A TREATY OF SYMMETRIC FUNCTION

AN APPROACH IN DERIVING GENERAL FORMULATION FOR SUMS OF POWER FOR AN ARBITRARY ARITHMETIC PROGRESSION

PART 1

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Sum of Power had gathered interest of many classical mathematicians for more than two thousand years ago. The quests of finding sum of power or discrete sum of numerical power can be traced back from the time of Archimedes in third BC then to Faulhaber in the sixteen century. Until today there is no closed form sums of power formulation for an arithmetic progression has been found. Many mathematicians were involved in this research and many approaches have been introduced but none is found to be conclusive. The generalized equation for sums of power discovered in this research has been compared to Faulhaber's sums of power for integers and it is found that this new generalized equation can be used for both integers and arithmetic progression, thus offering a new frontier in studying symmetric function, Fermat's last theorem, Riemman's Zeta function etc.

Keywords: Sums of Power, Faulhaber' sums of power, symmetric function, Sums of Numerical Power, Power Sum.

Scope: Pure Mathematics

1 Introduction.

The sums of power has gathered the interest of many mathematicians since the ancient time until today. The sum of integers for *n* term was formulated by Pythagoras [1] (570-500BC) and the formulation is given as follows:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2} \quad [1]$$

While the sum of square for integers formulation was discovered by Archimedes [2] (287-212BC), his formulation is given as follows:

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \quad [2]$$

The sum of cubes was first formulated by Indian mathematician by the name of Aryabhata who was born in 476 [3] his sum of cube formulation is given as follows:

$$1^3+2^3+3^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2 \quad [3]$$

Other mathematicians studied this formulation were Abu Bakr Al-Karaji [3] (953-1029) and Levi ben Gerson [4] (1288-1344) .

The sums of fourth power of integers was formulated by Abu Ali Al-Hassan ibn Al-Hassan ibn Al-Haytham [5] (965-1039) while he was in Egypt, his formulation can be seen as follows:

$$1^4+2^4+3^4+\cdots+n^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30} \quad [4]$$

Sum of power was first introduced into westerm world by Thomas Harriot [6] (1560-1621). His work concentrated on the sums of squares, cubes and fourth powers. The sums of power for integers for higher powers were formulated by Faulhaber [7] in 1617. He worked up equations for sums of power for integers up to17^{th} power. However, D.E. Knuth [8] has reported that, in Academia Algebra, Faulhaber managed to formulate equations for sums of power up to 23^{rd} power.

The general formulation of sums of power for arithmetic progression mainly for integers was formulated by Blaise Pascal [9] and it was written in words in his book Traite du Triangle. The relationship of this equation is given as follows:

$$(n+1)^{m+1}-\left(1+n+\binom{m+1}{2}\sum_{k=1}^{n}k^{m-1}+\binom{m+1}{3}\sum_{k=1}^{n}k^{m-2}+\cdots+(m+1)\sum_{k=1}^{n}k^{m}\right) \quad [5]$$
$$=(m+1)(1^m+2^m+3^m+\cdots+n^m)$$

This formula can be further written as follows:

$$(n+1)^{m+1}-(n+1)=\binom{m+1}{2}\sum_{k=1}^{n}k^{m-1}+\binom{m+1}{3}\sum_{k=1}^{n}k^{m-2}+\cdots+(m+1)\sum_{k=1}^{n}k^{m} \quad [6]$$

In 1713 Jakob Bernoulli in his book Ars Conjectandi which was published posthumously, derived the symbolic general formulation for sums of power for integers which makes the computation using generalizable formulation possible [10]. The generalize formulation is given as follows:

$$\sum_{k=1}^{n}k^p=\frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$$

By adopting Faulhaber's theorem. William et al [11] discovered the formulas for sums of odd powers by considering an arithmetic progression of the form as follows:

$$(x+1),(x+2)\ldots(x+n) \quad [7]$$

Let the sum of arithmetic series in equation [7] as follows:

$$\lambda = n(n+2x+1) \qquad [8]$$

The sum of power for this progression is gives as follows:

$$S_{2m-1} = (x+1)^{2m-1} + (x+2)^{2m-1} + \ldots + (x+n)^{2m-1} \qquad [9]$$

For some odd *p* (i.e. *p*=2*m*-1), the formulas for sums of power are given as follows:

$$S_3 = \frac{1}{4}[\lambda]^2 + \frac{1}{2}(x^2 + x)[\lambda] \qquad [10]$$

$$S_5 = \frac{1}{6}[\lambda]^3 + \frac{1}{12}(6x^2 + 6x + 1)[\lambda]^2 + \frac{1}{6}(3x^4 + 6x^3 + 2x^2 - x)[\lambda] \qquad [11]$$

$$S_7 = \frac{1}{8}[\lambda]^4 + \frac{1}{6}(3x^2 + 3x - 1)[\lambda]^3 + \frac{1}{12}(9x^4 + 18x^3 + 3x^2 - 6x + 1)[\lambda]^2 + \frac{1}{6}(3x^6 + 9x^5 + 6x^4 - 3x^3 - 2x^2 + x) \qquad [12]$$

Adopting Yoshinari Inaba's matrix method [12] for computing the *m*-th sum of power for the first *n* terms of arithmetic progression, N. Gauthier [13] derived a formula for computing the sum of *m*-th power of *n* successive terms of an arithmetic sequence. The formulation is given as follows:
Let the sum of power or an arithmetic terms as follows:

$$S_m = b^m + (a+b)^m + (2a+b)^m + \cdots + ((n-1)a+b)^m \qquad [13]$$

His result for *m*=2 is given as follows:

$$S_2 = \frac{1}{3}\left[a^2n^3 + 3a\left(1 - \frac{1}{2}a\right)n^2 + \left(3 - 3a + \frac{1}{2}a^2\right)n\right] \qquad [14]$$

The search of a simpler general formulation for sum of power for arithmetic progression had attracted many mathematicians and different methods had been proposed to represent the summation for years [1]-[13]. This paper is to present an elegant method for the sum of power of *p*-th for first *n* term of arithmetic progression. The purpose of this method is to construct a simpler equation.

2 An Alternative Derivation and Formulation of the Sum of Power for *p*-th Arithmetic Progression.

The idea of this paper is to expand the sum of power term into basic symmetric function $\left[\sum_{i=1}^{n} x_i\right]$ with repetitious coefficients. The generalized equation is proposed to be in the form as follows:

$$\sum_{i=1}^{n} x_i^p = \sum_{m=0}^{u}\left[\phi_m s^{2m} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2m}}{n^{p-(2m+1)}}\right] \qquad [15]$$

Where: $p-(2m+1) \geq -1$ if p is even, $p-(2m+1) \geq 0$ if p is odd, $s = x_{i+1} - x_i$, ϕ_m is a coefficient and $\phi_0 = 1$

and $$u = \begin{cases} \frac{p-1}{2} \; for\_odd\_p \\ \frac{p}{2} \; for\_even\_p \end{cases} \qquad [16]$$

By expanding the general equation [15] for first *p*=10, yields

$$\left[\sum_{i=1}^{n} x_i\right] = \phi_0 \left[\sum_{i=1}^{n} x_i\right] \quad [17]$$

$$\left[\sum_{i=1}^{n} x_i^2\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_1 n \quad [18]$$

$$\left[\sum_{i=1}^{n} x_i^3\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_1 \left[\sum_{i=1}^{n} x_i\right] \quad [19]$$

$$\left[\sum_{i=1}^{n} x_i^4\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_2 n \quad [20]$$

$$\left[\sum_{i=1}^{n} x_i^5\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_2 \left[\sum_{i=1}^{n} x_i\right] \quad [21]$$

$$\left[\sum_{i=1}^{n} x_i^6\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_3 n \quad [22]$$

$$\left[\sum_{i=1}^{n} x_i^7\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_3 \left[\sum_{i=1}^{n} x_i\right] \quad [23]$$

$$\left[\sum_{i=1}^{n} x_i^8\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_4 n \quad [24]$$

$$\left[\sum_{i=1}^{n} x_i^9\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^9}{n^8} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_4 \frac{\left[\sum_{i=1}^{n} x_i\right]}{n} \quad [25]$$

$$\left[\sum_{i=1}^{n} x_i^{10}\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^{10}}{n^9} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_4 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_5 n \quad [21]$$

2.1 Data Analysis Method.

This method is about data analysis and using the result to construct the equation needed for each of *p*-th term. Let this equation below applies:

$$\sum_{i=1}^{n} x_i^2 = a\left[\sum_{i=1}^{n} x_i\right]^2 + b \qquad [21]$$

Tabulating some values of *n* for 2 and 3 yields:

Table 1 Data for *n*=2

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2
1	2	3	1	4	5
2	3	5	4	9	13
3	4	7	9	16	25
4	5	9	16	25	41
5	6	11	25	36	61
6	7	13	36	49	85
7	8	15	49	64	113
8	9	17	64	81	145
9	10	19	81	100	181
10	11	21	100	121	221
11	12	23	121	144	265
12	13	25	144	169	313
13	14	27	169	196	365
14	15	29	196	225	421
15	16	31	225	256	481

Table 2 Data for *n*=3

x_1	x_2	x_3	Sum(x_i)	x_1^2	x_2^2	x_3^2	Sum(x_i^2)_n=3
1	2	3	6	1	4	9	14
2	3	4	9	4	9	16	29
3	4	5	12	9	16	25	50
4	5	6	15	16	25	36	77
5	6	7	18	25	36	49	110
6	7	8	21	36	49	64	149
7	8	9	24	49	64	81	194
8	9	10	27	64	81	100	245
9	10	11	30	81	100	121	302
10	11	12	33	100	121	144	365
11	12	13	36	121	144	169	434
12	13	14	39	144	169	196	509
13	14	15	42	169	196	225	590
14	15	16	45	196	225	256	677
15	16	17	48	225	256	289	770

By considering several values of *n* and plotting Sum(x) versus Sum(x^2) for these values of "*n*" yields the graph as follows:

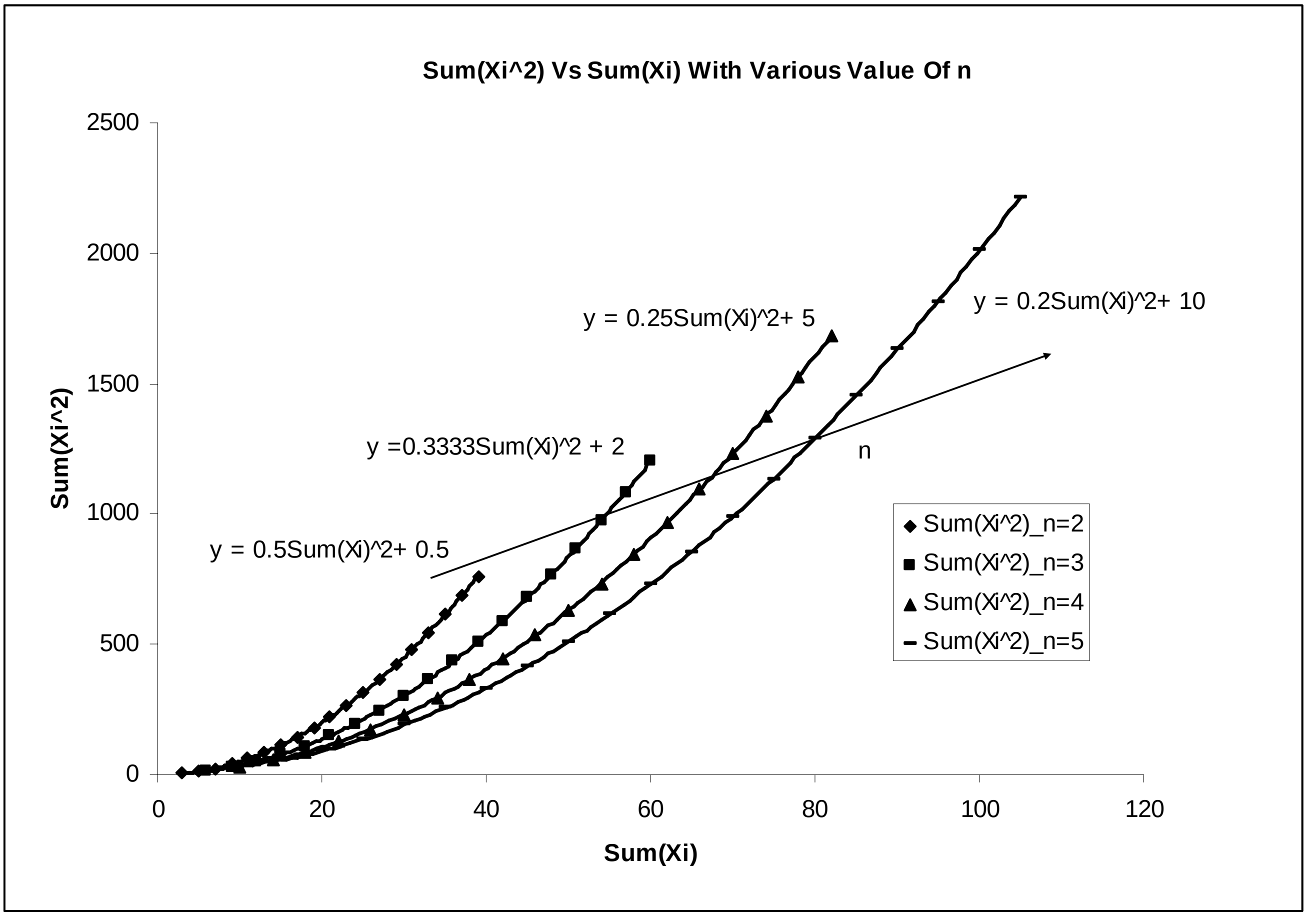

Figure 1.0 The curve for Sum(x^2) versus sum(x).

By collecting the coefficients of *a* and *b* for each *n* and tabulating them and plotting them, yields Figure 2.0 and Figure 3.0.

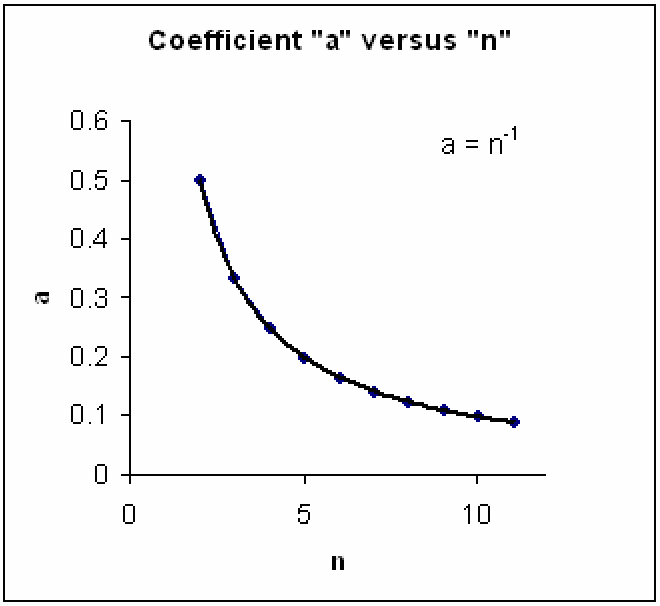

Figure 2.0 Curve for *a* versus *n.*

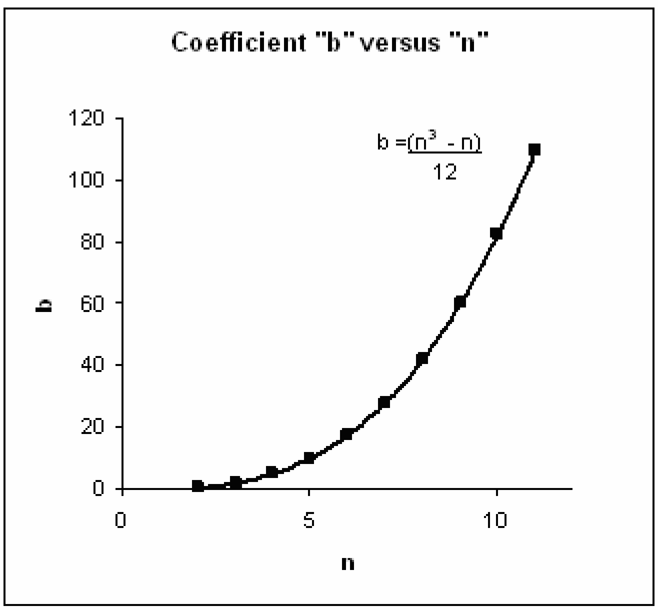

Figure 3.0 Curve for *b* versus *n.*

Table 3 Coefficient for *a* and *b* at various *n*

n	a	b
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$	2
4	$\frac{1}{4}$	5
5	$\frac{1}{5}$	10
6	$\frac{1}{6}$	$\frac{35}{2}$
7	$\frac{1}{7}$	28
8	$\frac{1}{8}$	42
⋮	⋮	⋮
n	$\frac{1}{n}$	$\frac{n(n^2-1)}{12}$

Therefore, simplifying all the coefficients for sum of power for p=2 yields:

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)}{12} \qquad [26]$$

This equation is only applicable for the integers. The equation for sum of power for arbitrary arithmetic progression for p=2 can be obtained by tabulating the data of the arithmetic progression x_i with difference *s*.

Now consider this equations:

$$\sum_{i=1}^{n} x_i^2 = a\left[\sum_{i=1}^{n} x_i\right]^2 + u \qquad [27]$$

By varying the values of *s* and making the value *n* fixed (i.e. *n*=2) and tabulating the data for various values of *s* yield Table 4 to Table 6.

Table 4 Tabulated data for *s*=1.

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=1
1	2	3	1	4	5
2	3	5	4	9	13
3	4	7	9	16	25
4	5	9	16	25	41
5	6	11	25	36	61
6	7	13	36	49	85
7	8	15	49	64	113
8	9	17	64	81	145
9	10	19	81	100	181
10	11	21	100	121	221
11	12	23	121	144	265
12	13	25	144	169	313
13	14	27	169	196	365
14	15	29	196	225	421
15	16	31	225	256	481

Table 5 Tabulated data with *s*=10

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=11
1	12	13	1	144	145
2	13	15	4	169	173
3	14	17	9	196	205
4	15	19	16	225	241
5	16	21	25	256	281
6	17	23	36	289	325
7	18	25	49	324	373
8	19	27	64	361	425
9	20	29	81	400	481
10	21	31	100	441	541
11	22	33	121	484	605
12	23	35	144	529	673
13	24	37	169	576	745
14	25	39	196	625	821
15	26	41	225	676	901

Table 6 Tabulated data with *s*=26

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=26
1	27	28	1	729	730
2	28	30	4	784	788
3	29	32	9	841	850
4	30	34	16	900	916
5	31	36	25	961	986
6	32	38	36	1024	1060
7	33	40	49	1089	1138
8	34	42	64	1156	1220

9	35	44	81	1225	1306
10	36	46	100	1296	1396
11	37	48	121	1369	1490
12	38	50	144	1444	1588
13	39	52	169	1521	1690
14	40	54	196	1600	1796
15	41	56	225	1681	1906

By plotting the data from the Table 4 to Table 6, yields the curves as in Figure 4 for various values of *s*.

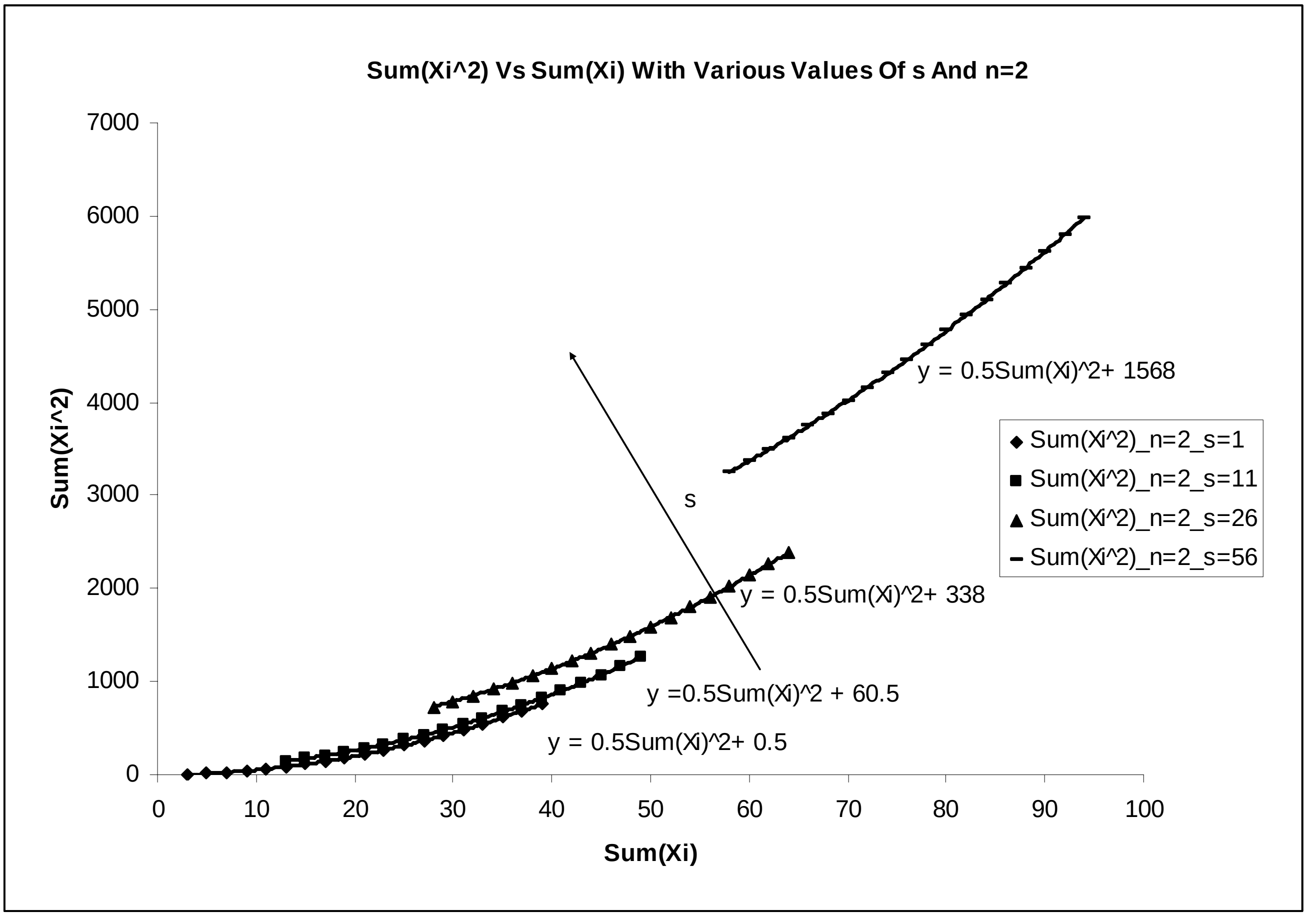

Figure 4.0 The curve for Sum(x_i^2) versus Sum(x) for various values of *s* and *n*=2.

Table 7 Coefficients "*a*" and "*u*" at various value of "*s*" with *n*=2.

s	*a*	*u*	$c = \frac{n(n^2-1)}{12}$	$\frac{u}{c}$	$\frac{u}{c} = s^2$
1	0.5	0.5	0.5	1	1
2	0.5	2	0.5	4	4
3	0.5	4.5	0.5	9	9
4	0.5	8	0.5	16	16
5	0.5	12.5	0.5	25	25
6	0.5	18	0.5	36	36
7	0.5	24.5	0.5	49	49
11	0.5	60.5	0.5	121	121
26	0.5	338	0.5	676	676
56	0.5	1568	0.5	3136	3136

Since $u = cs^2$ and substituting this value into equation [27] yields:

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)s^2}{12} \qquad [28]$$

This data analysis method can be expressed in a matrix form given as follows:

For even *p*

$$\sum_{i=1}^{n} x_i^p = \alpha_1\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_i\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}} s^p \qquad [29]$$

$$\sum_{i=1}^{n} x_{i+1}^p = \alpha_1\left(\sum_{i=1}^{n} x_{i+1}\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}} s^p \qquad [30]$$

$$\sum_{i=1}^{n} x_i^p = \alpha_1\left(\sum_{i=1}^{n} x_{i+2}\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_{i+2}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_{i+2}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}} s^p \qquad [31]$$

$$\vdots$$

$$\sum_{i=1}^{m} x_{i+\frac{p}{2}}^p = \alpha_1\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^p + \alpha_2\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}} s^p \qquad [32]$$

In the matrix form:

$$\begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x_{i+\frac{p}{2}}^p \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & s^2\left(\sum_{i=1}^{n} x_i\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_i\right)^{p-4} & \cdots & s^p \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} & \cdots & s^p \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-4} & \cdots & s^p \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{p+2}{2}} \end{pmatrix} \qquad [33]$$

Solving for the coefficients yields:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{p+2}{2}} \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & s^2\left(\sum_{i=1}^{n} x_i\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_i\right)^{p-4} & \cdots & s^p \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} & \cdots & s^p \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-4} & \cdots & s^p \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x_{i+\frac{p}{2}}^p \end{pmatrix} \qquad [34]$$

By varying the values of *n* we get various values of coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{p+2}{2}}$ in the forms of *n*.

For odd *p*

$$\sum_{i=1}^{n} x_i^p = \alpha_1\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_i\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}}\left(\sum_{i=1}^{n} x_i\right) s^{p-1} \qquad [35]$$

$$\sum_{i=1}^{n} x_{i+1}^p = \alpha_1\left(\sum_{i=1}^{n} x_{i+1}\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}}\left(\sum_{i=1}^{n} x_{i+1}\right) s^{p-1} \qquad [36]$$

$$\sum_{i=1}^{n} x_i^p = \alpha_1\left(\sum_{i=1}^{n} x_{i+2}\right)^p + \alpha_2\left(\sum_{i=1}^{n} x_{i+2}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{n} x_{i+2}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}}\left(\sum_{i=1}^{n} x_{i+2}\right) s^{p-1} \qquad [37]$$

$$\vdots$$

$$\sum_{i=1}^{m} x^{p}_{i+\frac{p}{2}} = \alpha_1\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^{p} + \alpha_2\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^{p-2} s^2 + \alpha_2\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right)^{p-4} s^4 + \ldots + \alpha_{\frac{p+2}{2}}\left(\sum_{i=1}^{m} x_{i+\frac{p}{2}}\right) s^{p-1} \quad [38]$$

In the matrix form:

$$\begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x^p_{i+\frac{p}{2}} \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & s^2\left(\sum_{i=1}^{n} x_i\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_i\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_i\right) \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_{i+1}\right) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right) \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{p+2}{2}} \end{pmatrix} \quad [39]$$

Solving for the coefficients yields:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{p+2}{2}} \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & s^2\left(\sum_{i=1}^{n} x_i\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_i\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_i\right) \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_{i+1}\right) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^p & s^2\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-2} & s^4\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right)^{p-4} & \cdots & s^{p-1}\left(\sum_{i=1}^{n} x_{i+\frac{p}{2}}\right) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x^p_{i+\frac{p}{2}} \end{pmatrix} \quad [40]$$

Again by varying the values of *n* we get various values of coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{p+2}{2}}$ in the forms of *n*.

This method can be used to generate arithmetic *p-th* terms for any value of *p*. However, the larger the value of *p* the more tedious the calculation would be. Since Microsoft Excel having maximum precision of 15 digits, the error in calculation will occurs for numbers more than 15 digits. In order to overcome this problem an “addon” should be installed on the Microsoft Excel, this research was done using Xnumbers [14] which leads to precession of up to 200 digits.

2.0 Algebraic Manipulation Method.

For small *p* the sum of power can be derived using simple algebraic manipulation of arithmetic terms. The formulation for some small *p* can be obtained as follows:

For $p=2$ and $n=2$

Let $(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$ [41]

and $(x_2 - x_1)^2 = x_1^2 + x_2^2 - 2x_1x_2$ [42]

Since the series is an arithmetic progression, thus

$$(x_2 - x_1) = s \quad [43]$$

Substituting [42] into [41], yields

$$2x_1x_2 = x_1^2 + x_2^2 - s^2 \quad [44]$$

Substituting [44] into [41] yields

$$(x_1 + x_2)^2 = 2(x_1^2 + x_2^2) - s^2 \quad [45]$$

Rearranging [45], yields

$$(x_1^2 + x_2^2) = \frac{(x_1 + x_2)^2}{2} + \frac{s^2}{2} \quad [46]$$

or

$$\sum_{i=1}^{2} x_i^2 = \frac{\left[\sum_{i=1}^{2} x_i\right]^2}{2} + \frac{s^2}{2} \qquad [47]$$

Now consider *p=2* and *n*=3,

$$(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2) + 2x_3x_1 + 2x_3x_2 + 2x_1x_2 \qquad [48]$$

Since $s = (x_3 - x_2) = (x_2 - x_1)$

repeating for term $(x_3 - x_2)$, yields

$$(x_3 - x_2)^2 = x_3^2 + x_2^2 - 2x_3x_2 \qquad [49]$$

Therefore, $2x_3x_2 = x_3^2 + x_2^2 - s^2$ [50]

Since $s = (x_3 - x_2) = (x_2 - x_1)$ then

$$(x_3 - x_2) = s \text{ and} \qquad [51]$$

adding [51] and [43], yields

$$(x_3 - x_1) = 2s \qquad [52]$$

squaring both sides [52] and rearranging it, yields

$$2x_3x_1 = x_3^2 + x_1^2 - 4s^2 \qquad [53]$$

Substituting [53], [50] and [44] into [48], yields

$$(x_1 + x_2 + x_3)^2 = 3(x_1^2 + x_2^2 + x_3^2) - 6s^2 \qquad [54]$$

rearranging [54], yields

$$(x_1^2 + x_2^2 + x_3^2) = \frac{(x_1 + x_2 + x_3)^2}{3} + 2s^2 \qquad [55]$$

or

$$\sum_{i=1}^{3} x_i^2 = \frac{\left[\sum_{i=1}^{3} x_i\right]^2}{3} + 2s^2 \qquad [56]$$

repeating the same procedure for terms from 2 to *n*, and by considering a general formulation for $p = 2$ of this form

$$\sum_{i=1}^{n} x_i^2 = a\left[\sum_{i=1}^{n} x_i\right]^2 + bs^2 \qquad [57]$$

and then calculating for some *n* and tabulating the data in a table, the table can be seen as table [8]. The curves constructed from the tabulated data can be seen as in figure [5] and figure [6].

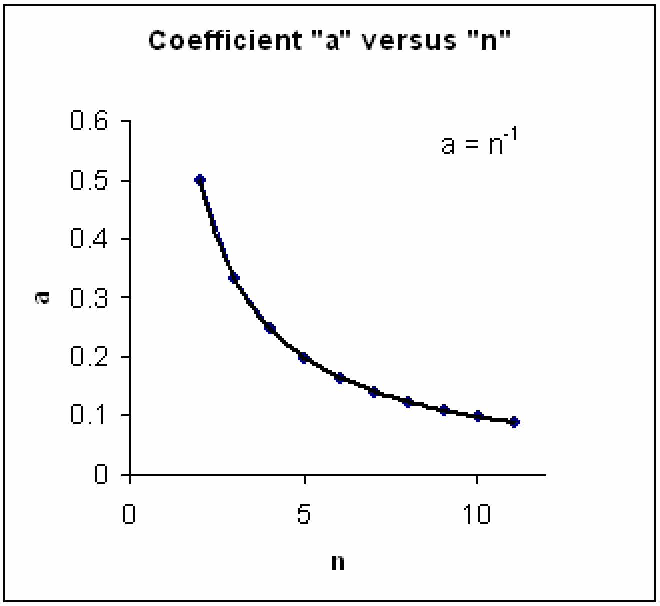

Figure 5.0 Curve for *a* versus *n*.

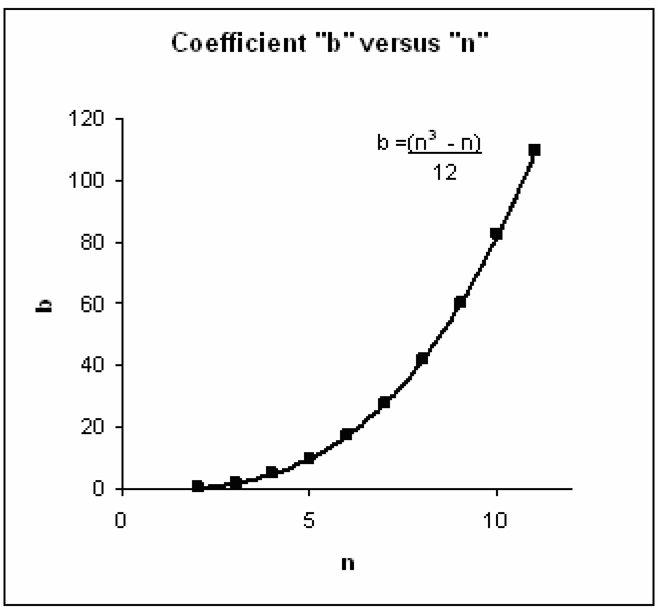

Figure 6.0 Curve for *b* versus *n*.

Table 8 Coefficient for *a* and *b* at various *n*.

n	a	b
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$	2
4	$\frac{1}{4}$	5
5	$\frac{1}{5}$	10
6	$\frac{1}{6}$	$\frac{35}{2}$
7	$\frac{1}{7}$	28
8	$\frac{1}{8}$	42
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n(n^2-1)}{12}$

Consequently,

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)}{12}s^2 \qquad [58]$$

Now consider *p*=*3* and *n*=2, thus:

$$(x_1 + x_2)^3 = (x_1^3 + x_2^3) + 3x_1x_2(x_1 + x_2) \qquad [59]$$

since $(y - x) = s$, then

$$2x_1x_2 = x_1^2 + x_2^2 - s^2 \qquad [60]$$

multiplying both sides [60] with $(x_1 + x_2)$, yields,

$$2x_1x_2(x_1+x_2)=(x_1^2+x_2^2)(x_1+x_2)-s^2(x_1+x_2) \quad [61]$$

multiplying both sides [59] with 2 and substituting [61] into the equation, yields

$$2(x_1+x_2)^3=5(x_1^3+x_2^3)+3x_1x_2(x_1+x_2)-3s^2(x_1+x_2) \quad [62]$$

subtracting equation [62] with equation [59], yields

$$(x_1+x_2)^3=4(x_1^3+x_2^3)-3s^2(x_1+x_2) \quad [63]$$

rearranging [63], yields

$$(x_1^3+x_2^3)=\frac{(x_1+x_2)^3}{4}+\frac{3s^2(x_1+x_2)}{4} \quad [64]$$

or $$\sum_{i=1}^{2}x_i^3=\frac{\left[\sum_{i=1}^{2}x_i\right]^3}{4}+\frac{3s^2}{4}\left[\sum_{i=1}^{2}x_i\right] \quad [65]$$

For $p=3$ and n=3,

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3x_2(x_1^2+x_3^2)+3x_1(x_2^2+x_3^2)+3x_3(x_1^2+x_2^2)+6x_1x_2x_3 \quad [66]$$

Since, $$x_3-x_2=s \quad [67]$$

$$x_2-x_1=s \quad [68]$$

Adding [67] to [68], yields:

$$x_3-x_1=2s \quad [69]$$

Squaring both sides of equations [67] to [69] and rearrange their terms, yields:

$$\left(x_3^2+x_2^2\right)=s^2+2x_3x_2 \quad [70]$$

$$\left(x_2^2+x_1^2\right)=s^2+2x_2x_1 \quad [71]$$

$$\left(x_3^2+x_1^2\right)=4s^2+2x_3x_1 \quad [72]$$

Substituting equations [70] to [72] into [66] and simplifying it, yields

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3s^2(4x_2+x_1+x_3)+24x_1x_2x_3 \quad [73]$$

Manipulating equation [73], yields

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3s^2(3x_2+(x_1+x_2+x_3))+24x_1x_2x_3 \quad [74]$$

Since,

$$x_2=\frac{\left(x_1+x_2+x_3\right)}{3} \quad [75]$$

Substituting equation [75] into [74] and simplifying it, yields:

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+6s^2(x_1+x_2+x_3)+24x_1x_2x_3 \quad [76]$$

Now consider Product Identity for arithmetic progression for n=3 and it is given as follows:

$$\prod_{i=1}^{3}x_i=x_1x_2x_3=\frac{1}{3^3}(x_1+x_2+x_3)\big((x_1+x_2+x_3)-3s\big)\big((x_1+x_2+x_3)+3s\big) \quad [77]$$

Substituting equation [77] into [76] and rearranging the terms, yields

$$(x_1^3+x_2^3+x_3^3)=\frac{(x_1+x_2+x_3)^3}{9}+2s^2(x_1+x_2+x_3) \quad [78]$$

or $$\sum_{i=1}^{3}x_i^3\ \frac{\left[\sum_{i=1}^{3}x_1\right]^3}{9}+2s^2\sum_{i=1}^{3}x_i \quad [79]$$

repeating the same procedure for terms from 2 to n, and by considering a general formulation for $p=3$ of this form

$$\sum_{i=1}^{n} x_i^3 = a\left[\sum_{i=1}^{n} x_i\right]^3 + bs^2\left[\sum_{i=1}^{n} x_i\right] \qquad [80]$$

and then calculating for some n terms yields:

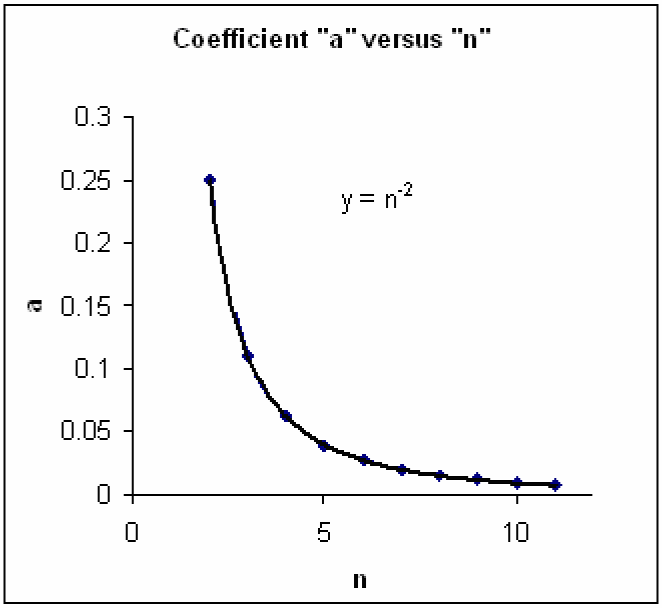

Figure 7.0 Curve for *a* versus *n*.

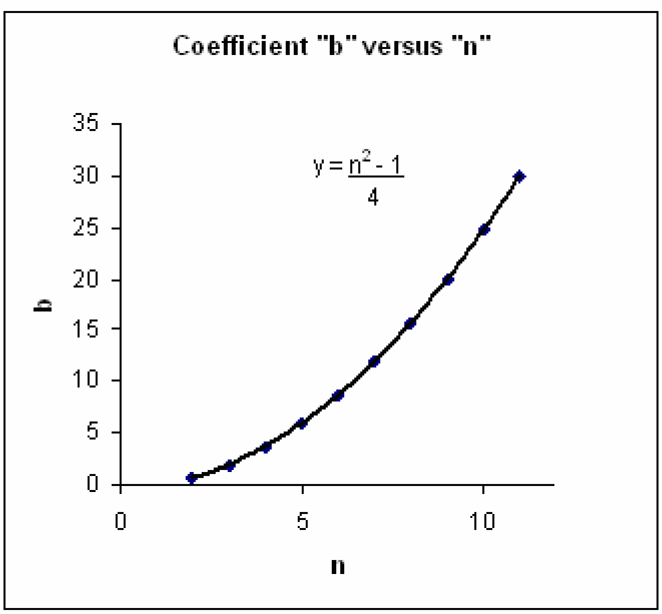

Figure 8.0 Curve for *b* versus *n*.

Table 9 Coefficient for *a* and *b* at various *n*

n	a	b
2	$\frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{9}$	2
4	$\frac{1}{16}$	$\frac{15}{4}$
5	$\frac{1}{25}$	6
6	$\frac{1}{36}$	$\frac{35}{4}$
7	$\frac{1}{49}$	12
8	$\frac{1}{64}$	$\frac{63}{4}$
9	$\frac{1}{81}$	20
⋮	⋮	⋮
n	$\frac{1}{n^2}$	$\frac{(n^2-1)}{4}$

As a result,

$$\sum_{i=1}^{n} x_i^3 = \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \frac{(n^2-1)s^2}{4}\left[\sum_{i=1}^{n} x_i\right] \qquad [81]$$

For $p=4$ and n=2,

Let $$\left(x_1 + x_2\right)^4 = x_1^4 + 4x_2x_1^3 + 6x_1^2x_2^2 + 4x_1x_2^3 + x_2^4 \qquad [82]$$

and rearranging [82] into symmetric function form, yields:

$$\left(x_1 + x_2\right)^4 = \left(x_1^4 + x_2^4\right) + 4x_1x_2\left(x_1^2 + x_2^2\right) + 6\left(x_1x_2\right)^2 \qquad [83]$$

From equation [58] when n=2, it gives

$$\sum_{i=1}^{2} x_i^2 = \left(x_1^2 + x_2^2\right) = \frac{\left[\sum_{i=1}^{2} x_1\right]^2}{2} + \frac{s^2}{2} \qquad [84]$$

Using product identity for arithmetic progression yields:

$$\prod_{i=1}^{2} x_i = \frac{1}{2^2}\left[\left[\sum_{i=1}^{2} x_i\right] - s^2\right] \qquad [85]$$

Substituting equation [85] into [83] and expressing them in a summation notations yields:

$$\left[\sum_{i=1}^{2} x_i\right]^4 = \left[\sum_{i=1}^{2} x_1^4\right] + \frac{1}{2}\left(\left[\sum_{i=1}^{2} x_i\right]^2 - s^2\right)\left(\left[\sum_{i=1}^{2} x_i\right]^2 + s^2\right) + \frac{3}{8}\left(\left[\sum_{i=1}^{2} x_i\right]^2 - s^2\right)^2 \qquad [86]$$

Simplifying and rearranging the equation [86], yields:

$$\left[\sum_{i=1}^{2} x_i^4\right] = \frac{1}{8}\left[\sum_{i=1}^{2} x_i\right]^4 + \frac{3s^2}{4}\left[\sum_{i=1}^{2} x_i\right]^2 + \frac{s^4}{8} \qquad [87]$$

Calculating the other terms and simplifying for term n, yields:

$$\left[\sum_{i=1}^{n} x_i^4\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + (n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{2n} + \frac{n(3n^2-7)(n^2-1)s^4}{240} \qquad [88]$$

Calculating the coefficients for the rest of the equations yields

$$\left[\sum_{i=1}^{n} x_i^5\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + 5(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{6n^2} + \frac{(3n^2-7)(n^2-1)s^4}{48}\left[\sum_{i=1}^{n} x_i\right] \qquad [89]$$

$$\left[\sum_{i=1}^{n} x_i^6\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + 5(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{4n^3} + (3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{16n} + \frac{n(3n^4-18n^2+31)(n^2-1)s^6}{1344} \qquad [90]$$

$$\left[\sum_{i=1}^{n} x_i^7\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + 7(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^5}{4n^4} + 7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{48n^2} + (3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]}{192}$$

[91]

$$\left[\sum_{i=1}^{n} x_i^8\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + 7(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{3n^5} + 7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{24n^3} + (3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{48n}$$

$$+\frac{n(5n^6-55n^4+239n^2-381)(n^2-1)s^8}{11520}$$

[92]

$$\left[\sum_{i=1}^{n} x_i^9\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^9}{n^8} + 3(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + 21(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^5}{40n^4} + (3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{16n^2}$$

$$+(5n^6-55n^4+239n^2-381)(n^2-1)s^8\frac{\left[\sum_{i=1}^{n} x_i\right]}{1280}$$

[93]

$$\left[\sum_{i=1}^{n} x_i^{10}\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^{10}}{n^9} + 15(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^8}{4n^7} + 7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{8n^5} + 5(3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{32n^3}$$

$$+(5n^6-55n^4+239n^2-381)(n^2-1)s^8\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{256n} + \frac{n(3n^{10}-55n^8+462n^6-2046n^4+4191n^2-2555)s^{10}}{33792}$$

[94]

3 Expressing Product of Arithmetic Terms in Form of the Most Basic Elementary Symmetric Function (i.e. $\sum_{i=1}^{n} x_i$).

From the Newton's formulas, Girard and Waring [15] formulated sums of power in the form of elementary symmetric functions $a_1, a_2, a_3 \cdots$. The function is given as follows:

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n \quad [95]$$

Factorizing [83] yields

$$f(x) = (x-x_1)(x-x_2)\cdots(x-x_n) \quad [96]$$

The elementary symmetric functions $a_1, a_2, a_3 \cdots$ of the roots are defined as follows:

$$\sum_{1\le i\le n} x_i = x_1 + x_2 + \cdots + x_n = a_1 \quad [97]$$

$$\sum_{1\le i<j\le n} x_i x_j = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 \cdots = -a_2 \quad [98]$$

$$\sum_{1\le i<j<k\le n} x_i x_j x_k = x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + \cdots = -a_3 \quad [99]$$

$$\vdots$$

$$x_1x_2x_3\cdots x_n = (-1)^n a_n \quad [100]$$

. The p^{th} power sums of the roots of above function is given as follows:

$$S_p = \sum_{1<i<n} x_i^p = x_1^p + x_2^p + x_3^p \cdots x_n^p \qquad [101]$$

Where

$$S_1 + a_1 = 0 \qquad [102]$$

$$S_2 + a_1 S_1 + 2a_2 = 0 \qquad [103]$$

$$S_3 + a_1 S_2 + a_2 S_1 + 2a_3 = 0 \qquad [104]$$

$$\vdots$$

$$S_n + a_1 S_{n-1} + a_2 S_{n-2} + \cdots + n a_n = 0 \qquad [105]$$

$$S_{n+1} + a_1 S_{n-1} + a_2 S_{n-2} + \cdots + n a_{n+1} = 0 \qquad [106]$$

These equations are expressed in the terms of monomials. In this research, it is found that when an arithmetic terms involved, the monomials can be expressed using the most basic elementary symmetric function or sums of an arithmetic term. A theorem has been found and its proof is given as follows:

Theorem 1.0

Let $$P_n = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i = \frac{1}{n^n} \prod_{t=0}^{\frac{n-2}{2}} \left(\left(\sum_{i=1}^{n} x_i \right)^2 - \left(\frac{n}{2}(1+2t)s \right)^2 \right) \text{ for even } n. \qquad [107]$$

and $$P_n = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i \frac{1}{n^n} \prod_{t=1}^{\frac{n-1}{2}} \left(\left(\sum_{i=1}^{n} x_i \right)^2 - (nts)^2 \right) \text{ for odd } n. \qquad [108]$$

Proof: By considering an arithmetic summation of *n* terms (i.e. $\sum_{i=1}^{n} x_i = \frac{n(2x_1 + (n-1)s))}{2}$), and by rearranging it, we get:

$$x_1 = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2} \right) \qquad [109]$$

Since $$x_n = (x_1 - (n-1)s) \qquad [110]$$

Substituting [109] into [110] yields:

$$x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2} \right) \qquad [111]$$

Also,

$$x_{n-1} = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2} \right) - s = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2} \right) \qquad [112]$$

By taking product of x_1 to x_n for even *n* yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right) \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{s}{2}\right) \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) \quad [113]$$

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n}\left(\sum_{i=1}^{n} x_i - \frac{n(n-1)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i - \frac{n(n-3)s}{2}\right) \cdots \left(\sum_{i=1}^{n} x_i - \frac{ns}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{ns}{2}\right) \cdots \left(\sum_{i=1}^{n} x_i + \frac{n(n-3)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{n(n-1)s}{2}\right) \quad [114]$$

Simplifying [114], yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n} \prod_{t=0}^{\frac{n-2}{2}} \left(\left(\sum_{i=1}^{n} x_i \right)^2 - \left(\frac{n}{2}(1+2t)s \right)^2 \right) \text{ for even } n. \quad [115]$$

and product of x_1 to x_n for odd n yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right) \cdots \frac{\sum_{i=1}^{n} x_i}{n} \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) \quad [116]$$

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n}\left(\sum_{i=1}^{n} x_i - \frac{n(n-1)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i - \frac{n(n-3)s}{2}\right) \cdots \sum_{i=1}^{n} x_i \cdots \left(\sum_{i=1}^{n} x_i + \frac{n(n-3)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{n(n-1)s}{2}\right) \quad [117]$$

Simplifying [117], yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \sum_{i=1}^{n} x_i \frac{1}{n^n} \prod_{t=1}^{\frac{n-1}{2}} \left(\left(\sum_{i=1}^{n} x_i \right)^2 - (nts)^2 \right) \text{ for odd } n. \quad [118]$$

3.1 Elementary Symmetric Function for Alternating Permutation of Arithmetic Terms Through Quantitative Method.

Since Sum of Power is the basic building blocks for symmetric polynomials, therefore it can always be expressed as product and sum of symmetric functions with rational coefficients. Consider a set of symmetric functions of arbitrary arithmetic terms as follows:

$$(x_1, x_2, \cdots, x_{n-1}, x_n)$$

The elementary symmetric polynomials of n variables in form of n and symmetric function $\sum_{i=1}^{n} x_i$ are given as follows:

1st Order

$$O_1(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1 + x_2 + \cdots + x_{n-1} + x_n = \sum_{i=1}^{n} x_i \quad [119]$$

The second order can be calculated using quantitative method as follows:

Let the second order be

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1 x_2 + x_1 x_3 + \cdots + x_i x_j = \sum_{i<j}^{n} x_i x_j$$

Consider an arithmetic term with *s*=1 and *n*=2, the tabulated data is given as follows:

Table 10 The values of $\sum_{1\le i<j}^{2} x_i x_j$ when *n*=2

x_1	x_2	$\sum_{i=1}^{2} x_i$	$\sum_{1\le i<j}^{2} x_i x_j$ _n=2
1	2	3	2
2	3	5	6
3	4	7	12
4	5	9	20
5	6	11	30
6	7	13	42
7	8	15	56
8	9	17	72
9	10	19	90
10	11	21	110
11	12	23	132
12	13	25	156
13	14	27	182
14	15	29	210
15	16	31	240
16	17	33	272
17	18	35	306
18	19	37	342
19	20	39	380

Plotting the data for some values of *n* yields graph as in Figure 9.

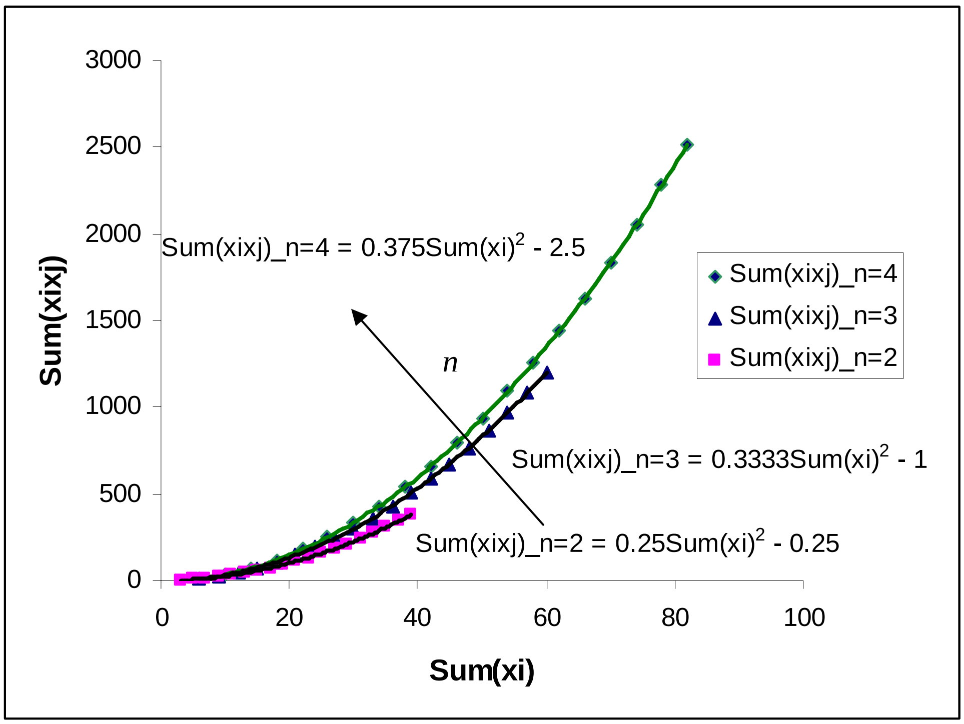

Figure 9 Graph of $\sum_{i<j}^{n} x_i x_j$ versus $\sum_{i=1}^{n} x_i$ for some values of *n*

Let the 2nd Order be as follows:

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j}^{n} x_i x_j = \phi_1 \left(\sum_{i=1}^{n} x_i \right)^2 - \phi_2$$

Collecting the coefficients of $\sum_{i<j}^{n} x_i x_j$ for some values of *n* yields Table 11.

Table 11 The values of ϕ_1 and ϕ_2 at various values of *n*.

n	ϕ_1	ϕ_2
2	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{9}$	$\frac{1}{3}$
4	$\frac{1}{16}$	$\frac{5}{12}$
5	$\frac{1}{25}$	$\frac{1}{2}$
6	$\frac{1}{36}$	$\frac{7}{12}$
7	$\frac{1}{49}$	$\frac{2}{3}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{(n+1)}{12}$

Therefore the 2nd Order for *s*=1 can be written as follows:

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j}^{n} x_i x_j = \frac{1}{n^2} \left(\sum_{i=1}^{n} x_i \right)^2 - \frac{(n+1)}{12}$$

Repeating the same process for various "*s*" yields:

Table 12 The value of coefficient ϕ_1 with various value of "*s*".

n	ϕ_1 (*s*=1)	ϕ_1 (*s*=2)	ϕ_1 (*s*=3)	ϕ_1 (*s*=4)
2	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
5	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
7	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$

Table 13 The value of coefficient ϕ_1 with various values of "*s*".

n	ϕ_2 (*s*=1)	ϕ_2 (*s*=2)	ϕ_2 (*s*=3)	ϕ_2 (*s*=4)
2	$\frac{1}{4}$	1	$\frac{9}{4}$	4
3	$\frac{1}{3}$	$\frac{4}{3}$	3	$\frac{16}{3}$
4	$\frac{5}{12}$	$\frac{5}{3}$	$\frac{15}{4}$	$\frac{20}{3}$
5	$\frac{1}{2}$	2	$\frac{9}{2}$	8
6	$\frac{7}{12}$	$\frac{7}{3}$	$\frac{21}{4}$	$\frac{28}{3}$
7	$\frac{2}{3}$	$\frac{8}{3}$	6	$\frac{32}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n+1)}{12}$	$\frac{(n+1)2^2}{12}$	$\frac{(n+1)3^2}{12}$	$\frac{(n+1)4^2}{12}$

From the Table 13, it can be deduced that ϕ_2 can be written as follows

$$\phi_2 = \frac{(n+1)s^2}{12}$$

Therefore, the second order can be written as follows:

2nd Order

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1x_2 + x_1x_3 + \cdots + x_ix_j = \sum_{i<j}^{n} x_ix_j = \binom{n}{2}\left(\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 - \frac{(n+1)s^2}{12}\right) \quad [120]$$

3rd Order

$$O_3(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_ix_jx_k = \sum_{i<j<k}^{n} x_ix_jx_k = \binom{n}{3}\left(\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 - \frac{(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right) \quad [121]$$

4th Order

$$O_4(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j<k<l}^{n} x_ix_jx_kx_l = \binom{n}{4}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^4 - \frac{(n+1)s^2}{2}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 + \frac{(n+1)(5n+7)s^4}{240}\right] \quad [122]$$

5th Order

$$O_5(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j<k<l<m}^{n} x_i x_j x_k x_l x_m = \binom{n}{5}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^5 - \frac{5(n+1)s^2}{6}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 + \frac{(n+1)(5n+7)s^4}{48}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right] \quad [123]$$

6th Order

$$O_6(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j<k<l<m<n}^{n} x_i x_j x_k x_l x_m x_n = \binom{n}{6}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^6 - \frac{5(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^4 + \frac{(n+1)(5n+7)s^4}{16}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 - \frac{(n+1)(35 \cdot n^2 + 112 \cdot n + 93)s^6}{4032}\right] \quad [124]$$

7th Order

$$O_7(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j<k<l<m<n<p}^{n} x_i x_j x_k x_l x_m x_n x_p = \binom{n}{7}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^7 - \frac{7(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^5 + \frac{7(n+1)(5n+7)s^4}{48}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 - \frac{(n+1)(35 \cdot n^2 + 112 \cdot n + 93)s^6}{576}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right] \quad [125]$$

8th Order

$$O_8(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j<k<l<m<n<o<p}^{n} x_i x_j x_k x_l x_m x_n x_o x_p = \binom{n}{8}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^8 - \frac{7(n+1)s^2}{3}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^6 + \frac{7(n+1)(5n+7)s^4}{24}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^4 - \frac{(n+1)(35 \cdot n^2 + 112 \cdot n + 93)s^6}{144}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 + \frac{(n+1)(5n+9)(35 \cdot n^2 + 126\, n + 127)s^8}{34560}\right] \quad [126]$$

9th Order can be calculated by using the same coefficients used in Order 8th, it is given as follows:

$$O_9(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{\text{i<j<k<l<m<n<o<p<q}}^{n} x_i x_j x_k x_l x_m x_n x_o x_p x_q = \binom{n}{9}\left[\begin{array}{l}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^9 - Q_1(n+1)s^2\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^7 + Q_2(n+1)(5n+7)s^4\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^5 \\ -Q_3(n+1)(35\cdot n^2+112\cdot n+93)s^6\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 + \\ Q_4(n+1)(5n+9)(35\cdot n^2+126\cdot n+127)s^8\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\end{array}\right] \qquad [127]$$

Coefficients $[Q_1 \cdots Q_2]$ can be calculated by using product identity of an arithmetic progression when n=9, the calculation is given as follows:

$$P_9 = x_1\cdot x_2\cdots x_9 = \frac{1}{9^9}\left(\left(\sum_{i=1}^{n} x_i\right)^2-(9s)^2\right)\cdot\left(\left(\sum_{i=1}^{n} x_i\right)^2-(18s)^2\right)\left(\left(\sum_{i=1}^{n} x_i\right)^2-(27s)^2\right)\left(\left(\sum_{i=1}^{n} x_i\right)^2-(36s)^2\right)\sum_{i=1}^{n} x_i \qquad [128]$$

$$P_9 = x_1\cdot x_2\cdots x_9 = \frac{\left(\sum_{i=1}^{n} x_i\right)^9}{387420489} - \frac{10\left(\sum_{i=1}^{n} x_i\right)^7 s^2}{1594323} + \frac{91\left(\sum_{i=1}^{n} x_i\right)^5 s^4}{19683} - \frac{820\left(\sum_{i=1}^{n} x_i\right)^3 s^6}{729} + 64\left(\sum_{i=1}^{n} x_i\right)s^8 \qquad [129]$$

Comparing the coefficients yields:

$$Q_1 = 3, Q_2 = \frac{21}{40},\ Q_3 = \frac{1}{48} \text{ and } Q_4 = \frac{1}{3840}$$

Basically,

$$Q_m = \left(\frac{1}{2^{2m}}\right)\left(\frac{1}{2m+1}\right)\left(\frac{1}{m+1}\right)\binom{p}{2m} \qquad [130]$$

The 10th Order can be calculated as follows:

$$O_{10}(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{i<j<k<l<m<n<o<p<q<r}^{n} x_i x_j\cdots x_q x_r=\binom{n}{10}\left[\begin{array}{l}\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{10}-Q_1(n+1)s^2\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{8}+Q_2(n+1)(5n+7)s^4\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{6}\\ -Q_3(n+1)(35\cdot n^2+112\cdot n+93)s^6\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{4}+\\ Q_4(n+1)(5n+9)(35\cdot n^2+126n+127)s^8\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{2}-T_k\end{array}\right] \quad [131]$$

The Generalised Equation can be written as follows:

$$O_p(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{i<j<\cdots<z}^{n} x_i x_j\cdots x_z=\binom{n}{p}\sum_{m=0}^{k}\left[(-1)^m Q_m T_m s^{2m}\left[\frac{\sum_{i=1}^{n}x_i}{n}\right]^{(p-2m)}\right] \quad [132]$$

Where coefficient $Q_0=1$ and $k=\begin{cases}\frac{p-1}{2} & for\_odd\_p\\ \frac{p}{2} & for\_even\_p\end{cases}$ [133]

The last coefficient T_k can be calculated by performing least square method analysis on various values of "n".
Let the last coefficient in this form:

$$T_k(n)=\sum_{j=0}^{k}a_j n^{k-j} \quad [134]$$

The value of T_k at various values of n can be calculated as follows:

$$T_k(n)=\frac{\sum_{m=0}^{\frac{p-2}{2}}\left[(-1)^m Q_m T_m s^{2m}\left(\frac{\sum_{i=0}^{n}x_i}{n}\right)^{(p-2m)}\right]-\frac{O_p}{\binom{n}{p}}}{Q_k s^p} \quad \text{for even } m \quad [135]$$

$$T_k(n)=\frac{\sum_{m=0}^{\frac{p-3}{2}}\left[(-1)^m Q_m T_m s^{2m}\left(\frac{\sum_{i=0}^{n}x_i}{n}\right)^{(p-2m)}\right]-\frac{O_p}{\binom{n}{p}}}{Q_k s^{p-1}\left(\frac{\sum_{i=0}^{n}x_i}{n}\right)} \qquad \text{for odd } m \qquad [136]$$

Where $n \geq p$ and Q_k is a coefficient given as follows:

$$Q_k=\left(\frac{1}{2^{2k}}\right)\left(\frac{1}{k+1}\right)\left(\frac{1}{2k+1}\right)\binom{p}{2k}$$

For some value of *n*, we can construct the matrix to get the coefficients a_k for solving $T_k(n)$ below:

$$T_k(n)=\left(a_0 n^k + a_1 n^{k-1} + a_2 n^{k-2} \cdots a_{k-1}n + a_k\right) \qquad [137]$$

Let the equations at various values of n arranged in a matrix form as follows:

$$\begin{pmatrix} T_k(n) \\ T_k(n+1) \\ \vdots \\ T_k(n+k-1) \\ T_k(n+k) \end{pmatrix} = \begin{pmatrix} n^k & n^{k-1} & \cdots & n & 1 \\ (n+1)^k & (n+1)^{k-1} & \cdots & (n+1) & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (n+k-1)^k & (n+k-1)^{k-1} & \cdots & (n+k-1) & 1 \\ (n+k)^k & (n+k)^{k-1} & \cdots & (n+k) & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \end{pmatrix} \qquad [138]$$

Where,

$$T=\begin{pmatrix} T_k(n) \\ T_k(n+1) \\ \vdots \\ T_k(n+k-1) \\ T_k(n+k) \end{pmatrix},\ N=\begin{pmatrix} n^k & n^{k-1} & \cdots & n & 1 \\ (n+1)^k & (n+1)^{k-1} & \cdots & (n+1) & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (n+k-1)^k & (n+k-1)^{k-1} & \cdots & (n+k-1) & 1 \\ (n+k)^k & (n+k)^{k-1} & \cdots & (n+k) & 1 \end{pmatrix} \text{ and } a=\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \end{pmatrix} \qquad [139]$$

Therefore,

$$T(n)=N\cdot a \qquad [140]$$

Thus,

$$a=N^{-1}\cdot T \qquad [141]$$

These orders are useful to construct sums of power for the arithmetic progression, as the sums of power for the arithmetic progression can always be expressed into elementary symmetric polynomials.

3.2.1 Using Multinomial Theorem and Product Arithmetic Terms in Generating Sums of Power for an Arbitrary Arithmetic term.

The Multinomial Theorem states that if *p* is nonnegative integers then

$$(x_1 + x_2 + \cdots + x_k)^p = \sum \binom{p}{r_1, r_2 \ldots r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} \tag{142}$$

In this research it is proposed that The Multinomial Theorem for arbitrary arithmetic progression can be expressed as the power of arithmetic sum descending by 2 for each subsequent term (i.e *p*-2*j*). The equation is given as follows:

$$(x_1 + x_2 + \cdots + x_k)^p = \sum \binom{p}{r_1, r_2 \ldots r_k} x_1^{r_1} x_2^{r_2} \cdots x = \sum_{j=0}^{k} \left[\phi_k s^{2k} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \tag{143}$$

This relationship is actually the building block for sum of power of arbitrary arithmetic progression. The sum of powers can be calculated directly from this relationship; however for larger *p* the calculation would be tedious. Every monomial term in the multinomial can also be expressed as follows:

$$\sum_{i<j<\cdots<n}^{n} x_i^{r_1} x_j^{r_2} \cdots x_n^{r_n} = \sum_{j=0}^{k} \left[\ddot{\phi}_k s^{2k} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \tag{144}$$

Where: $p-(2j+1) \geq -1$ if p is even, $p-(2j+1) \geq 0$ if p is odd, $s = x_{i+1} - x_i$, ϕ_k is a coefficient,

$$k = \begin{cases} \frac{p-1}{2} \; for\_odd\_p \\ \frac{p}{2} \; for\_even\_p \end{cases} \quad \text{and} \quad p = \sum_{i=1}^{n} r_i \tag{145}$$

Therefore let's consider the equation below for *p*=2,

$$\sum_{i<j}^{n} x_i x_j = \alpha_1 \left(\sum_{i=1}^{n} x_i \right)^2 + \alpha_2 s^2 \tag{146}$$

Solving the coefficients yields,

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^2 & s^2 \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^2 & s^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i<j}^{n} x_i x_j \\ \sum_{i<j}^{n} x_{i+1} x_{j+1} \end{pmatrix} \qquad [147]$$

When $n = 2$, the solution is given as follows:

$$\sum_{i<j}^{2} x_i x_j = \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{1}{4}s^2 \qquad [148]$$

Calculating for some values of *n* and the tabulated data is given as follows:

Table 14 The values of α_1 and α_2 at various values of *n*.

n	α_1	α_2
2	$\frac{1}{4}$	$-\frac{1}{4}$
3	$\frac{1}{3}$	-1
4	$\frac{3}{8}$	$-\frac{5}{2}$
5	$\frac{2}{5}$	-5
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n-1)}{2n}$	$-\frac{n(n^2-1)}{24}$

Therefore, for all *n* the equation is given as follows:

$$\sum_{i<j}^{n} x_i x_j = \frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24}s^2 \qquad [149]$$

When *p*=3, consider the equation below,

$$\sum_{i<j}^{n} x_i x_j^2 = \beta_1\left(\sum_{i=1}^{n} x_i\right)^3 + \beta_2\left(\sum_{i=1}^{n} x_i\right)s^2 \qquad [150]$$

Solving the coefficients yields,

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^3 & \left(\sum_{i=1}^{n} x_i\right)s^2 \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^3 & \left(\sum_{i=1}^{n} x_{i+1}\right)s^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i<j}^{n} x_i x_j^2 \\ \sum_{i<j}^{n} x_{i+1} x_{j+1}^2 \end{pmatrix} \qquad [151]$$

When $n = 2$, the solution is given as follows:

$$\sum_{i<j}^{2} x_i x_j^2 = \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)s^2 \qquad [152]$$

Calculating for some values of *n* and the tabulated data is given as follows:

Table 15 The values of β_1 and β_2 at various values of *n*.

n	β_1	β_2
2	$\frac{1}{4}$	$-\frac{1}{4}$
3	$\frac{2}{9}$	0
4	$\frac{3}{16}$	1.25
5	$\frac{4}{25}$	4
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n-1)}{n^2}$	$\frac{(n^2-1)(n-3)}{12}$

Therefore, for all *n* the equation is given as follows:

$$\sum_{i<j}^{n} x_i x_j^2 = \frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2 \qquad [153]$$

Applying the same procedures, we get equations as follows:

$$\sum_{i<j}^{n} x_i x_j^3 = \frac{(n-1)}{n^3}\left(\sum_{i=1}^{n} x_i\right)^4 + \frac{(n^2-1)(n-2)}{4n}\left(\sum_{i=1}^{n} x_i\right)s^2 - \frac{n(n^2-1)(3n^2-7)}{240}s^4 \qquad [154]$$

$$\sum_{i<j}^{n} x_i x_j^4 = \frac{(n-1)}{n^4}\left(\sum_{i=1}^{n} x_i\right)^5 + \frac{(n^2-1)(3n-5)}{6n^2}\left(\sum_{i=1}^{n} x_i\right)^3 s^2 + \frac{(n^2-1)(3n^2-7)(n-5)}{240}\left(\sum_{i=1}^{n} x_i\right) s^4 \qquad [155]$$

$$\sum_{i<j}^{n} x_i x_j^5 = \frac{(n-1)}{n^5}\left(\sum_{i=1}^{n} x_i\right)^6 + \frac{5(n^2-1)(2n-3)}{12n^3}\left(\sum_{i=1}^{n} x_i\right)^4 s^2 + \frac{(n^2-1)(3n^2-7)(n-3)}{48n}\left(\sum_{i=1}^{n} x_i\right)^2 s^4 - \frac{n(n^2-1)(3n^4-18n^2+31)}{1344} s^6 \qquad [156]$$

$$\sum_{i<j}^{n} x_i x_j^6 = \frac{(n-1)}{n^6}\left(\sum_{i=1}^{n} x_i\right)^7 + \frac{(n^2-1)(5n-7)}{4n^4}\left(\sum_{i=1}^{n} x_i\right)^5 s^2 + \frac{(n^2-1)(3n^2-7)(3n-7)}{48n^2}\left(\sum_{i=1}^{n} x_i\right)^3 s^4 + \frac{n(n^2-1)(3n^4-18n^2+31)(n-7)}{1344}\left(\sum_{i=1}^{n} x_i\right) s^6 \qquad [157]$$

Combining the results for all *n* yields

For odd *q*,

$$\sum_{i<j}^{n} x_i x_j^q = \frac{(n-1)}{n^q}\left(\sum_{i=1}^{n} x_i\right)^{q+1} + \frac{\phi_1}{n^{q-2}}\left(n - \frac{q+1}{q-1}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-1} s^2 + \frac{\phi_2}{n^{q-4}}\left(n - \frac{q+1}{q-3}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-3} s^4 + \cdots + \frac{\phi_{\frac{q-1}{2}}}{n}\left(n - \frac{q+1}{2}\right)\left(\sum_{i=1}^{n} x_i\right)^2 s^{q-1} - n\phi_{\frac{q+1}{2}} s^{q+1} \qquad [158]$$

$$\sum_{i<j}^{n} x_i x_j^q = \sum_{j=0}^{\frac{q-1}{2}}\left[\frac{\phi_j s^{2j}}{n^{q-2j}}\left(n - \frac{(q+1)}{q+(1-2j)}\right)\left(\sum_{i=1}^{n} x_i\right)^{q+(1-2j)}\right] - n\phi_{\frac{q+1}{2}} s^{q+1} \qquad [159]$$

Or

For even *q*,

$$\sum_{i<j}^{n} x_i x_j^q = \frac{(n-1)}{n^q}\left(\sum_{i=1}^{n} x_i\right)^{q+1} + \frac{\phi_1}{n^{q-2}}\left(n - \frac{q+1}{q-1}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-1} s^2 + \frac{\phi_2}{n^{q-4}}\left(n - \frac{q+1}{q-3}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-3} s^4 + \cdots + \phi_{\frac{q}{2}}\left(n-(q+1)\right)\left(\sum_{i=1}^{n} x_i\right) s^q \qquad [160]$$

$$\sum_{i<j}^{n} x_i x_j^q = \sum_{j=0}^{\frac{q}{2}}\left[\frac{\phi_j s^{2j}}{n^{q-2j}}\left(n - \frac{(q+1)}{q+(1-2j)}\right)\left(\sum_{i=1}^{n} x_i\right)^{q+(1-2j)}\right] \qquad [161]$$

Where, ϕ_j is a function of Bernoulli numbers and $\left(q+(1-2j)\right) \neq 0$ (i.e. the denominator of $(q+1)$ is not zero) and if denominator is zero, the expansion of the term takes the last forms of $n\phi_{\frac{q+1}{2}} s^{q+1}$ or $\phi_{\frac{q}{2}}\left(n-(q+1)\right)\left(\sum_{i=1}^{n} x_i\right) s^q$ for odd and even *q* respectively.

Consider when *p*=2,

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(x_1 + x_2 + \cdots + x_{n-1} + x_n\right)^2 = \left(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2\right) + \binom{2}{1\ \ 1}\left(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n\right)$$

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(x_1 + x_2 + \cdots + x_{n-1} + x_n\right)^2 = \sum_{i=1}^{n} x_i^2 + \binom{2}{1\ \ 1}\sum_{i<j}^{n} x_i x_j \quad [162]$$

Rearranging the equation [162], yields

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - \binom{2}{1\ \ 1}\sum_{i<j}^{n} x_i x_j$$

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\sum_{i<j}^{n} x_i x_j \quad [163]$$

Since, $\sum_{i<j}^{n} x_i x_j = \frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24}s^2$, then [164]

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\left(\frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24}s^2\right)$$

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - \left(\frac{(n-1)}{n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{12}s^2\right)$$

$$\sum_{i=1}^{n} x_i^2 = \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} + \frac{n(n^2-1)}{12}s^2 \quad [165]$$

Consider when p=3,

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + \binom{3}{1\ \ 2}\sum_{i<j}^{n} x_i x_j^2 + \binom{3}{1\ \ 1\ \ 1}\sum_{i<j<k}^{n} x_i x_j x_k \quad [166]$$

Since, $\sum_{i<j}^{n} x_i x_j^2 = \frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2$ and $\sum_{i<j<k}^{n} x_i x_j x_k$ is the elementary symmetric function of 3rd order, then

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + 3\left(\frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2\right) +$$

$$6\binom{n}{3}\left(\frac{1}{n^3}\left(\sum_{i=1}^{n} x_i\right)^3 - \frac{(n+1)s^2}{4n}\left(\sum_{i=1}^{n} x_i\right)\right)$$

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + \frac{(n^2-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 - \frac{(n^2-1)}{4}s^2\left(\sum_{i=1}^{n} x_i\right) \quad [167]$$

Rearranging equation [167], yields

$$\left(\sum_{i=1}^{n} x_i^3\right) = \frac{\left(\sum_{i=1}^{n} x_i\right)^3}{n^2} + \frac{(n^2-1)s^2}{4}\left(\sum_{i=1}^{n} x_i\right) \qquad [168]$$

The multinomial also can be expressed as follows:

$$\left(\sum_{i=1}^{n} x_i\right)^p = \left(\sum_{i=1}^{n} x_i^p\right) + \sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} + \sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} + \cdots + \sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_k^{r_3} \qquad [169]$$

Rearranging equation [169], yields

$$\left(\sum_{i=1}^{n} x_i^p\right) = \left(\sum_{i=1}^{n} x_i\right)^p - \left[\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} + \sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} + \cdots + \sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_k^{r_3}\right] \qquad [170]$$

The proposed conjecture reiterates that for all monomials for an arbitrary arithmetic progression can always be expanded as follows:

$$\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} = \alpha_0\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \alpha_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right) s^{p-1} \text{ for odd } p \qquad [171]$$

$$\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} = \alpha_0\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \alpha_{\frac{p}{2}} s^p \text{ for even } p \qquad [172]$$

$$\sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} = \beta_0\left(\sum_{i=1}^{n} x_i\right)^p + \beta_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \beta_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right) s^{p-1} \text{ for odd } p \qquad [173]$$

$$\sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} = \beta_0\left(\sum_{i=1}^{n} x_i\right)^p + \beta_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \beta_{\frac{p}{2}} s^p \text{ for even } p \qquad [174]$$

$$\sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_m^{r_k} = \gamma_0\left(\sum_{i=1}^{n} x_i\right)^p + \gamma_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \gamma_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right) s^{p-1} \text{ for odd } p \qquad [175]$$

$$\sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_m^{r_k} = \gamma_0\left(\sum_{i=1}^{n} x_i\right)^p + \gamma_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \gamma_{\frac{p}{2}} s^p \text{ for even } p \qquad [176]$$

Collecting the coefficients yields:

$$\left(\sum_{i=1}^{n} x_i^p\right) = \left(1-(\alpha_0+\beta_0+\cdots+\gamma_0)\right)\left(\sum_{i=1}^{n} x_i\right)^p - \left(\alpha_1+\beta_1+\cdots+\gamma_1\right)\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 - \cdots - \left(\alpha_{\frac{p-1}{2}}+\beta_{\frac{p-1}{2}}+\cdots+\gamma_{\frac{p-1}{2}}\right)\left(\sum_{i=1}^{n} x_i\right) s^{p-1} \text{ for odd } p \qquad [177]$$

$$\left(\sum_{i=1}^{n} x_i^p\right) = \left(1-(\alpha_0+\beta_0+\cdots+\gamma_0)\right)\left(\sum_{i=1}^{n} x_i\right)^p - \left(\alpha_1+\beta_1+\cdots+\gamma_1\right)\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 - \cdots - \left(\alpha_{\frac{p}{2}}+\beta_{\frac{p}{2}}+\cdots+\gamma_{\frac{p}{2}}\right)s^p \quad \text{for even } p \qquad [178]$$

Which is in the form of equation [15] when simplified.

4.0 Formulating the Generalize Equation for Sums of Power.

The coefficients ϕ_m involved in the polynomials up to p=12 can be simplified as follows

$$\phi_0 = 1 \qquad [179]$$

$$\phi_1 = \frac{p(p-1)}{24}(n^2-1) \qquad [180]$$

$$\phi 2 = \frac{p(p-1)(p-2)(p-3)}{24^2(10)}(3n^2-7)(n^2-1) \qquad [181]$$

$$\phi_3 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{24^3(70)}(3n^4-18n^2+31)(n^2-1) \qquad [182]$$

$$\phi_4 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{24^4(1400)}(5n^6-55n^4+239n^2-381)(n^2-1) \qquad [183]$$

$$\phi_5 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)}{24^5(15400)}(3n^{10}-55n^8+462n^6-2046n^4+4191n^2-2555) \text{ or}$$

$$\phi_5 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)}{24^5(15400)}(3n^6-37n^4+225n^2-511)(n^2-5)(n^2-1) \qquad [184]$$

$$\phi_6 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)(p-10)(p-11)}{24^6(400400)}(105n^{12}-2730n^{10}+35035n^8-265980n^6+1144143n^4-2325050n^2+1414477) \qquad [185]$$

or in binomial expansion forms

$$\phi_1 = \frac{1}{4}\frac{1}{(2m+1)}\binom{p}{2m}(n^2-1) \qquad [186]$$

$$\phi 2 = \frac{1}{48}\frac{1}{(2m+1)}\binom{p}{2m}(3n^2-7)(n^2-1) \qquad [187]$$

$$\phi_3 = \frac{1}{192}\frac{1}{(2m+1)}\binom{p}{2m}(3n^4-18n^2+31)(n^2-1) \qquad [188]$$

$$\phi_4 = \frac{1}{1280}\frac{1}{(2m+1)}\binom{p}{2m}(5n^6-55n^4+239n^2-381)(n^2-1) \qquad [189]$$

$$\phi_5 = \frac{1}{3072}\frac{1}{(2m+1)}\binom{p}{2m}(3n^6-37n^4+225n^6-511)(n^2-5)(n^2-1) \qquad [190]$$

$$\phi_6 = \frac{1}{430080}\frac{1}{(2m+1)}\binom{p}{2m}(105n^{10}-2625n^8+32410n^6-233570n^4+910573n^2-1414477)(n^2-1) \qquad [191]$$

The generalize form of ϕ_m can be written as follows

$$\phi_m = \frac{1}{c_1 2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}\sum_{t=0}^{m} c_{t+1} n^{2(m-t)}(-1)^t \qquad [192]$$

Or

$$\phi_m = \frac{1}{2^{2m}} \frac{1}{(2m+1)} \binom{p}{2m} P_m \qquad [193]$$

where $(m-t) \geq 0$. The polynomials can be expressed as $P_m = \sum_{t=0}^{m} C_{t+1} n^{2(m-t)} (-1)^t$. In order to identify how the coefficients are formed, each term in the polynomial is tabulated in a table. The tabulated data is gives as in the Table 15.

Table 15 The terms values for P_m.

P_m	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term	7^{th} term		
P_1	n^2	-1							
P_2	$3n^4$	$-10n^2$	7						
P_3	$3n^6$	$-21n^4$	$49n^2$	-31					
P_4	$5n^8$	$-60n^6$	$294n^4$	$-620n^2$	381				
P_5	$3n^{10}$	$-55n^8$	$462n^6$	$-2046n^4$	$4191n^2$	-2555			
P_6	$105n^{12}$	$-2730n^{10}$	$35035n^8$	$-265980n^6$	$1144143n^4$	$-2325050n^2$	1414477		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
P_m	C_1n^{2m}	C_2n^{2m-2}	C_3n^{2m-4}	C_4n^{2m-6}	C_5n^{2m-8}	C_6n^{2m-19}	C_7n^{2m-12}	….	C_m

By tabulating the value of $\frac{\left|\sum_{t=0}^{m} C_{t+1}\right|}{C_1}$, yields new data and it is given as in the Table 16.

Table 16 The terms normalized values for P_m.

P_m	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term	7^{th} term		
P_1	1	1							
P_2	1	$\frac{10}{3}$	$\frac{7}{3}$						
P_3	1	7	$\frac{49}{3}$	$\frac{31}{3}$					
P_4	1	12	$\frac{294}{5}$	124	$\frac{381}{5}$				
P_5	1	$\frac{55}{3}$	154	682	1397	$\frac{2555}{3}$			
P_6	1	26	$\frac{1001}{3}$	$\frac{17732}{7}$	$\frac{54483}{5}$	$\frac{66430}{3}$	$\frac{1414477}{3}$		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
P_m	1	$\frac{C_2}{C_1}$	$\frac{C_3}{C_1}$	$\frac{C_4}{C_1}$	$\frac{C_5}{C_1}$	$\frac{C_6}{C_1}$	$\frac{C_7}{C_1}$	….	$\frac{C_m}{C_1}$

Plotting the P_m curves for some *m* yields,

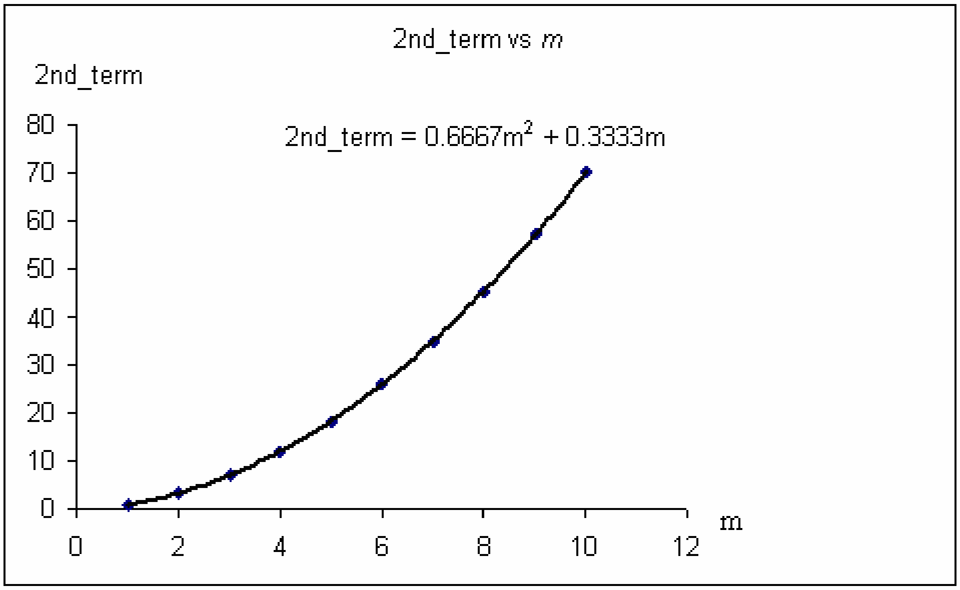

Figure 10 Graph of 2nd term versus *m*.

From this curve we can deduce that

$$2nd\_term = \frac{m}{3}(2m+1) \qquad [194]$$

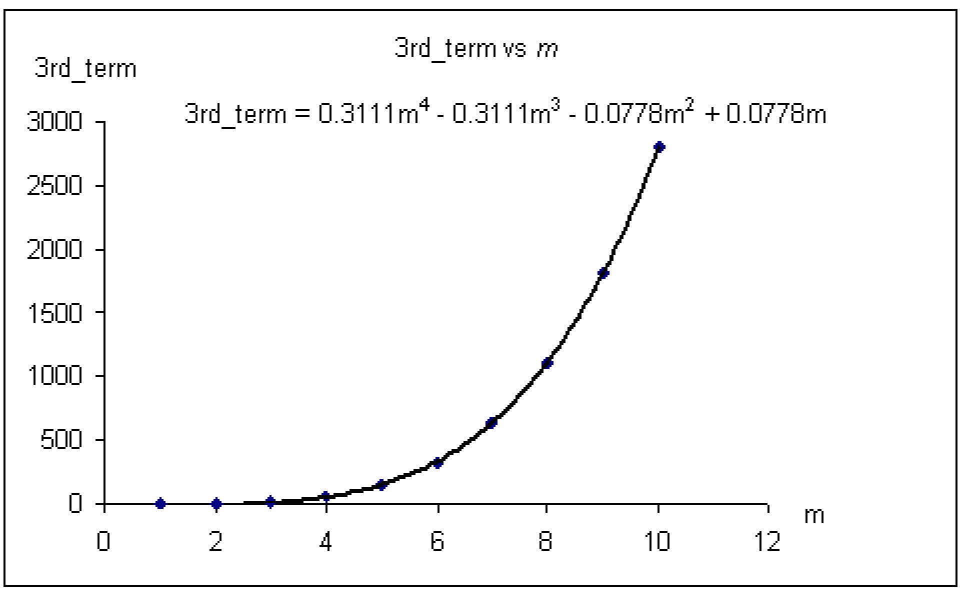

Figure 11 Graph of 3rd term versus *m*.

From this curve we can deduce that

$$3nd\_term = \frac{7m}{3^2 x10}(m-1)(2m+1)(2m-1) \qquad [195]$$

The term can be rewritten as $T_m = \gamma_m \cdot f(m)$

where $f(m)$ is a function of *m* and γ_m is a coefficient which depends on the Bernoulli number and $\left(2^{2m-1} - 1\right)$.

Analyzing γ_m for some terms yields:

Table 17 The values of γ_m and B_m at various values of *m*.

m	γ_m	B_m	$\gamma_m = \zeta(2^{2m-1}-1)B_m$
0	1	-1	$-(2^{-1}-1)\cdot 2B_0$
1	$\frac{1}{3}$	$\frac{1}{6}$	$(2^{1}-1)\cdot 2B_1$
2	$\frac{7}{90}$	$-\frac{1}{30}$	$-(2^{3}-1)\cdot \frac{B_2}{3}$
3	$\frac{31}{1890}$	$\frac{1}{42}$	$(2^{5}-1)\cdot \frac{B_3}{45}$
4	$\frac{381}{113400}$	$-\frac{1}{30}$	$-(2^{7}-1)\cdot \frac{B_4}{1260}$
5	$\frac{2555}{3742200}$	$\frac{5}{66}$	$(2^{9}-1)\cdot \frac{B_5}{56700}$
6	$\frac{1414477}{3^6\cdot 35\cdot 400400}$	$-\frac{691}{2730}$	$-(2^{11}-1)\cdot \frac{B_6}{3742200}$
7	$\frac{860055}{3^7\cdot 35\cdot 400400}$	$\frac{7}{6}$	$(2^{13}-1)\cdot \frac{B_7}{340540200}$
8	$\frac{118518239}{3^7\cdot 35\cdot 400400\cdot 680}$	$-\frac{3617}{510}$	$-(2^{15}-1)\cdot \frac{B_8}{40864824000}$
9	$\frac{5749691557}{3^9\cdot 35\cdot 18088\cdot 400400}$	$\frac{43867}{798}$	$(2^{17}-1)\cdot \frac{B_9}{6252318072000}$
10	$\frac{1922471824497}{3^{10}\cdot 35\cdot 400400\cdot 9948400}$	$-\frac{174611}{330}$	$-(2^{19}-1)\cdot \frac{B_{10}}{1187940433680000}$
11	$\frac{8960213962315}{3^{10}\cdot 35\cdot 400400\cdot 228813200}$	$\frac{854513}{138}$	$(2^{21}-1)\cdot \frac{B_{11}}{274414240180080000}$
12	$\frac{1982765468311237}{3^{10}\cdot 35\cdot 400400\cdot 19752096\cdot 12650}$	$-\frac{236364091}{2730}$	$-(2^{23}-1)\cdot \frac{B_{12}}{7573833028970208000}$

From the Table 17, apparently P_m can be formulated as follows:

$$f(m) = -2\sum_{t=1}^{m}\left[(2t+1)\binom{m}{t} n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(2m-2j+1)}{\prod_{j=0}^{t}(2t-2j+1)}\right] \quad [196]$$

Since 1st term is n^{2m} then,

$$P_m = n^{2m} + T(m) \quad [197]$$

Therefore, P_m is given as follows:

$$P_m = \left[n^{2m} - 2\sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] \right] \qquad [198]$$

Where, B_t is the Bernoulli's number and $m \geq 1$.

When *m*=1

$$P_1 = \left[n^{2(1)} - 2\sum_{t=1}^{1} \left[(2t+1)(2^{2t-1}-1)\binom{1}{t} B_t n^{2(1-t)} \frac{\prod_{j=0}^{t-1}(1+2(1-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] \right]$$

$$= n^{2(1)} - 2(2(1)+1)(2^{2(1)-1}-1)\binom{1}{1} B_1 n^{2(1-1)} \frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)}$$

$$= n^2 - 6\left(\frac{1}{6}\right) \frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)} = n^2 - \frac{(3)}{(3)\cdot(1)} = n^2 - 1 \qquad [199]$$

When *m*=2

$$P_2 = \left[n^{2(2)} - 2\sum_{t=1}^{2} \left[(2t+1)(2^{2t-1}-1)\binom{2}{t} B_t n^{2(2-t)} \frac{\prod_{j=0}^{t-1}(1+2(2-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] \right]$$

$$= \left[\begin{array}{l} n^4 - 2(2(1)+1)(2^{2(1)-1}-1)\binom{2}{1} B_1 n^{2(2-1)} \frac{\prod_{j=0}^{0}(1+2(2-j))}{\prod_{j=0}^{1}(1+2(t-j))} - \\ 2(2(2)+1)(2^{2(2)-1}-1)\binom{2}{2} B_2 n^{2(2-2)} \frac{\prod_{j=0}^{1}(1+2(2-j))}{\prod_{j=0}^{2}(1+2(2-j))} \end{array} \right]$$

$$= \left[n^4 - 12\left(\frac{1}{6}\right) n^2 \frac{\prod_{j=0}^{0}(5-2j)}{\prod_{j=0}^{1}(3-2j)} - 70\left(-\frac{1}{30}\right) n^0 \frac{\prod_{j=0}^{1}(5-2j)}{\prod_{j=0}^{2}(5-2j)} \right]$$

$$= \left[n^4 - 2n^2 \frac{(5)}{(3)(1)} + \frac{7}{3}\frac{(5)(3)}{(5)(3)(1)} \right] = n^4 - \frac{10}{3}n^2 + \frac{7}{3} = \frac{1}{3}\left(3n^4 - 10n^2 + 7\right) \qquad [200]$$

Therefore,

$$\phi_m = \frac{1}{2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}P_m$$

$$= \frac{1}{2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}\left[n^{2m} - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t n^{2(m-t)}\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] \quad [201]$$

4.1 The Derivation of Bernoulli Number for the Generalized Equation for Sums of Power.

It is known that the generalized equation P_m is zero when n=1. Therefore, the coefficients P_m can be used to find Bernoulli's number. Few Bernoulli's numbers calculation can be seen as follows:

Consider,

$$P_m = \left[n^{2m} - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t n^{2(m-t)}\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right]$$

Since $P_m = 0$ when n=1.

$$\left[1^{2m} - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t (1)^{2(m-t)}\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] = 0$$

$$\left[1 - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] = 0 \quad [202]$$

Rewritting the equation [202] yields

$$f(m) = \sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right] - \frac{1}{2} = 0 \text{ for all } m \in N \quad [203]$$

When $m = 1$,

$$f(1) = \sum_{t=1}^{1}\left[(2t+1)(2^{2t-1}-1)\binom{1}{t}B_t \frac{\prod_{j=0}^{t-1}(1+2(1-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right] - \frac{1}{2} = 0$$

$$(2(1)+1)(2^{2(1)-1}-1)\binom{1}{1}B_1\frac{\prod_{j=0}^{1-1}(1+2(1-j))}{\prod_{j=0}^{1}(1+2(1-j))}-\frac{1}{2}=0$$

$$3B_1\frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)}-\frac{1}{2}=0$$

$$3B_1\frac{3}{(3)(1)}-\frac{1}{2}=0$$

$$3B_1-\frac{1}{2}=0$$

$$B_1=\frac{1}{6} \qquad [204]$$

When $m=2$,

$$f(2)=\sum_{t=1}^{2}\left[(2t+1)(2^{2t-1}-1)\binom{2}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(2-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]-\frac{1}{2}=0$$

$$(2(1)+1)(2^{2(1)-1}-1)\binom{2}{1}B_1\frac{\prod_{j=0}^{1-1}(1+2(2-j))}{\prod_{j=0}^{1}(1+2(1-j))}+(2(2)+1)(2^{2(2)-1}-1)\binom{2}{2}B_2\frac{\prod_{j=0}^{2-1}(1+2(2-j))}{\prod_{j=0}^{2}(1+2(2-j))}-\frac{1}{2}=0$$

$$6B_1\frac{\prod_{j=0}^{0}(5-2j)}{\prod_{j=0}^{1}(3-2j)}+35B_2\frac{\prod_{j=0}^{1}(5-2j)}{\prod_{j=0}^{2}(5-2j)}-\frac{1}{2}=0$$

$$6\left(\frac{1}{6}\right)\frac{5}{(3)(1)}+35B_2\frac{(5)(3)}{(5)(3)(1)}-\frac{1}{2}=0$$

$$\frac{5}{3}+35B_2-\frac{1}{2}=0$$

$$B_2=-\frac{1}{30} \qquad [205]$$

The Bernoulli's number for when $m>1$ can be calculated by rewriting the equation [203]. The derivation is given as follows:

Expanding equation [203] yields:

$$f(m) = (2(1)+1)(2^{2(1)-1}-1)\binom{m}{1}B_1\frac{\prod_{j=0}^{0}(1+2(m-j))}{\prod_{j=0}^{1}(1+2((1)-j))} + (2(2)+1)(2^{2(2)-1}-1)\binom{m}{2}B_1\frac{\prod_{j=0}^{1}(1+2(m-j))}{\prod_{j=0}^{2}(1+2((2)-j))}$$

$$+\cdots+(2(m)+1)(2^{2(m)-1}-1)\binom{m}{m}B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2((m)-j))}-\frac{1}{2}=0$$

[206]

Rewritting equation [206] yields:

$$\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]+(2(m)+1)(2^{2(m)-1}-1)\binom{m}{m}B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2((m)-j))}-\frac{1}{2}=0$$

$$(2m+1)(2^{2m-1}-1)B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]$$

$$B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{1}{(2m+1)(2^{2m-1}-1)}\left[\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right]$$

Since,

$$\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{(2m+1)(2m-1)(2m-3)\cdots(3)}{(2m+1)(2m-1)(2m-3)\cdots(3)(1)}=1$$

Therefore,

$$B_m=\frac{1}{(2m+1)(2^{2m-1}-1)}\left[\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] \qquad [207]$$

5.0 Derivation of Sums of Power for Integers using Generalized Equation for Sums of Power.

Setting $x_1 = 1$ and s=1, this equation [15] reduces into classical Faulhaber's Sum of Power for integers.

Power sums for Integers generalize equation is given as follows:

$$\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} i^p = \sum_{j=0}^{m} \left[\phi_m \frac{\left[\sum_{i=1}^{n} i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \qquad [208]$$

Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, equation [208] becomes:

$$\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} i^p = \sum_{j=0}^{m} \left[\phi_j \frac{\left[\frac{n(n+1)}{2}\right]^{p-2j}}{n^{p-(2j+1)}} \right]$$

$$= n\sum_{j=0}^{m} \left[\phi_j \left[\frac{(n+1)}{2}\right]^{p-2j} \right] \qquad [209]$$

Where, $\phi_0 = 1$ and $m = \begin{cases} \frac{p-1}{2} \; for\_odd\_p \\ \frac{p}{2} \; for\_even\_p \end{cases}$

For p=2,

$$\sum_{i=1}^{n} i^2 = n\sum_{j=0}^{1} \left[\phi_j \left[\frac{(n+1)}{2}\right]^{2-2j} \right]$$

$$= n\left[\phi_0 \left(\frac{n+1}{2}\right)^2 + \phi_1 \right]$$

$$= n\left[\left(\frac{n+1}{2}\right)^2 + \left(\frac{n^2-1}{12}\right) \right]$$

$$= \frac{n^2(n+1)^2}{4n} + \frac{n(n^2-1)}{12} = \frac{n(n+1)}{12}\left(3(n+1)+(n-1)\right)$$

$$= \frac{n(n+1)}{12}(4n+2) = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(2n^3+3n^2+n\right)$$

or

$$\sum_{i=1}^{n} i^2 = \frac{\left[\sum_{i=1}^{n} i\right]^2}{n} + \frac{n(n^2-1)}{12} \qquad [210]$$

Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Equation [210] reduces to:

$$\sum_{i=1}^{n} i^2 = \frac{n^2(n+1)^2}{4n} + \frac{n(n^2-1)}{12} = \frac{n(n+1)}{12}\left(3(n+1)+(n-1)\right)$$
$$= \frac{n(n+1)}{12}(4n+2) = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(2n^3+3n^2+n\right) \qquad [211]$$

6.0 The Verification of Sums of Power General Formulation and the Derivation of the Sums of Power Through Binomials Expansion of Elementary Arithmetic Terms.

By taking sums of power of elementary arithmetic terms from x_1 to x_n for even and odd *n* yields:

For even *n*

$$\sum_{i=1}^{n} x_i^p = x_1 + \cdots + x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right)^p + \cdots + \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{s}{2}\right)^p + \cdots$$
$$+ \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right)^p \qquad [212]$$

For odd *n*

$$\sum_{i=1}^{n} x_i^p = x_1 + \cdots + x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right)^p + \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \cdots + \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{s}{2}\right)^p + \cdots$$
$$+ \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right)^p + \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right)^p \qquad [213]$$

By considering Binomial expansion of $(x+y)^p$ which is given as follows:

$$(x+y)^p = x^p + \binom{p}{1} x^{p-1} y + \binom{p}{2} x^{p-2} y^2 + \cdots + y^p \qquad [214]$$

and substituting $x = \frac{\sum_{i=1}^{n} x_i}{n}$ and $y = \frac{(n-q)s}{2}$ into both equations above, where *q* is an odd numbers, yields the:

Even *n*

$$\sum_{i=1}^{n} x_i^p = \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p + \binom{p}{1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-1}\left(-\frac{(n-1)s}{2}\right) + \binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left(-\frac{(n-1)s}{2}\right)^2 + \binom{p}{3}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-3}\left(-\frac{(n-1)s}{2}\right)^3 + \cdots + \left(-\frac{(n-1)s}{2}\right)^p$$
$$+ \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p + \binom{p}{1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-1}\left(-\frac{(n-3)s}{2}\right) + \binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left(-\frac{(n-3)s}{2}\right)^2 + \binom{p}{3}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-3}\left(-\frac{(n-3)s}{2}\right)^3 + \cdots + \left(-\frac{(n-3)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(-\frac{(n-5)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(-\frac{(n-5)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(-\frac{(n-5)s}{2}\right)^3+\cdots+\left(-\frac{(n-5)s}{2}\right)^p$$

$$\vdots \qquad [215]$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(\frac{(n-5)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(\frac{(n-5)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(\frac{(n-5)s}{2}\right)^3+\cdots+\left(\frac{(n-5)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(\frac{(n-3)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(\frac{(n-3)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(\frac{(n-3)s}{2}\right)^3+\cdots+\left(\frac{(n-3)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(\frac{(n-1)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(\frac{(n-1)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(\frac{(n-1)s}{2}\right)^3+\cdots+\left(\frac{(n-1)s}{2}\right)^p$$

Simplifying [215], yields:

$$\sum_{i=1}^{n}x_i^p=n\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left[\left(-\frac{(n-1)s}{2}\right)^2+\left(-\frac{(n-3)s}{2}\right)^2+\cdots+\left(\frac{(n-1)s}{2}\right)^2+\left(\frac{(n-1)s}{2}\right)^2\right]$$

$$+\binom{p}{4}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-4}\left[\left(-\frac{(n-1)s}{2}\right)^4+\left(-\frac{(n-3)s}{2}\right)^4+\cdots+\left(\frac{(n-1)s}{2}\right)^4+\left(\frac{(n-1)s}{2}\right)^4\right]$$

$$+\binom{p}{6}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-6}\left[\left(-\frac{(n-1)s}{2}\right)^6+\left(-\frac{(n-3)s}{2}\right)^6+\cdots+\left(\frac{(n-1)s}{2}\right)^6+\left(\frac{(n-1)s}{2}\right)^6\right]+$$

$$\vdots \qquad [216]$$

$$+\left[\left(-\frac{(n-1)s}{2}\right)^p+\left(-\frac{(n-3)s}{2}\right)^p+\cdots+\left(\frac{(n-1)s}{2}\right)^p+\left(\frac{(n-1)s}{2}\right)^p\right]$$

Odd n

$$\sum_{i=1}^{n}x_i^p=\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(-\frac{(n-1)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(-\frac{(n-1)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(-\frac{(n-1)s}{2}\right)^3+\cdots+\left(-\frac{(n-1)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(-\frac{(n-3)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(-\frac{(n-3)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(-\frac{(n-3)s}{2}\right)^3+\cdots+\left(-\frac{(n-3)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-1}\left(-\frac{(n-5)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(-\frac{(n-5)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-3}\left(-\frac{(n-5)s}{2}\right)^3+\cdots+\left(-\frac{(n-5)s}{2}\right)^p$$

$$\vdots \qquad [217]$$

$$+$$

$$\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p$$

$$+$$

$$\vdots$$

$$+\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-1}\left(\frac{(n-5)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left(\frac{(n-5)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-3}\left(\frac{(n-5)s}{2}\right)^3+\cdots+\left(\frac{(n-5)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-1}\left(\frac{(n-3)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left(\frac{(n-3)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-3}\left(\frac{(n-3)s}{2}\right)^3+\cdots+\left(\frac{(n-3)s}{2}\right)^p$$

$$+\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p+\binom{p}{1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-1}\left(\frac{(n-1)s}{2}\right)+\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left(\frac{(n-1)s}{2}\right)^2+\binom{p}{3}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-3}\left(\frac{(n-1)s}{2}\right)^3+\cdots+\left(\frac{(n-1)s}{2}\right)^p$$

Simplifying both equations, yields:

$$\begin{aligned}\sum_{i=1}^{n} x_i^p = n\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p &+\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\left[\left(-\frac{(n-1)s}{2}\right)^2+\left(-\frac{(n-3)s}{2}\right)^2+\cdots+\left(\frac{(n-1)s}{2}\right)^2+\left(\frac{(n-1)s}{2}\right)^2\right]\\ &+\binom{p}{4}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-4}\left[\left(-\frac{(n-1)s}{2}\right)^4+\left(-\frac{(n-3)s}{2}\right)^4+\cdots+\left(\frac{(n-1)s}{2}\right)^4+\left(\frac{(n-1)s}{2}\right)^4\right]\\ &+\binom{p}{6}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-6}\left[\left(-\frac{(n-1)s}{2}\right)^6+\left(-\frac{(n-3)s}{2}\right)^6+\cdots+\left(\frac{(n-1)s}{2}\right)^6+\left(\frac{(n-1)s}{2}\right)^6\right]+\\ &\vdots\\ &+\left[\left(-\frac{(n-1)s}{2}\right)^p+\left(-\frac{(n-3)s}{2}\right)^p+\cdots+\left(\frac{(n-1)s}{2}\right)^p+\left(\frac{(n-1)s}{2}\right)^p\right]\end{aligned} \quad [218]$$

For even *p* simplifying [215], yields:

$$\sum_{i=1}^{n} x_i^p = n\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p+2\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^2 s^2}{2}\right]+2\binom{p}{4}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-4}\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^4 s^4}{2^3}\right]+\cdots+2\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^p s^p}{2^{p-1}}\right] \quad [219]$$

For odd *p* simplifying [215], yields:

$$\begin{aligned}\sum_{i=1}^{n} x_i^p = n\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^p &+2\binom{p}{2}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-2}\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^2 s^2}{2}\right]+2\binom{p}{4}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{p-4}\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^4 s^4}{2^3}\right]+\cdots\\ &+2\binom{p}{p-1}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^{p-2} s^{p-1}}{2^{p-2}}\right]\end{aligned} \quad [220]$$

It is found through Mathcad Symbolic Operation that,

$$2\sum_{j=1}^{\frac{n}{2}}\left[\frac{(n-(2j-1))^m}{2^p}\right]=2\sum_{j=1}^{\frac{n-1}{2}} j^m=\frac{1}{2^{2m}}\frac{1}{(2m+1)}\left[n^{2m}-2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t n^{2(m-t)}\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] \quad [221]$$

Thus,

For odd p

$$\sum_{i=1}^{n}x_i^p=n\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+2s^2\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\left(\sum_{j=1}^{\frac{n-1}{2}}j^2\right)+2s^4\binom{p}{4}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-4}\left(\sum_{j=1}^{\frac{n-1}{2}}j^4\right)+\cdots+2s^{p-1}\binom{p}{p-1}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\left(\sum_{j=1}^{\frac{n-1}{2}}j^{p-1}\right) \quad [222]$$

And For odd p

$$\sum_{i=1}^{n}x_i^p=n\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^p+2s^2\binom{p}{2}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-2}\sum_{j=1}^{\frac{n-1}{2}}j^2+2s^2\binom{p}{4}\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^{p-4}\sum_{j=1}^{\frac{n-1}{2}}j^2+\cdots+2s^p\sum_{j=1}^{\frac{n-1}{2}}j^p \quad [223]$$

Simplifying both equations [222] and [22] yields

$$\sum_{i=1}^{n}x_i^p=\sum_{m=0}^{u}\left[\phi_m s^{2m}\frac{\left[\sum_{i=1}^{n}x_i\right]^{p-2m}}{n^{p-(2m+1)}}\right] \quad [224]$$

where $u=\begin{cases}\frac{p-1}{2} & for\_odd\_p\\ \frac{p}{2} & for\_even\_p\end{cases}$ [225]

Which is identical to equation [15].

Conclusion.

The general equation for Sum of Power presented in this paper can be extended on many other uses due to its simplicity and elegant formulation. This formula includes Faulhaber's sum of power and most of other formulae derived for sum of power because of its expression in form of the most basic elementary symmetric function of an arithmetic progression. Since integer is part of arithmetic progression, it offers new form of sum of power as an alternative to Faulhaber formulation. Apart from that, this generalized equation can be extended to real number powers which are useful for extending the sums of power to Riemann's Zeta function and numerical analysis of summation for rational and irrational power. When n is set to 2, the generalized equation reduces to Fermat's Last Theorem and expressing Fermat's Last Theorem in a polynomial form of symmetric function. Thus offering new insight in the Fermat's Last Theorem studies. The rest of papers related to this research can be found at the references [16]-[20].

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