A Discussion on Gauge Symmetry and Charge Conservation

Gabriel Di Lemos Santiago Lima[1,*]

[1] *Centro Brasileiro de Pesquisas Físicas,*

Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil

Abstract

This work is intended to rediscuss the relation between gauge symmetry and current conservation.

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*Electronic address: `glslima@cbpf.br`

I. INTRODUCTION

In previous works, it was suggested that effective actions of anomalous models could be mapped into gauge invariant ones by some mathematical manipulations over the functional integral [1], [2]. On the other hand, similar developments may be used to map the non-gauge Proca model [3] into the gauge invariant Stueckelberg one [4], which is shown to be unitary and renormalizable [5]∼[7]. To be more precise, two formulations may be raised in order to achieve a gauge invariant effective action: the *standard* formulation, first developed by Fadeev and Shatashvilli [8] and, then, derived from such mapping by Harada and Tsutsui [2], and a kind of generalization of the Stueckelberg trick, that may be also obtained by the Harada and Tsutsui procedure, called the *enhanced* formulation [9].

However, for the simplest case of the Jackiw-Rajaraman model [10], it was particularly shown that the standard formulation does not coincide physically with the original one unless we restrict the gauge fields by a constraint that cancels the anomaly out [11]. On the other hand, it can be shown that if we impose the equations of motions of the abelian gauge field, and accept that the anomaly may cancel out as a subsidiary condition, then the *enhanced* version of the Harada and Tsutsui procedure may be equivalent to the original one, since a simple gauge choice, which means to impose the equation that cancels out the anomaly, reduces the enhanced formulation to the original one. This naturally brings a kind of equivalence between gauge and non-gauge models, since one is reducible to the other by a gauge choice that do not change the physical results [12].

At this point, one question deserves special attention: if current may be conserved by imposing the equations of motion of the gauge field as a kind of subsidiary condition, and a gauge theory may be equivalent to a non-gauge one, what is the relation between gauge symmetry and current conservation?

This work is intended to rediscuss this quite important question. In this sense, section II is intended to establish the classical relation between Noether symmetry and current conservation. In section III, we bring this discussion to the quantum environment, where the anomaly case shows up quite naturally. We discuss the alternative point-of-view in which the anomaly may be canceled out as a subsidiary condition in section IV. In section V, it is shown that the enhanced Stueckelberg formalism may be used to verify the consistence of anomalous models. Finally, the conclusion is presented in section VI.

II. NOETHER SYMMETRY AND CLASSICAL CURRENT CONSERVATION

Let us consider the classical theory defined by the action

$$I[\psi,\bar{\psi},A_\mu] = I_M[\psi,\bar{\psi},A_\mu] + I_G[A_\mu] \tag{1}$$

where $I_M[\psi,\bar{\psi},A_\mu]$ is the matter action defined by fermionic fields ψ and $\bar{\psi}$ that carries any $SU\left(N\right)$ representation, and by gauge fields defined by

$$A_\mu = A_\mu^a T_a, \tag{2}$$

where T_a are the $SU\left(N\right)$ generators acting on the matter fields and satisfying

$$[T_a,T_b] = if_{abc}T_c, \qquad \text{tr}\,(T_aT_b) = -\frac{1}{2}\delta_{ab}. \tag{3}$$

We *define* the *local* gauge transformations as

$$A_\mu^g = gA_\mu g^{-1} - \frac{i}{e}\left(\partial_\mu g\right)g^{-1}, \tag{4}$$

$$\psi^g = g\psi, \tag{5}$$

$$\bar{\psi}^g = \bar{\psi}g^{-1}, \tag{6}$$

with g being a local and arbitrary $SU(N)$ element, defined by

$$g = \exp(i\theta\left(x\right)), \qquad \theta(x) \equiv \theta^a\left(x\right)T_a. \tag{7}$$

The entire action (1) as well as each of its part is suposed to be invariant under the referred transformations, that is

$$I[\psi^g,\bar{\psi}^g,A_\mu^g] = I[\psi,\bar{\psi},A_\mu], \tag{8}$$

$$I_M[\psi^g,\bar{\psi}^g,A_\mu^g] = I_M[\psi,\bar{\psi},A_\mu], \tag{9}$$

$$I_G[A_\mu^g] = I_G[A_\mu]. \tag{10}$$

We shall analyze each part of the action separetly, starting from the matter action. The following development was inspired by ref. [13] brought to this context. Since the matter action I_M is invariant by the *local* gauge transformations above (9), it is also invariant under *global* gauge transformations

$$\begin{aligned} \psi &\to \exp(i\theta)\,\psi, \\ \bar{\psi} &\to \bar{\psi}\exp(-i\theta), \\ A_\mu &\to \exp(i\theta)\,A_\mu\exp(-i\theta), \end{aligned} \tag{11}$$

being $\theta \equiv \theta^a T_a$ an arbitrary linear conbination of the gauge group generators (3) with $\partial_\mu\theta^a = 0$. If we change the transformation law (11) from global to a *modified* local one

$$\begin{aligned} \psi &\to \exp(i\theta(x))\,\psi, \\ \bar{\psi} &\to \bar{\psi}\exp(-i\theta(x)), \\ A_\mu &\to \exp(i\theta(x))\,A_\mu\exp(-i\theta(x)), \end{aligned} \tag{12}$$

which, in the infinitesimal form, reduces to

$$\begin{aligned} \psi(x) &\to \psi(x) + i\theta(x)\psi(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x) - i\bar{\psi}(x)\theta(x), \\ A_\mu &\to A_\mu - i\left[A_\mu, \theta(x)\right], \end{aligned} \tag{13}$$

$$\begin{aligned} A_\mu^a T_a \to A_\mu^a T_a - iA_\mu^b\theta^c(x)\left[T_b, T_c\right] &= A_\mu^a T_a + A_\mu^b\theta^c(x) f_{bca} T_a, \\ \Longrightarrow A_\mu^a &\to A_\mu^a + f_{abc}A_\mu^b\theta^c(x), \end{aligned} \tag{14}$$

we will obtain the local gauge transformations (9) *without* the inhomogeneous part of the gauge field and, therefore, we will have no symmetry. On the other hand, if $\theta(x)$ is constant, we get back to the global gauge transformations (11). Therefore, the noninvariance of the action by (12) must only depend on $\partial_\mu\theta^a(x)$. Thus, if we perform the infinitesimal transformations (13), we shall obtain

$$\begin{aligned} & I_M[\psi^g, \bar{\psi}^g, gA_\mu g^{-1}] \\ &= I_M[\psi, \bar{\psi}, A_\mu] + \int dx\theta^a(x)\left(\frac{\delta I_M}{\delta\psi(x)} iT_a\psi(x) - i\bar{\psi}(x)T_a\frac{\delta I_M}{\delta\bar{\psi}(x)} + f_{bac}A_\mu^b\frac{\delta I_M}{\delta A_\mu^c(x)}\right) \\ &\equiv I_M[\psi, \bar{\psi}, A_\mu] + \int dx\partial_\mu\theta^a(x)J_a^\mu(x); \end{aligned} \tag{15}$$

$$\begin{aligned} &\Rightarrow \int dx\theta^a(x)\left(\frac{\delta I_M}{\delta\psi(x)} iT_a\psi(x) - i\bar{\psi}(x)T_a\frac{\delta I_M}{\delta\bar{\psi}(x)} + f_{bac}A_\mu^b\frac{\delta I_M}{\delta A_\mu^c(x)}\right) \\ &= -\int dx\theta^a(x)\partial_\mu J_a^\mu(x). \end{aligned} \tag{16}$$

The infinitesimal gauge parameter $\theta^a(x)$ being arbitrary, we conclude that

$$\int dx\theta^a(x)\left(\frac{\delta I_M}{\delta\psi(x)}iT_a\psi(x) - i\bar{\psi}(x)T_a\frac{\delta I_M}{\delta\bar{\psi}(x)}\right) \tag{17}$$
$$= -\int dx\theta^a(x)\left\{\partial_\mu J_a^\mu(x) + f_{bac}A_\mu^b\frac{\delta I_M}{\delta A_\mu^c(x)}\right\}.$$

If we use the equations of motion of the matter field

$$\frac{\delta I}{\delta\psi(x)} = \frac{\delta I_M}{\delta\psi(x)} = 0, \tag{18}$$
$$\frac{\delta I}{\delta\bar{\psi}(x)} = \frac{\delta I_M}{\delta\bar{\psi}(x)} = 0,$$

then, we just find

$$\partial_\mu J_a^\mu(x) + f_{cab}A_\mu^c\frac{\delta I_M}{\delta A_\mu^b(x)} = 0. \tag{19}$$

One may notice that this result is independent of local gauge invariance of any part of the action. Indeed, this was achieved only by global invariance and the equations of motion of the matter fields. In the abelian case, we see that we have a conserved current

$$\partial_\mu J^\mu(x) = 0 \tag{20}$$

with *no mention* to local gauge symmetry.

Now, we explore the local gauge invariance of the matter action, that is,

$$I_M[\psi^g, \bar{\psi}^g, A_\mu] = I_M[\psi, \bar{\psi}, A^{g^{-1}}] = I_M[\psi, \bar{\psi}, g^{-1}A_\mu g - \frac{i}{e}\left(\partial_\mu g^{-1}\right)g]. \tag{21}$$

if, instead of A we use

$$A'_\mu = gA_\mu g^{-1}, \tag{22}$$

in the matter action, then we will just obtain

$$I_M[\psi^g, \bar{\psi}^g, A'_\mu] = I_M[\psi^g, \bar{\psi}^g, gA_\mu g^{-1}] \tag{23}$$
$$= I_M[\psi, \bar{\psi}, \left(gA_\mu g^{-1}\right)^{g^{-1}}]$$
$$= I_M[\psi, \bar{\psi}, A_\mu - \frac{i}{e}\left(\partial_\mu g^{-1}\right)g].$$

In the infinitesimal form, $(gA_\mu g^{-1})^{g^{-1}}$ reduces to

$$\left(gA_\mu g^{-1}\right)^{g^{-1}} = A_\mu + \frac{i}{e}\left[\partial_\mu\left(1 - i\theta\right)\right]\left(1 + i\theta\right) = A_\mu - \frac{1}{e}\partial_\mu\theta. \tag{24}$$

Therefore, for θ *infinitesimal*,

$$\begin{aligned} I_M[\psi^g, \bar{\psi}^g, gA_\mu g^{-1}] &= I_M[\psi, \bar{\psi}, A_\mu^a - \frac{1}{e}\partial_\mu\theta^a] \\ &= I_M[\psi, \bar{\psi}, A_\mu] - \frac{1}{e}\int dx \partial_\mu\theta^a(x)\frac{\delta I_M}{\delta A_\mu^a(x)}. \end{aligned} \tag{25}$$

Comparing (25) with (15), we just find

$$\int dx \partial_\mu\theta^a(x) J_a^\mu(x) = \int dx \partial_\mu\theta^a(x)\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu^a(x)}\right), \tag{26}$$

and, since $\theta^a(x)$ is arbitrary and we did not make use of any boundary condition at infinite, we may ensure that

$$J_a^\mu(x) = -\frac{1}{e}\frac{\delta I_M}{\delta A_\mu^a(x)}. \tag{27}$$

Thus, we see that the local gauge symmetry of the matter action shows a coupling of the gauge field with the matter action in such a way that the current obays the expression (27) above. Thus, using (27) in (19), we arrive at

$$D_{\mu b}^a J_b^\mu(x) = 0, \tag{28}$$

where

$$D_{\mu b}^a \equiv \delta_b^a\partial_\mu + ef_{cab}A_\mu^c. \tag{29}$$

Therefore, we see that, while in the abelian case local gauge symmetry does not interfere whith the current conservation law, in the Yang-Mills one the symmetry of the *matter action* is important in the sense that current may be obtained from (27), which ensures that current is covariantly conserved. On the other hand, it is clear that gauge symmetry of any other part of the action is irrelevant.

We now turn our attention to a pure gauge field term, which may be exemplied by $I_G[A_\mu]$. Performing an infinitesimal gauge transformation on $I_G[A_\mu]$ and making use of its gauge invariance we arrive at

$$\begin{aligned} I_G\left[A_\mu + \frac{1}{e}D_\mu\theta\right] &= I_G\left[A_\mu\right] + \int dx \frac{\delta I_G[A]}{\delta A_\mu^a(x)}\frac{1}{e}D_{\mu b}^a\theta^b(x) \\ &= I_G\left[A_\mu\right] - \int dx \theta^a(x) D_{\mu b}^a\left(\frac{1}{e}\frac{\delta I_G[A]}{\delta A_\mu^b(x)}\right), \\ \Rightarrow D_{\mu b}^a&\left(-\frac{1}{e}\frac{\delta I_G[A]}{\delta A_\mu^b(x)}\right) = 0. \end{aligned} \tag{30}$$

which is nothing but a *Noether* identity, as can be exemplified by the kinect term of the abelian vector model $I_G\left[A\right] = -\frac{1}{4}\int d^n x F^{\mu\nu}F_{\mu\nu}$.

Thus, current conservation in the classical abelian case is achieved only by global gauge invariance *and* the classical equations of motion of the matter fields. In the non-abelian case, on the other hand, it is necessary that the matter action possesses local gauge symmetry in order to arrive at the covariantly conserved Yang-Mills current (28).

On the other hand, if instead of $I_G\left[A\right]$ we have a non-gauge action, that is, if

$$I[\psi,\bar{\psi},A_\mu] = I_M[\psi,\bar{\psi},A_\mu] + I_A[A_\mu], \tag{31}$$

being $I_A[A^g_\mu] \neq I_A[A_\mu]$, then we will not have the Noether identity (30) above and, thus, $D^a_{\mu b}\left(-\frac{1}{e}\frac{\delta I_A[A]}{\delta A^b_\mu(x)}\right) \neq 0$, but the current is still conserved by gauge invariance of the matter action, thus, if we get the equation of motion of the gauge field, we will arrive at

$$\frac{\delta I}{\delta A_\mu} = \frac{\delta I_M}{\delta A_\mu} + \frac{\delta I_A}{\delta A_\mu} = 0. \tag{32}$$

Taking the covariant divergence of (32) and using the fact that current is conserved, we obtain

$$D^a_{\mu b}\left(-\frac{1}{e}\frac{\delta I_A[A]}{\delta A^b_\mu(x)}\right) = 0. \tag{33}$$

This is the generalization of the classical subsidiary condition that appears in the Proca model. As it was seen, this is not an identity since $I_A[A]$ is not gauge invariant. This means that eq. (33) imposes restrictions over the gauge field. One must check, however, the consistence of such constraints with the specific theory under consideration.

III. NOETHER SYMMETRY AND QUANTUM CURRENT CONSERVATION

We now consider the case where the fermion fields are quantized. This is achieved by functional integration over the fermions, defining the effective action

$$\exp\left(iW[A_\mu]\right) \equiv \int d\psi d\bar{\psi} \exp\left(iI[\psi,\bar{\psi},A_\mu]\right). \tag{34}$$

Now, we follow analogue procedure as done in previous section, performing an infinitesimal transformation only over the *matter* action and changing the global transformation law as in (13). Then, we will obtain a modified transformed action which will differ from the original one by $\partial_\mu \theta^a(x)$, since it is invariant for $\partial_\mu \theta^a = 0$. Using this in (34) and expanding it to the first order, we will have

$$\Delta \{\exp(iW)\} = i \int d^n x \theta^a(x) \int d\psi d\overline{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) \left(\frac{\delta I}{\delta \psi(x)} i T_a \psi(x) \right.$$
$$\left. - i\overline{\psi}(x) T_a \frac{\delta I}{\delta \overline{\psi}(x)} + f_{bac} A^b_\mu \frac{\delta I_M}{\delta A^c_\mu(x)} \right)$$
$$= -i \int d^n x \theta^a(x) \int d\psi d\overline{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) \partial_\mu J^\mu_a. \tag{35}$$

At this point, just as in classical case, we may notice that local gauge invariance of the matter action is important in non-abelian case, in the sense that it allow us to use the identification (27) to arrive at

$$\int d\psi d\overline{\psi} D^a_{\mu b} J^\mu_b \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right)$$
$$= \int d\psi d\overline{\psi} \left(i\overline{\psi}(x) T_a \frac{\delta I_M}{\delta \overline{\psi}(x)} - \frac{\delta I_M}{\delta \psi(x)} i T_a \psi(x) \right) \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right). \tag{36}$$

Now comes the subtle difference between the classical and quantum case. If the quantum fermionic measure is invariant under *local* gauge transformations, that is, if

$$d\psi^\theta d\overline{\psi}^\theta = d\psi d\overline{\psi}, \tag{37}$$

then, to *any* functional $F\left[\psi, \overline{\psi}, A_\mu\right]$, we will have

$$\int d\psi d\overline{\psi} F\left(\psi, \overline{\psi}, A_\mu\right) = \int d\psi^\theta d\overline{\psi}^\theta F\left(\psi^\theta, \overline{\psi}^\theta, A_\mu\right)$$
$$= \int d\psi d\overline{\psi} F\left(\psi + \delta\psi, \overline{\psi} + \delta\overline{\psi}, A_\mu\right)$$
$$= \int d\psi d\overline{\psi} F\left(\psi, \overline{\psi}, A_\mu\right) + \int d\psi d\overline{\psi} dA_\mu \left(\frac{\delta F}{\delta \psi} \delta\psi + \delta\overline{\psi} \frac{\delta F}{\delta \overline{\psi}} \right), \tag{38}$$

$$\Rightarrow \int d\psi d\overline{\psi} \left(\frac{\delta F}{\delta \psi} \delta\psi + \delta\overline{\psi} \frac{\delta F}{\delta \overline{\psi}} \right) = 0. \tag{39}$$

If we take

$$F\left(\psi, \overline{\psi}, A_\mu\right) \equiv \exp\left(iI_M[\psi, \overline{\psi}, A_\mu]\right) \text{ and } \delta\psi = i\theta(x)\psi(x); \delta\overline{\psi} = -i\overline{\psi}(x)\theta(x), \tag{40}$$

then, since $\theta(x)$ is arbitrary, we find

$$\int d\psi d\overline{\psi} \left(\frac{\delta I_M}{\delta \psi} i T_a \psi - i\overline{\psi} T_a \frac{\delta I_M}{\delta \overline{\psi}} \right) \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right)$$
$$\equiv - \int d\psi d\overline{\psi} D^a_{\mu b} J^\mu_b \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) = 0, \tag{41}$$

which is the quantum version of current conservation law.

We may notice that, at this point of discussion, the *local* gauge invariance of $d\psi d\overline{\psi}$ was important to achieve (41). On the other hand, there is no distinct discussion about the importance of local gauge invariance of the action. Indeed, just as in classical case, it seems important only in the non-abelian case, due to the coupling (27), to arrive at the current covariant derivative in (19).

If, instead of (37), we have

$$d\psi^{\theta}d\overline{\psi}^{\theta}=\exp\left(\alpha_{1}\left[A_{\mu},\theta\right]\right)d\psi d\overline{\psi}, \tag{42}$$

which, in infinitesimal form turns to be

$$d\psi^{\theta}d\overline{\psi}^{\theta}=\left(1+\left.\frac{\delta\alpha_{1}}{\delta\theta}\right|_{\theta=0}\theta(x)\right)d\psi d\overline{\psi}, \tag{43}$$

then, performing the same development as in (38), but using (43) we arrive at

$$-\int d\psi d\overline{\psi}\left(\frac{\delta F}{\delta\psi}\delta\psi+\delta\overline{\psi}\frac{\delta F}{\delta\overline{\psi}}\right)=\int d\psi d\overline{\psi}\left.\frac{\delta\alpha_{1}}{\delta\theta}\right|_{\theta=0}F\left(\psi,\overline{\psi},A_{\mu}\right).$$

Identifying our generic functional with (40), we find

$$\frac{\int d\psi d\overline{\psi}D_{\mu b}^{a}J_{b}^{\mu}\exp\left(iI[\psi,\overline{\psi},A_{\mu}]\right)}{\int d\psi d\overline{\psi}\exp\left(iI[\psi,\overline{\psi},A_{\mu}]\right)}=\left.\frac{\delta\alpha_{1}\left[A,\theta\right]}{\delta\theta^{a}(x)}\right|_{\theta=0}. \tag{44}$$

We, thus, *define* an anomalous model as the one whose normalized covariant current divergence, after integrated out all other fields besides the gauge one, is not *identically* null, and its anomaly as being

$$\mathcal{A}_{a}\left[A\right]\equiv\frac{\int d\varphi d\psi d\overline{\psi}D_{\mu b}^{a}J_{b}^{\mu}\exp\left(iI[\psi,\overline{\psi},A_{\mu},\varphi]\right)}{\int d\varphi d\psi d\overline{\psi}\exp\left(iI[\psi,\overline{\psi},A_{\mu},\varphi]\right)}, \tag{45}$$

and we see that, in the case where we have a non-trivial fermionic Jacobian, the theory exhibits an anomaly that may potentially break current conservation at quantum level, given by

$$\mathcal{A}_{a}\left[A\right]=\left.\frac{\delta\alpha_{1}\left[A,\theta\right]}{\delta\theta^{a}(x)}\right|_{\theta=0}. \tag{46}$$

It is easy to notice that such appearance of an anomaly in (44) may be *particularly* explained by an effective action gauge invariance breakdown of a classically gauge invariant model. To see this, we write

$$
\begin{aligned}
\exp\left(iW[A_\mu]\right) &\equiv \int d\psi d\bar{\psi} \exp\left(iI[\psi,\bar{\psi},A_\mu]\right) \\
&= \int d\psi^{-\theta} d\bar{\psi}^{-\theta} \exp\left(iI[\psi^{-\theta},\bar{\psi}^{-\theta},A_\mu]\right) \\
&= \int d\psi d\bar{\psi} \exp\left(iI[\psi,\bar{\psi},A^\theta_\mu] - i\alpha_1\left[A,\theta\right]\right) \qquad (47)\\
&= \exp\left(iW\left[A^\theta\right] - i\alpha_1\left[A,\theta\right]\right) \qquad (48)
\end{aligned}
$$

which leads to the Wess-Zumino action [14]

$$\alpha_1\left[A,\theta\right] = W\left[A^\theta\right] - W[A_\mu]. \qquad (49)$$

Using $\theta(x)$ infinitesimal and expanding the exponential of the gauge transformed effective action to the first order in both, left and right-hand sides, it is straightforward to find

$$\int dx D^a_{\mu b}\theta^b(x)\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A^a_\mu}\exp\left(iW\left[A\right]\right) = \int dx D^a_{\mu b}\theta^b \int d\psi d\bar{\psi}\frac{1}{e}\frac{\delta I}{\delta A^a_\mu}\exp\left(iI[\psi,\bar{\psi},A_\mu]\right) \qquad (50)$$

$$\Rightarrow D^a_{\mu b}\left(-\frac{1}{e}\frac{W\left[A\right]}{\delta A^b_\mu}\right)\exp\left(iW\left[A\right]\right) = \int d\psi d\bar{\psi} D^a_{\mu b}\left(-\frac{1}{e}\frac{\delta I}{\delta A^b_\mu}\right)\exp\left(iI[\psi,\bar{\psi},A_\mu]\right). \qquad (51)$$

Since we are dealing with a gauge invariant starting action such as (1), then $D^a_{\mu b}\left(-\frac{1}{e}\frac{\delta I_G[A]}{\delta A^b_\mu}\right) \equiv 0$, thus

$$D^a_{\mu b}\left(-\frac{1}{e}\frac{W\left[A\right]}{\delta A^b_\mu}\right)\exp\left(iW\left[A\right]\right) = \int d\psi d\bar{\psi} D^a_{\mu b}\left(-\frac{1}{e}\frac{\delta I_M}{\delta A^b_\mu}\right)\exp\left(iI[\psi,\bar{\psi},A_\mu]\right). \qquad (52)$$

As the matter action is gauge invariant, we may identify the current with eq. (27) and, therefore, from (52), we may also write the anomaly as

$$\mathcal{A}_a\left[A\right] = D^a_{\mu b}\left(-\frac{1}{e}\frac{W\left[A\right]}{\delta A^b_\mu}\right). \qquad (53)$$

We shall recognize the necessity of using the gauge invariance of the matter action to write the anomaly as in (53), contrary to (44), even in the abelian case.

IV. QUANTUM NOETHER SYMMETRY AND SUBSIDIARY CONDITIONS

Now, we turn our attention to the full quantum theory, *i. e.*, the one defined after integrated the fermions *and* the gauge field. We start from the vacuum functional of a non-invariant model, however anomaly-free, described by (31)

$$
\begin{aligned}
Z &\equiv \int dA_\mu d\psi d\bar{\psi} \exp \left(iI_M[\psi, \bar{\psi}, A_\mu] + iI_A\,[A] \right) \\
&= \int dA_\mu \exp \left(iW_M\,[A] + iI_A\,[A] \right)
\end{aligned}
\tag{54}
$$

with its gauge symmetry broken by the free bosonic term $I_A\,[A]$.

Since the theory is not anomalous and the matter action is gauge invariant, then the integrated matter action is also gauge invariant, which leads us to conclude that

$$
D^a_{\mu b} \left(-\frac{1}{e} \frac{W_M\,[A]}{\delta A^b_\mu} \right) = 0, \tag{55}
$$

which ensures that current is covariantly conserved. Performing an infinitesimal change of variables on the gauge field $A_\mu \rightarrow A^\theta_\mu$ in (54) and using gauge invariance of the bosonic measure $dA^\theta_\mu = dA_\mu$, it is straightforward to find that

$$
\left\langle D^a_{\mu b} \left(-\frac{1}{e} \frac{I_A\,[A]}{\delta A^b_\mu} \right) \right\rangle = 0, \tag{56}
$$

which seems to be the quantum version of the subsidiary condition (33).

If, on the other hand, we keep the boson field being classical, we may use the variational principle on the effective action and take the covariant divergence of the equation of motion. Then, we obviously arrive at

$$
D^a_{\mu b} \left(-\frac{1}{e} \frac{I_A\,[A]}{\delta A^b_\mu} \right) = 0, \tag{57}
$$

as a subsidiary condition, just as in (33).

We now turn to the anomalous case; we saw that the non-invariance of the fermionic measure leads us to an effective action which is not gauge invariant. On the other hand, if we bring the above point-of-view to this context, and allow ourselves to use the gauge invariance of the bosonic measure, then we will also find

$$
\langle \mathcal{A}_a\,[A] \rangle = \left\langle D^a_{\mu b} \left(-\frac{1}{e} \frac{W\,[A]}{\delta A^b_\mu} \right) \right\rangle = 0, \tag{58}
$$

or, if we consider the gauge field as being classical, get its equation of motion from the effective action and take its covariant divergence, we arrive at

$$
D^a_{\mu b} \left(-\frac{1}{e} \frac{W\,[A]}{\delta A^b_\mu} \right) = 0 \tag{59}
$$

which turns to cancel the anomaly, in other words, to get null current divergence as a subsidiary condition.

The subtle difference between the two cases mentioned above relies in the fact that in the first one we obtain current conservation from the non-anomalous matter action, use gauge invariance of dA_μ and get the subsidiary condition as its consequence, while in the second one we use the gauge invariance of dA_μ to achieve the nullity of anomaly as the subsidiary condition.

From the point-of-view presented above, thus, we obtain two distinct ways to achieve current conservation at quantum level: by *global* gauge invariance of the matter action *and local* gauge invariance of the fermionic measure $d\psi d\overline{\psi}$ or, alternatively, by *local* gauge invariance of the classical matter action *and local* gauge invariance of the bosonic measure dA_μ. We notice that, unlike in classical case, local gauge invariance is of greater importance to the current conservation law, since it remains necessary even in the abelian case. However, its importance relies only in the fermionic measure or, alternatively, in the classical matter action and bosonic measure. It is not necessary a stronger condition which is that the entire effective action be gauge invariance, since such breakdown, as it was seen, may merely represent restrictions over the gauge fields. On the other hand, it remains to be verified the consistence of models subjected to such kind of restrictions.

V. GAUGE INVARIANT ENHANCED FORMULATION AND CONSISTENCE OF ANOMALOUS GAUGE MODELS

An interesting way to check the consistence of non-invariant gauge models is to notice if the theory under consideration may be thought as a gauge invariant one with the subsidiary condition being provided by a particular gauge choice. Indeed, this may be the case of the Proca model and the anomalous chiral Schwinger model. As can be shown, an inclusion of the Stueckelberg scalar in both models turns them to be gauge invariant.

Consider, for instance, the Proca model

$$I_P\left[A\right] \equiv \int d^n x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu\right). \tag{60}$$

If we use the Stueckelberg trick by doing

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu B, \tag{61}$$

we arrive at the Stueckelberg action

$$I_{Stueck}\left[A,B\right] = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x \left(A^\mu + \frac{1}{e}\partial^\mu B\right)\left(A_\mu + \frac{1}{e}\partial_\mu B\right) \tag{62}$$

which *is* gauge invariant. We see that the Stueckelberg model becomes the Proca one in the gauge choice where B is set constant.

On the other hand, if we get the equation of motion of the original Proca model (60), we shall have

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{63}$$

Taking the divergence of (63), we obtain

$$\partial_\mu A^\mu = 0 \tag{64}$$

as subsidiary condition.

Now, if we use the Stueckelberg's alternative and perform the functional integration over the Stueckelberg field at (62), we shall arrive at

$$W_{Stueck}\left[A\right] = \int d^nx \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu\right\} \tag{65}$$

which reduces to Proca by choosing the Lorentz gauge $\partial_\mu A^\mu = 0$ that just turns to be the subsidiary condition (64).

Let us now turn to the example of the anomalous chiral Schwinger model. Its effective action is given by

$$W\left[A\right] = \int d^2x \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2}{8\pi}A_\mu \left[a g^{\mu\nu} - \left(g^{\mu\alpha} + \epsilon^{\mu\alpha}\right)\frac{\partial_\alpha\partial_\beta}{\Box}\left(g^{\beta\nu} - \epsilon^{\beta\nu}\right)\right]A_\nu\right\}. \tag{66}$$

Its equation of motion gives us

$$\partial_\mu F^{\mu\nu} + \frac{e^2}{4\pi}\frac{a^2}{\left(a-1\right)}\left(\eta^{\mu\nu} - \frac{\partial^\nu\partial^\mu}{\Box}\right)A_\mu = 0,$$

with

$$\mathcal{A} = \left(a-1\right)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu = 0$$

as its subsidiary condition. We can use the same trick (61), including the Stueckelberg scalar, performing functional integration over it and we arrive at

$$W_{eff}\left[A\right] = \int d^2x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\frac{e^2}{4\pi}\frac{a^2}{(a-1)}A_\mu \left[g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box} \right] A_\nu \right\}. \tag{67}$$

We see that, analogue to Proca, this model reduces to the anomalous one (66) at the particular gauge choice where $(a-1)\,\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu = 0$.

These two examples bring up a useful guide to turn non-gauge models into gauge equivalent ones by the addition of the Stueckelberg field, and may be a road to the renormalizability of the so-called anomalous gauge models.

VI. CONCLUSION

This work was intended to raise some considerations to the old Noether idea about the relation between gauge symmetry and current conservation. It is common the point-of-view that current conservation is due to gauge symmetry, and it is indeed true. However, it is also common the rather stronger assumption that the whole model must be local gauge invariant so as to allow current conservation. As it was presented, we can face current conservation by an alternative point-of-view. At the classical level, the crucial role to achieve it is played by *global* gauge symmetry, or local gauge invariance of the matter action in nonabelian case.

At the quantum level, two alternative ingredients must be added: *either* the fermionic jacobian must be trivial *or* the matter action and also the bosonic jacobian must be local gauge invariant. In the first alternative, if we are dealing with a model whose action is gauge invariant, we arrive at an anomaly-free model with a current which is trivially conserved. In the second one, on the other hand, we arrive at an effective action with broken symmetry, and current conservation may be guaranteed by gauge invariance of dA_μ or, if we are dealing with classical vector fields, by the variational principle as a kind of subsidiary condition.

The consistence of the anomaly cancelation as subsidiary condition in anomalous models may be found if one proves that such models can be obtained from anomaly-free ones, as it is the case of the $2-D$ Jackiw-Rajaraman model, as shown in the previous section through the inclusion of the Stueckelberg scalar into the theory, just as it is done in Proca model.

Finally, we stress out that such technique of recovering gauge invariance through the inclusion of the Stueckelberg's field, and by proving its equivalence with the original non-gauge model, may be a road to the renormalizability of anomalous models, just as it was shown to be in the original Stueckelberg's example.

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