-1)^n\right)}{2}-\frac{1}{2^{(2j+1)}}Q_j\binom{3}{2j+1}s^{2j+1}\frac{\left[\sum_{i=1}^{n}x_i\right]^{3-(2j+1)}}{n^{3-(2j+2)}}\frac{\left(1+(-1)^n\right)}{2}\right]$$

$$=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^3}{n^3}+\frac{3O_1s^2\left[\sum_{i=1}^{n}x_i\right]}{4n}\right]\frac{\left(1-(-1)^n\right)}{2}-\left[\frac{3s\left[\sum_{i=1}^{n}x_i\right]^2}{2n}+\frac{Q_1ns^3}{8}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [10]$$

For p=4

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^4=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^4}{n^4}+\frac{3O_1s^2\left[\sum_{i=1}^{n}x_i\right]^2}{2n^2}+\frac{O_2s^4}{16}\right]\frac{\left(1-(-1)^n\right)}{2}-\left[\frac{2s\left[\sum_{i=1}^{n}x_i\right]^3}{n^2}+\frac{Q_1s^3\left[\sum_{i=1}^{n}x_i\right]}{2}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [11]$$

For p=5

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^5=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^5}{n^5}+\frac{5O_1s^2\left[\sum_{i=1}^{n}x_i\right]^3}{2n^3}+\frac{5O_2s^4\left[\sum_{i=1}^{n}x_i\right]}{16n}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{5s\left[\sum_{i=1}^{n} x_i\right]^4}{2n^3}+\frac{5Q_1 s^3\left[\sum_{i=1}^{n} x_i\right]^2}{4n}+\frac{Q_2 n s^5\left[\sum_{i=1}^{n} x_i\right]^2}{32}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [12]$$

For p=6

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^6=\left[\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^6}+\frac{15O_1 s^2\left[\sum_{i=1}^{n} x_i\right]^4}{4n^4}+\frac{15O_2 s^4\left[\sum_{i=1}^{n} x_i\right]^2}{16n^2}+\frac{O_3 s^6}{64}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{3s\left[\sum_{i=1}^{n} x_i\right]^5}{n^4}+\frac{5Q_1 s^3\left[\sum_{i=1}^{n} x_i\right]^3}{2n^2}+\frac{3Q_2 s^5\left[\sum_{i=1}^{n} x_i\right]}{16}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [13]$$

For p=7

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^7=\left[\frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^7}+\frac{21O_1 s^2\left[\sum_{i=1}^{n} x_i\right]^5}{4n^5}+\frac{35O_2 s^4\left[\sum_{i=1}^{n} x_i\right]^3}{16n^3}+\frac{7O_3 s^6\left[\sum_{i=1}^{n} x_i\right]}{64}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{7s\left[\sum_{i=1}^{n} x_i\right]^6}{2n^5}+\frac{35Q_1 s^3\left[\sum_{i=1}^{n} x_i\right]^4}{8n^3}+\frac{21Q_2 s^5\left[\sum_{i=1}^{n} x_i\right]^2}{32n}+\frac{Q_3 n s^7}{128}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [14]$$

For p=8

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^8=\left[\frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^8}+\frac{7O_1 s^2\left[\sum_{i=1}^{n} x_i\right]^6}{n^6}+\frac{35O_2 s^4\left[\sum_{i=1}^{n} x_i\right]^4}{8n^4}+\frac{7O_3 s^6\left[\sum_{i=1}^{n} x_i\right]^2}{16n^2}+\frac{O_4 s^8}{256}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{4s\left[\sum_{i=1}^{n} x_i\right]^7}{n^6}+\frac{7Q_1 s^3\left[\sum_{i=1}^{n} x_i\right]^5}{n^4}+\frac{7Q_2 s^5\left[\sum_{i=1}^{n} x_i\right]^3}{2n^2}+\frac{Q_3 s^7\left[\sum_{i=1}^{n} x_i\right]}{16}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [15]$$

For p=9

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^9=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^9}{n^9}+\frac{18O_1s^2\left[\sum_{i=1}^{n}x_i\right]^7}{n^7}+\frac{63O_2s^4\left[\sum_{i=1}^{n}x_i\right]^5}{8n^5}+\frac{21O_3s^6\left[\sum_{i=1}^{n}x_i\right]^3}{16n^3}+\frac{9O_4s^8\left[\sum_{i=1}^{n}x_i\right]}{512n}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{9s\left[\sum_{i=1}^{n}x_i\right]^8}{2n^7}+\frac{21Q_1s^3\left[\sum_{i=1}^{n}x_i\right]^6}{2n^5}+\frac{63Q_2s^5\left[\sum_{i=1}^{n}x_i\right]^4}{16n^3}+\frac{9Q_3s^7\left[\sum_{i=1}^{n}x_i\right]^2}{32n}+\frac{Q_4ns^9}{512}\right]\frac{\left(1+(-1)^n\right)}{2} \quad [16]$$

For p=10

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^{10}=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^{10}}{n^{10}}+\frac{45O_1s^2\left[\sum_{i=1}^{n}x_i\right]^8}{4n^8}+\frac{105O_2s^4\left[\sum_{i=1}^{n}x_i\right]^6}{8n^6}+\frac{105O_3s^6\left[\sum_{i=1}^{n}x_i\right]^4}{32n^4}+\frac{45O_4s^8\left[\sum_{i=1}^{n}x_i\right]^2}{256n^2}+\frac{O_5s^{10}}{1024}\right]\frac{\left(1-(-1)^n\right)}{2}$$

$$-\left[\frac{5s\left[\sum_{i=1}^{n}x_i\right]^9}{n^8}+\frac{15Q_1s^3\left[\sum_{i=1}^{n}x_i\right]^7}{n^6}+\frac{63Q_2s^5\left[\sum_{i=1}^{n}x_i\right]^5}{8n^4}+\frac{15Q_3s^7\left[\sum_{i=1}^{n}x_i\right]^3}{16n^2}+\frac{5Q_4s^9\left[\sum_{i=1}^{n}x_i\right]}{256}\right]\frac{\left(1+(-1)^n\right)}{2} \quad [17]$$

2 Qualitative Method Derivation of Alternating Sums of Power.

Consider when p=2 and n=2.

Table 1 Data for Alternating Sum of Power for p=2

x_1	x_2	$\sum_{i=1}^{2}x_i$	$\sum_{i=1}^{2}(-1)^{i+1}x_i^2$
1	2	3	-3
2	3	5	-5
3	4	7	-7
4	5	9	-9
5	6	11	-11
6	7	13	-13
7	8	15	-15
8	9	17	-17
9	10	19	-19
10	11	21	-21
11	12	23	-23
12	13	25	-25
13	14	27	-27
14	15	29	-29
15	16	31	-31

Plotting the curve of $\sum_{i=1}^{2} x_i$ versus $\sum_{i=1}^{2} (-1)^{i+1} x_i^2$ yields Figure 1.

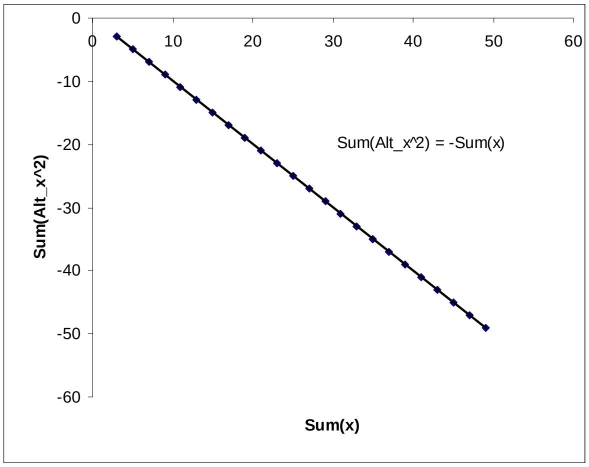

Figure 1 Curve of $\sum_{i=1}^{2} x_i$ versus $\sum_{i=1}^{2} (-1)^{i+1} x_i^2$

It can be seen that when n=2, the data can be related by a relationship given as follows:

$$\sum_{i=1}^{2} (-1)^{i+1} x_i^2 = -\sum_{i=1}^{2} x_i \qquad [18]$$

Consider when p=2 and n=3.

Table 2 Alternating Sum of Power for p=2 and n=3.

x_1	x_2	x_3	$\sum_{i=1}^{3} x_i$	$\sum_{i=1}^{3} (-1)^{i+1} x_i^2$
1	2	3	6	6
2	3	4	9	11
3	4	5	12	18
4	5	6	15	27
5	6	7	18	38
6	7	8	21	51
7	8	9	24	66
8	9	10	27	83
9	10	11	30	102
10	11	12	33	123
11	12	13	36	146
12	13	14	39	171
13	14	15	42	198
14	15	16	45	227
15	16	17	48	258

Plotting the curve of $\sum_{i=1}^{3} x_i$ versus $\sum_{i=1}^{3} (-1)^{i+1} x_i^2$ yields Figure 2.

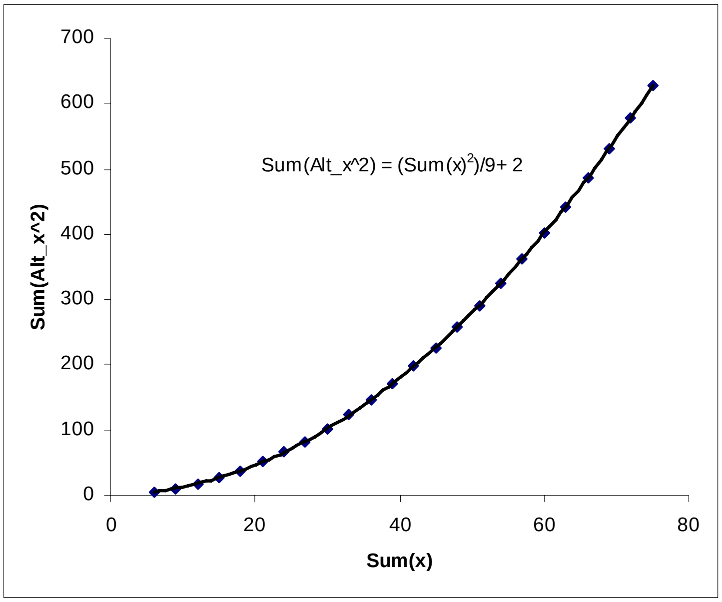

Figure 2 Curve of $\sum_{i=1}^{3} x_i$ versus $\sum_{i=1}^{3} (-1)^{i+1} x_i^2$

It can be seen that when n=3, the data can be related by a relationship given as follows:

$$\sum_{i=1}^{3} (-1)^{i+1} x_i^2 = \frac{\left[\sum_{i=1}^{2} x_i\right]^2}{9} + 2 \qquad [19]$$

Consider the coefficients of the alternating sums of power for p=2 are given as follows:

$$\sum_{i=1}^{n} (-1)^{i+1} x_i^2 = a\left[\sum_{i=1}^{2} x_i\right]^2 + b \quad \text{for odd } n \qquad [20]$$

$$\sum_{i=1}^{n} (-1)^{i+1} x_i^2 = -c\left[\sum_{i=1}^{2} x_i\right] \quad \text{for even } n \qquad [21]$$

Tabulating the values of alternating sum of power for p=2 for some odd and even terms n yield Table 3

Table 3 Coefficients involved for various values of n.

n	a	b	n	c
3	$\frac{1}{9}$	2	2	-1
5	$\frac{1}{25}$	6	4	-1
7	$\frac{1}{49}$	12	6	-1
9	$\frac{1}{81}$	20	8	-1
11	$\frac{1}{121}$	30	10	-1
13	$\frac{1}{169}$	42	12	-1
15	$\frac{1}{225}$	56	14	-1
17	$\frac{1}{289}$	72	16	-1
19	$\frac{1}{361}$	90	18	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{(n^2-1)}{4}$	n	-1

Therefore, the equation for alternating sums of power for p=2 is given as follows:

For even n

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = -\left[\sum_{i=1}^{2}x_i\right] \qquad [22]$$

For odd n

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = \frac{\left[\sum_{i=1}^{2}x_i\right]^2}{n^2} + \frac{(n^2-1)}{4} \qquad [23]$$

Combining both results yields:

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = \left[\frac{\left[\sum_{i=1}^{2}x_i\right]^2}{n^2} + \frac{(n^2-1)}{4}\right]\frac{(1-(-1)^n)}{2} - \left[\sum_{i=1}^{2}x_i\right]\frac{(1+(-1)^n)}{2} \qquad [24]$$

Equation [24] is valid when the difference between numbers is 1 (i.e. s=1). For various values of s, consider the table as follows:

Table 4 Alternating Sums of Power for some values of s (even n=2).

x_1	x_2	$\sum_{i=1}^{2} x_i$	$\sum_{i=1}^{2} (-1)^{i+1} x_i^2$	s	x_1	x_2	$\sum_{i=1}^{2} x_i$	$\sum_{i=1}^{2} (-1)^{i+1} x_i^2$	s
1	3	4	-8	2	1	4	5	-15	3
2	4	6	-12	2	2	5	7	-21	3
3	5	8	-16	2	3	6	9	-27	3
4	6	10	-20	2	4	7	11	-33	3
5	7	12	-24	2	5	8	13	-39	3
6	8	14	-28	2	6	9	15	-45	3
7	9	16	-32	2	7	10	17	-51	3
8	10	18	-36	2	8	11	19	-57	3
9	11	20	-40	2	9	12	21	-63	3
10	12	22	-44	2	10	13	23	-69	3
11	13	24	-48	2	11	14	25	-75	3
12	14	26	-52	2	12	15	27	-81	3
13	15	28	-56	2	13	16	29	-87	3
14	16	30	-60	2	14	17	31	-93	3
15	17	32	-64	2	15	18	33	-99	3

From the data in Table 4, equation [18] can be altered to

$$\sum_{i=1}^{n} (-1)^{i+1} x_i^2 = -s\left[\sum_{i=1}^{2} x_i\right] \qquad [25]$$

The data for odd n is given as in the Table 5.

Table 5 Alternating Sums of Power for some values of s (odd n=3).

x_1	x_2	x_3	$\sum_{i=1}^{2} x_i$	$\sum_{i=1}^{2} (-1)^{i+1} x_i^2$	s	x_1	x_2	x_3	$\sum_{i=1}^{2} x_i$	$\sum_{i=1}^{2} (-1)^{i+1} x_i^2$	s
1	3	5	9	17	2	1	4	7	12	34	3
2	4	6	12	24	2	2	5	8	15	43	3
3	5	7	15	33	2	3	6	9	18	54	3
4	6	8	18	44	2	4	7	10	21	67	3
5	7	9	21	57	2	5	8	11	24	82	3
6	8	10	24	72	2	6	9	12	27	99	3
7	9	11	27	89	2	7	10	13	30	118	3
8	10	12	30	108	2	8	11	14	33	139	3
9	11	13	33	129	2	9	12	15	36	162	3
10	12	14	36	152	2	10	13	16	39	187	3
11	13	15	39	177	2	11	14	17	42	214	3
12	14	16	42	204	2	12	15	18	45	243	3
13	15	17	45	233	2	13	16	19	48	274	3
14	16	18	48	264	2	14	17	20	51	307	3
15	17	19	51	297	2	15	18	21	54	342	3

From the data in Table 5, equation [23] can be altered to

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = \frac{\left[\sum_{i=1}^{n}x_i\right]^2}{n^2} + \frac{(n^2-1)s^2}{4} \qquad [26]$$

Combining both equations for even and odd *n* yields

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = \left[\frac{\left[\sum_{i=1}^{n}x_i\right]^2}{n^2} + \frac{(n^2-1)s^2}{4}\right]\frac{\left(1-(-1)^n\right)}{2} - s\left[\sum_{i=1}^{n}x_i\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [27]$$

Repeating the same procedure for some value of *p* yields these equations:

For *p*=3

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^3 == \left[\frac{\left[\sum_{i=1}^{n}x_i\right]^3}{n^3} + \frac{3(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]}{4n}\right]\frac{\left(1-(-1)^n\right)}{2} - \left[\frac{3s\left[\sum_{i=1}^{n}x_i\right]^2}{2n} + \frac{(n^2-3)ns^3}{8}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [28]$$

For *p*=4

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^4 = \left[\frac{\left[\sum_{i=1}^{n}x_i\right]^4}{n^4} + \frac{3(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]^2}{2n^2} + \frac{(n^2-1)(n^2-5)s^4}{16}\right]\frac{\left(1-(-1)^n\right)}{2}$$
$$- \left[\frac{2s\left[\sum_{i=1}^{n}x_i\right]^3}{n^2} + \frac{(n^2-3)s^3\left[\sum_{i=1}^{n}x_i\right]}{2}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [29]$$

For *p*=5

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^5 = \left[\frac{\left[\sum_{i=1}^{n}x_i\right]^5}{n^5} + \frac{5(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]^3}{2n^3} + \frac{5(n^2-1)(n^2-5)s^4\left[\sum_{i=1}^{n}x_i\right]}{16n}\right]\frac{\left(1-(-1)^n\right)}{2}$$
$$- \left[\frac{5s\left[\sum_{i=1}^{n}x_i\right]^4}{2n^3} + \frac{5(n^2-3)s^3\left[\sum_{i=1}^{n}x_i\right]^2}{4n} + \frac{(n^2-5)^2 ns^5\left[\sum_{i=1}^{n}x_i\right]^2}{32}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [30]$$

For p=6

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^6=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^6}{n^6}+\frac{15(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]^4}{4n^4}+\frac{15(n^2-1)(n^2-5)s^4\left[\sum_{i=1}^{n}x_i\right]^2}{16n^2}+\frac{(n^2-1)(n^4-14n^2+61)s^6}{64}\right]\frac{\left(1-(-1)^n\right)}{2}$$
$$-\left[\frac{3s\left[\sum_{i=1}^{n}x_i\right]^5}{n^4}+\frac{5(n^2-3)s^3\left[\sum_{i=1}^{n}x_i\right]^3}{2n^2}+\frac{3(n^2-5)^2s^5\left[\sum_{i=1}^{n}x_i\right]}{16}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [31]$$

For p=7

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^7=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^7}{n^7}+\frac{21(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]^5}{4n^5}+\frac{35(n^2-1)(n^2-5)s^4\left[\sum_{i=1}^{n}x_i\right]^3}{16n^3}+\frac{7(n^2-1)(n^4-14n^2+61)s^6\left[\sum_{i=1}^{n}x_i\right]}{64}\right]\frac{\left(1-(-1)^n\right)}{2}$$
$$-\left[\frac{7s\left[\sum_{i=1}^{n}x_i\right]^6}{2n^5}+\frac{35(n^2-3)s^3\left[\sum_{i=1}^{n}x_i\right]^4}{8n^3}+\frac{21(n^2-5)^2s^5\left[\sum_{i=1}^{n}x_i\right]^2}{32n}+\frac{(n^6-21n^4+175n^2-427)ns^7}{128}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [32]$$

For p=8

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^8=\left[\frac{\left[\sum_{i=1}^{n}x_i\right]^8}{n^8}+\frac{7(n^2-1)s^2\left[\sum_{i=1}^{n}x_i\right]^6}{n^6}+\frac{35(n^2-1)(n^2-5)s^4\left[\sum_{i=1}^{n}x_i\right]^4}{8n^4}+\frac{7(n^2-1)(n^4-14n^2+61)s^6\left[\sum_{i=1}^{n}x_i\right]^2}{16n^2}+\frac{(n^8-28n^6+350n^4-1708n^2+1385)s^8}{256}\right]\frac{\left(1-(-1)^n\right)}{2}$$
$$-\left[\frac{4s\left[\sum_{i=1}^{n}x_i\right]^7}{n^6}+\frac{7(n^2-3)s^3\left[\sum_{i=1}^{n}x_i\right]^5}{n^4}+\frac{7(n^2-5)^2s^5\left[\sum_{i=1}^{n}x_i\right]^3}{2n^2}+\frac{(n^6-15n^4+75n^2-61)s^7\left[\sum_{i=1}^{n}x_i\right]}{16}\right]\frac{\left(1+(-1)^n\right)}{2} \qquad [33]$$

The generalized equation for coefficients of O_j and Q_j can be formed through qualitative method by collecting all coefficients in a table. The procedure is done as follows:

For odd *n*

Table 6 Coefficients for odd *n*

j/m	0	1	2	3	4	5
1	n^2	-1				
2	n^4	$-6n^2$	5			
3	n^6	$-15n^4$	$75n^2$	-61		
4	n^8	$-28n^6$	$350n^4$	$-1708n^2$	1385	
5	n^{10}	$-45n^8$	$1050n^6$	$-12810n^4$	$62325n^2$	50521

Taking out all signs and any *n* term yields:

Table 7 Values of coefficients for odd *n*

m/j	0	1	2	3	4	5
1	1	1				
2	1	6	5			
3	1	15	75	61		
4	1	28	350	1708	1385	
5	1	45	1050	12810	62325	50521

Plotting the data of *m* versus O_m at *j*=1, yields Figure 3.

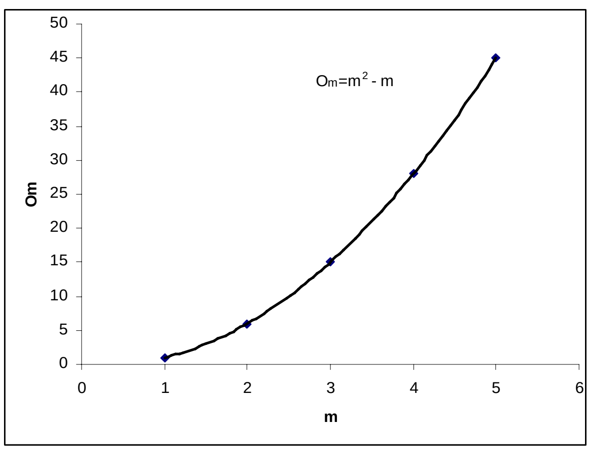

Figure 3 Curve of *m* versus O_m at *j*=1.

Thus, the equation for O_m at *j*=1 can be written as follows:

$$O_{m,\,j=1} = m(2m-1) \quad [34]$$

Repeating the same procedure we can get the coefficients O_m at various value of *j*. Some of the equations of O_m are given as follows:

$$O_{m,\,j=2} = \frac{5}{6}m(m-1)(2m-1)(2m-3) \quad [35]$$

$$O_{m,\,j=3} = \frac{61}{90}m(m-1)(m-2)(2m-1)(2m-3)(2m-5) \quad [36]$$

$$O_{m,\,j=4} = \frac{277}{504}m(m-1)(m-2)(m-3)(2m-1)(2m-3)(2m-5)(2m-7) \quad [37]$$

$$O_{m,j=5} = \frac{50521}{113400} m(m-1)(m-2)(m-3)(m-4)(2m-1)(2m-3)(2m-5)(2m-7)(2m-9) \quad [38]$$

From the Table 7, it can be seen that the coefficients involved in the progression of Alternating Sums of Power is depending on the Euler's or Secant/Zig numbers. The generating function of Secant numbers can be seen as follows:

$$\sec x = \sum_{j=0}^{\infty} \frac{E_j x^{2j}}{(2j)!}$$
$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{1385}{40320}x^8 + \frac{50521}{3628800}x^{10} + \cdots \quad [39]$$

Therefore,

For j=1, 2, 3,.. E_j =1, 5, 61, 1385, 50521,..

The list of secant numbers can be found by sloane's number index A000364 [3]. The generalized equation for coefficients O_m can be formed by summation of all coefficients at j's. The generalized equation is given as follows:

$$O_m = O_{m,j=0} + O_{m,j=1} + O_{m,j=2} + O_{m,j=3} + \cdots$$
$$= \sum_{j=0}^{m} O_{m,j}$$

Therefore,

$$O_m = n^{2m} + \sum_{j=1}^{m}\left[(-1)^j \binom{m}{j}(2j+1)E_j n^{2(m-j)} \frac{\prod_{k=0}^{j-1}(-1+2(m-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]$$

Consider when m=1

$$O_1 = n^{2(1)} + \sum_{j=1}^{1}\left[(-1)^j \binom{1}{j}(2j+1)E_j n^{2(1-j)} \frac{\prod_{k=0}^{j-1}(-1+2(1-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]$$

$$= n^{2(1)} + (-1)^1 \binom{1}{1}(2(1)+1)E_1 n^{2(1-1)} \frac{\prod_{k=0}^{1-1}(-1+2(1-k))}{\prod_{k=0}^{1}(1+2(1-k))}$$

$$= n^2 - 3E_1 \frac{(1)}{(3)(1)} = n^2 - 1$$

When m=2

$$O_2 = n^{2(2)} + \sum_{j=1}^{2}\left[(-1)^j \binom{2}{j}(2j+1)E_j n^{2(2-j)} \frac{\prod_{k=0}^{j-1}(-1+2(2-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]$$

$$= \left[\begin{array}{l} n^{2(2)} + (-1)^1 \binom{2}{1} (2(1)+1) E_1 n^{2(2-1)} \dfrac{\prod_{k=0}^{1-1} (-1+2(2-k))}{\prod_{k=0}^{1} (1+2(1-k))} + \\ (-1)^2 \binom{2}{2} (2(2)+1) E_2 n^{2(2-2)} \dfrac{\prod_{k=0}^{2-1} (-1+2(2-k))}{\prod_{k=0}^{2} (1+2(2-k))} \end{array} \right]$$

$$= \left[n^4 - 6E_1 n^2 \frac{(3)}{(3)(1)} + 5E_2 \frac{(3)(1)}{(5)(3)(1)} \right]$$

$$= \left[n^4 - 6n^2 + 5 \right]$$

When m=3

$$O_3 = n^{2(3)} + \sum_{j=1}^{3} \left[(-1)^j \binom{3}{j} (2j+1) E_j n^{2(3-j)} \frac{\prod_{k=0}^{j-1} (-1+2(3-k))}{\prod_{k=0}^{j} (1+2(j-k))} \right]$$

$$= \left[\begin{array}{l} n^{2(3)} + (-1)^1 \binom{3}{1} (2(1)+1) E_1 n^{2(3-1)} \dfrac{\prod_{k=0}^{1-1} (-1+2(3-k))}{\prod_{k=0}^{1} (1+2(1-k))} + \\ (-1)^2 \binom{3}{2} (2(2)+1) E_2 n^{2(3-2)} \dfrac{\prod_{k=0}^{2-1} (-1+2(3-k))}{\prod_{k=0}^{2} (1+2(2-k))} + \\ (-1)^3 \binom{3}{3} (2(3)+1) E_3 n^{2(3-3)} \dfrac{\prod_{k=0}^{3-1} (-1+2(3-k))}{\prod_{k=0}^{3} (1+2(3-k))} \end{array} \right]$$

$$= \left[n^6 - 9E_1 n^4 \frac{(5)}{(3)(1)} + 15E_2 n^2 \frac{(5)(3)}{(5)(3)(1)} - 7E_3 \frac{(5)(3)(1)}{(7)(5)(3)(1)} \right]$$

$$= \left[n^6 - 15n^4 + 75n^2 - 61 \right]$$

For even n, the procedure to get the generalized equation can be formed using the same procedure as in odd n. Firstly, the coefficients are tabulated in a table given as follows:

Table 8 Coefficients for even n

j/m	0	1	2	3	4	5
1	n^2	-3				
2	n^4	$-10n^2$	25			
3	n^6	$-21n^4$	$175n^2$	-427		
4	n^8	$-36n^6$	$630n^4$	$-5124n^2$	12465	
5	n^{10}	$-55n^8$	$1650n^6$	$-28182n^4$	$228525n^2$	555731

Taking out all signs and any n term yields:

Table 9 Coefficients for even n

j/m	0	1	2	3	4	5
1	1	3				
2	1	10	25			
3	1	21	175	427		
4	1	36	630	5124	12465	
5	1	55	1650	28182	228525	555731

Plotting the data of m versus Q_m at j=1, yields figure [3].

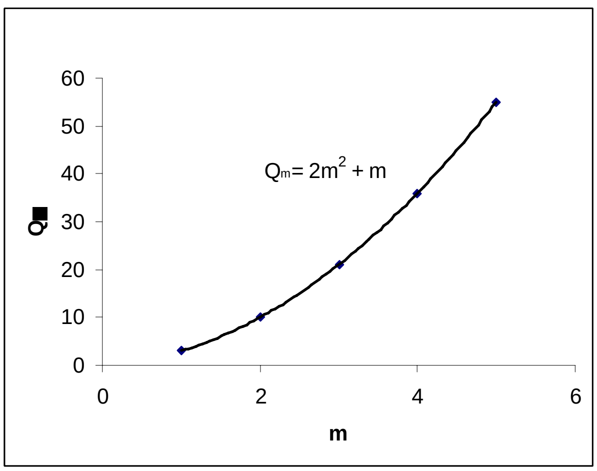

Figure 4 Curve of m versus Q_m at j=1.

Thus, the equation for Q_m at j=1 can be written as follows:

$$Q_{m,j=1} = m(2m+1) \qquad [40]$$

Repeating the same procedure we can get the coefficients Q_m at various value of j. Some of the equations of Q_m are given as follows:

$$Q_{m,j=2} = \frac{5}{6} m(m-1)(2m+1)(2m-1) \qquad [41]$$

$$Q_{m,j=3} = \frac{61}{50} m(m-1)(m-2)(2m+1)(2m-1)(2m-3) \qquad [42]$$

$$Q_{m,j=4} = \frac{277}{504} m(m-1)(m-2)(m-3)(2m+1)(2m-1)(2m-3)(2m-5) \qquad [43]$$

$$Q_{m,j=5} = \frac{50521}{113400} m(m-1)(m-2)(m-3)(m-4)(2m+1)(2m-1)(2m-3)(2m-5)(2m-7) \qquad [44]$$

From these equations we can form the generalized equation for the coefficient Q_m. It is given as follows:

$$Q_m = n^{2m} + \sum_{j=1}^{m}\left[(-1)^j \binom{m}{j}(2j+1)E_j n^{2(m-j)} \frac{\prod_{k=0}^{j-1}(1+2(m-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right] \quad [45]$$

Consider *m*=1

$$Q_1 = n^{2(1)} + \sum_{j=1}^{1}\left[(-1)^j \binom{1}{j}(2j+1)E_j n^{2(1-j)} \frac{\prod_{k=0}^{j-1}(1+2(1-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]$$

$$= n^{2(1)} + (-1)^1 \binom{1}{1}(2(1)+1)E_1 n^{2(1-1)} \frac{\prod_{k=0}^{1-1}(1+2(1-k))}{\prod_{k=0}^{1}(1+2(1-k))}$$

$$= n^2 - 3E_1 \frac{(3)}{(3)(1)} = n^2 - 3 \quad [46]$$

Consider *m*=2

$$Q_2 = \left[n^{2(2)} + \sum_{j=1}^{2}\left[(-1)^j \binom{2}{j}(2j+1)E_j n^{2(2-j)} \frac{\prod_{k=0}^{j-1}(1+2(2-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]\right]$$

$$= \left[\begin{array}{l} n^4 + (-1)^1 \binom{2}{1}(2(1)+1)E_1 n^{2(2-1)} \frac{\prod_{k=0}^{1-1}(1+2(2-k))}{\prod_{k=0}^{1}(1+2(1-k))} + \\ (-1)^2 \binom{2}{2}(2(2)+1)E_2 n^{2(2-2)} \frac{\prod_{k=0}^{2-1}(1+2(2-k))}{\prod_{k=0}^{2}(1+2(2-k))} \end{array}\right]$$

$$= \left[n^4 - 6E_1 n^2 \frac{(5)}{(3)(1)} + 5E_2 \frac{(5)(3)}{(5)(3)(1)}\right]$$

$$= \left[n^4 - 10n^2 + 25\right] \quad [47]$$

Consider *m*=3

$$Q_3 = \left[n^{2(3)} + \sum_{j=1}^{3}\left[(-1)^j \binom{3}{j}(2j+1)E_j n^{2(3-j)} \frac{\prod_{k=0}^{j-1}(1+2(3-k))}{\prod_{k=0}^{j}(1+2(j-k))}\right]\right]$$

$$
= \begin{bmatrix} n^6 + (-1)^1 \binom{3}{1} (2(1)+1) E_1 n^{2(3-1)} \dfrac{\prod_{k=0}^{1-1} (1+2(3-k))}{\prod_{k=0}^{1} (1+2(1-k))} + \\ (-1)^2 \binom{3}{2} (2(2)+1) E_2 n^{2(3-2)} \dfrac{\prod_{k=0}^{2-1} (1+2(3-k))}{\prod_{k=0}^{2} (1+2(2-k))} + \\ (-1)^3 \binom{3}{3} (2(3)+1) E_3 n^{2(3-3)} \dfrac{\prod_{k=0}^{3-1} (1+2(3-k))}{\prod_{k=0}^{3} (1+2(3-k))} \end{bmatrix}
$$

$$
= \left[n^6 - 9E_1 n^4 \frac{(7)}{(3)(1)} + 15E_2 n^2 \frac{(7)(5)}{(5)(3)(1)} - 7E_3 \frac{(7)(5)(3)}{(7)(5)(3)(1)} \right]
$$

$$
= \left[n^6 - 21n^4 + 175n^2 - 427 \right] \qquad [48]
$$

3 Alternating Sum of Power for Integers.

Let,

$$
x_i = 1,2,3\ldots n \qquad [49]
$$

Therefore, the summation of basic elementary symmetric function is given as follows:

$$
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad [50]
$$

For p=1

$$
\sum_{i=1}^{n} (-1)^{i+1} x_i = \frac{\left[\sum_{i=1}^{n} x_i\right]}{n} \frac{\left(1-(-1)^n\right)}{2} - \frac{ns}{2} \frac{\left(1+(-1)^n\right)}{2}
$$

$$
= \frac{\left[\frac{n(n+1)}{2}\right]}{n} \frac{\left(1-(-1)^n\right)}{2} - \frac{n}{2} \frac{\left(1+(-1)^n\right)}{2}
$$

$$
= \frac{(n+1)}{2} \frac{\left(1-(-1)^n\right)}{2} - \frac{n}{2} \frac{\left(1+(-1)^n\right)}{2}
$$

$$
= \frac{1}{4}\left((2n+1)(-1)^n - 1\right)
$$

For p=2

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^2 = \left[\frac{\left[\sum_{i=1}^{n}x_i\right]^2}{n^2} + \frac{(n^2-1)s^2}{4}\right]\frac{\left(1-(-1)^n\right)}{2} - s\left[\sum_{i=1}^{n}x_i\right]\frac{\left(1+(-1)^n\right)}{2}$$

$$= \left[\left[\frac{n+1}{2}\right]^2 + \frac{(n^2-1)}{4}\right]\frac{\left(1-(-1)^n\right)}{2} - \left[\frac{n(n+1)}{2}\right]\frac{\left(1+(-1)^n\right)}{2}$$

$$= -\frac{n(n+1)(-1)^n}{2}$$

For p=3

$$\sum_{i=1}^{n}(-1)^{i+1}x_i^3 = \left[\left(\frac{n+1}{2}\right)^3 + \frac{3(n^2-1)(n+1)}{8}\right]\frac{\left(1-(-1)^n\right)}{2} - \left[\frac{3n(n+1)^2}{8} + \frac{(n^2-3)n}{8}\right]\frac{\left(1+(-1)^n\right)}{2}$$

$$= -\frac{1}{8}\left((4n^3+6n^2-1)(-1)^n+1\right)$$

4 Discussion and Conclusion.

The general equation for alternating Sum of Power presented in this paper uses Euler’s or Zig/Secant numbers instead of Bernoulli’s numbers. This equation works on all arithmetic progression, thus offers a simple and elegant way to calculate the alternative sums of power. It is also highly believed that this equation also might work with real and complex numbers power *p-th*, which would be included in the future papers.

References

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[2] “Series Math Study” http://www.seriesmathstudy.com/sms/finitealternativeodd

[3] http://www.research.att.com/~njas/sequences/A000364