The Equivalence Between Gauge and Non-Gauge Abelian Models

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Abstract

This work is intended to estabilish the equivalence between gauge and non-gauge abelian models. Following a technique proposed by Harada and Tsutsui, it is shown that the Proca and chiral Schwinger models may be equivalent to correspondent gauge invariant ones. Finally, it is shown that a gauge invariant version of the chiral Schwinger model, after integrated out the fermions, can be identified with the 2-D Stueckelberg model without the gauge fixing term.

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I. INTRODUCTION

It is well known that anomalous gauge theories usually spoils unitarity and renormalizability due to the quantum breakdown of gauge symmetry at quantum level [1], [2]. Yet, it is also believed that the gauge anomaly breaks current conservation. In view of this, gauge anomalous models are usually considered as being inconsistents.

Contrary to this idea, a group of authors has shown that gauge symmetry may be restored by the addition of extra degrees of freedom. Indeed, the work of Fadeev and Shatashvilli [3] restores gauge symmetry of the final effective action by adding a Wess-Zumino term to the fundamental action. Soon after, the works of Babelon, Shaposnik and Viallet [4] and Harada and Tsutsui [5] showed, independently, that such Wess-Zumino term could be derived through algebraic manipulations over the functional integral. Then, it became clear that such way of deriving gauge invariant models from anomalous ones did not need to be limited to the case of anomalous models, but it could be also applied to non anomalous ones that do not exhibit classical gauge symmetry, like the Proca model, also analyzed by the last cited authors [6], thus, leading to a natural generalization of their techinque.

To understand the role played by the emerging extra field in abelian case, a recent work has shown that the gauge invariant formulation of the Proca model may be identified with the Stueckelberg theory [7], leading to the interpretation of such field as being the Stueckelberg scalar [8]. This may conduct to the idea that anomalous models are analogous to any theory that breaks gauge invariance and, thus, may be treated in the same way.

At this point, one may ask, in general, if both formulations can be taken as being physically equivalent and, in particular, whether the current of anomalous models in both formulations is conserved or not. Yet, only in the context of anomalous models, the Harada-Tsutsui technique may lead to two distinct ways of achieving the same gauge invariant effective theory before the integration over the matter fields: the one which adds up the Wess-Zumino term, known as the *standard* formulation [5], and another one which also couples the extra degrees of freedom with the matter fields, called the *enhanced* formulation [8]. In this sense one could ask whether both ways are redundant or if the informations contained in each of them are physically distinguishable.

This work is intended to elucidate these questions for the case of abelian gauge models, and the relation between original abelian anomalous models, the standard formulation and the enhanced one is analyzed, as well as the relation between the Proca and Stueckelberg's models. In this sense, in section I, the *enhanced* version of Harada-Tsutsui gauge invariant mapping is derived, as well as the original *standard* one. In section II, we rederive the Stueckelberg model from the Proca's one, and an analysis of both formulations shows their equivalence. In section III, the same kind of analysis is done, but comparing the enhanced version of abelian anomalous gauge models with the original ones. It is shown that if one alternatively considers that the current is conserved by the equation of motion of the gauge field, as an analogue to the subsidiary condition arising in the Proca model, then both formulations may become equivalent, since the first may be reduced to the second by a gauge condition which represents the anomaly cancelation of the original model. The chiral Schwinger model is used as an example. It is also shown that the enhanced formulation of abelian anomalous models is free from anomalies.

Then, the two examples analyzed lead us, naturally, to an equivalence statement related to gauge and non-gauge theories, which is done in section IV. Yet in this section, it is shown that if the anomaly is not gauge invariant, it still remains in the standard formulation, and that this one may be equivalent to the other formulations only if the anomaly is gauge invariant. Finally, in section V, it is shown that, after integrated out the fermions, a gauge invariant formulation of the chiral Schwinger model may be identified with the original Stueckelberg's model in $2-D$. We, thus, conclude this work in section VI.

II. ENHANCED VERSION OF HARADA-TSUTSUI GAUGE INVARIANT MAPPING

We consider an *anomalous* generic abelian effective action, defined by

$$\exp\left(iW[A_\mu]\right) = \int d\psi d\bar{\psi} \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right), \tag{1}$$

where $I[\psi, \overline{\psi}, A_\mu]$ is invariant under local gauge transformations

$$A_\mu^\theta = A_\mu + \frac{1}{e}\partial_\mu \theta(x), \tag{2}$$

$$\psi^\theta = \exp\left(i\theta(x)\right)\psi, \tag{3}$$

$$\bar{\psi}^\theta = \exp\left(-i\theta(x)\right)\bar{\psi}, \tag{4}$$

that is,

$$I\left[\psi^{\theta},\bar{\psi}^{\theta},A^{\theta}\right]=I\left[\psi,\bar{\psi},A\right], \tag{5}$$

while, by definition,

$$W[A_{\mu}^{\theta}]\neq W[A_{\mu}]. \tag{6}$$

The formulation with the addition of the Wess-Zumino term, first proposed by Fadeev and Shatashvilli [3], and then derived by Harada and Tsutsui [5], arises when one goes to the full quantum theory by redefining the vacuum functional

$$Z=\int dAd\psi d\overline{\psi}\exp\left(iI[\psi,\overline{\psi},A_{\mu}]\right)=\int dA_{\mu}\exp\left(iW[A_{\mu}]\right) \tag{7}$$

multiplying it by the gauge volume

$$Z=\int d\theta dAd\psi d\overline{\psi}\exp\left(iI[\psi,\overline{\psi},A_{\mu}]\right)=\int d\theta dA\exp\left(iW[A_{\mu}]\right). \tag{8}$$

We, then, change variables in the gauge field so that

$$A_{\mu}\rightarrow A_{\mu}^{\theta};\ dA_{\mu}\rightarrow dA_{\mu}^{\theta}, \tag{9}$$

and use translational invariance dA_{μ}, so that

$$dA_{\mu}^{\theta}=dA_{\mu}, \tag{10}$$

to reach the final gauge invariant effective action, which takes the $\theta-field$ into account, defined by

$$\exp\left(iW_{eff}[A_{\mu}]\right)\equiv\int d\theta\exp\left(iW[A_{\mu}^{\theta}]\right). \tag{11}$$

Using (1), it is evident that

$$\exp\left(iW[A_{\mu}^{\theta}]\right)=\int d\psi d\overline{\psi}\exp\left(iI_{st}[\psi,\overline{\psi},A_{\mu},\theta]\right), \tag{12}$$

where

$$I_{st}[\psi,\overline{\psi},A_{\mu},\theta]\equiv I[\psi,\overline{\psi},A_{\mu}]+\alpha_{1}\left[A,\theta\right] \tag{13}$$

is called the standard action and

$$\alpha_{1}\left[A,\theta\right]\equiv W[A_{\mu}^{\theta}]-W[A_{\mu}] \tag{14}$$

is known as the Wess-Zumino term [9]. It can be seen that, besides the final effective action is gauge invariant, the starting one (13) is not, since the Wess-Zumino term breaks gauge symmetry. On the other hand, we may raise an alternative gauge invariant initial action by noticing that (11) can be also obtained by

$$\exp\left(iW[A_\mu^\theta]\right) = \int d\psi d\bar{\psi} \exp\left(iI_{en}[\psi,\overline{\psi},A_\mu,\theta]\right), \tag{15}$$

where

$$I_{en}[\psi,\overline{\psi},A_\mu,\theta] \equiv I[\psi,\overline{\psi},A_\mu^\theta]. \tag{16}$$

This simplifies and systematizes the Harada-Tsutsui procedure by noticing that one needs only to make the substitution $A_\mu \to A_\mu^\theta$ on the fundamental action, as it becomes clear in the example of the massive vector theory. It is also evident that, to reach such a really gauge invariant formulation, we do not even need to proceed such substitution to the entire action. Indeed, one needs only to add up a gradient of a scalar to the gauge field in the parts of the initial action that does not remain gauge invariant *after* integrated out the fermions.

The inclusion of the $\theta - field$ in the enhanced formulation also transforms it into a modified gauge theory, even before the integration over the scalar. To see this, we notice that such formulation is invariant under Pauli's transformations [10]

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\Lambda \\ \theta &\to \theta - \Lambda. \end{aligned} \tag{17}$$

as also noticed by the authors in the massive vector case [6]. We shall distinguish between the scalar provided by the standard action from the one associated to the enhanced formulation, calling the first Wess-Zumino field and the second, Stueckelberg's one.

III. EQUIVALENCE BETWEEN THE PROCA AND STUECKELBERG MODELS

Consider a Proca field interacting with fermionic ones, whose action is

$$I_P[\psi,\overline{\psi},A_\mu] \equiv I_M[\psi,\overline{\psi},A_\mu] + W_P\left[A\right], \tag{18}$$

where $I_M[\psi,\overline{\psi},A_\mu]$ is the matter action minimally coupled to the abelian field A_μ, that exhibits local gauge symmetry, and $W_P\left[A\right]$ is the pure Proca action, defined by

$$W_P\left[A\right] \equiv \int d^nx \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu\right).$$

Evidently, the action above has no gauge symmetry, since the massive term breaks it. The classical equations of motion lead us to

$$\frac{\delta I_M}{\delta\psi} = \frac{\delta I_M}{\delta\bar{\psi}} = 0 \tag{19}$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = eJ^\nu, \tag{20}$$

where

$$J^\mu = -\frac{1}{e}\frac{\delta I_M}{\delta A^\mu} \tag{21}$$

is the conserved matter current obtained by global invariance. If we take the divergence of eq. (20), then we just arrive with

$$\partial_\mu A^\mu = 0 \tag{22}$$

as a *subsidiary* condition.

On the other hand, one could apply the Harada-Tsutsui technique by gauge transforming only the massive part of the action to obtain

$$I_{P(en)}\left[\psi,\overline{\psi},A,\theta\right] = I_M[\psi,\overline{\psi},A] + W_{P(en)}\left[A,\theta\right], \tag{23}$$

where $W_{P(en)}\left[A\right]$ is just the pure enhanced Proca action, given by

$$W_{P(en)}\left[A,\theta\right] \equiv W_P\left[A^\theta\right] = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x\left(A^\mu + \frac{1}{e}\partial^\mu\theta\right)\left(A_\mu + \frac{1}{e}\partial_\mu\theta\right). \tag{24}$$

It is easy to notice that $W_{P(en)}\left[A\right]$ is just the Stueckelberg action. To see this, we notice that if we rename the $\theta - field$ so as

$$B(x) \equiv \frac{m}{e}\theta(x), \tag{25}$$

then (24) takes the exact expression of the Stueckelberg action [7]

$$W_{Stueck}\left[A,B\right] = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\int d^4x\left(mA^\mu + \partial^\mu B\right)\left(mA_\mu + \partial_\mu B\right). \tag{26}$$

It is clear that the Stueckelberg model is reducible to the original Proca's one by the gauge choice where the Stueckelberg field is set constant. But the result that is of our interest would be to show the equivalence between the Proca's model and its gauge invariant version after integrated out the $\theta - field$. To this end, we may integrate (24) over the gauge orbits to find the gauge invariant version of Proca model coupled to the fermions

$$\exp\left(iI'_P\left[\psi,\bar{\psi},A\right]\right) \equiv \exp\left(iI_M\left[\psi,\bar{\psi},A\right]\right)\int d\theta\exp\left(iW_{P(en)}\left[A,\theta\right]\right). \tag{27}$$

To do this, we notice that

$$\int d\theta \exp\left(iW_{P(en)}\left[A,\theta\right]\right) = \exp\left(iW_P\left[A\right]\right)\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right), \quad (28)$$

and that

$$\begin{aligned} &\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right) \\ &= \exp\left(-\frac{i}{2}m^2\int d^nxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right)\int d\theta \exp\left(-i\frac{m^2}{2e}\int d^nx\left[\left(\frac{e}{\Box}\partial^\mu A_\mu+\theta\right)\Box\left(\frac{e}{\Box}\partial^\nu A_\nu+\theta\right)\right]\right). \end{aligned} \quad (29)$$

Performing the following change of variables in the $\theta - field$:

$$\theta \to \theta' = \theta + \frac{e}{\Box}\partial^\mu A_\mu, \quad (30)$$

we find

$$\int d\theta \exp\left(i\int \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{m^2}{e}A^\mu\partial_\mu\theta\right) \sim \exp\left(-\frac{i}{2}m^2\int d^nxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right), \quad (31)$$

and, thus

$$I'_P\left[\psi,\bar{\psi},A\right] = I_M\left[\psi,\bar{\psi},A\right] + \int d^nx\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2A_\mu\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu\right\}. \quad (32)$$

Although we went far away going to the full quantum model to derive (32), we now use its classical version and derive the equations of motion. Then, we just obtain

$$\frac{\delta I_M}{\delta\psi} = \frac{\delta I_M}{\delta\bar{\psi}} = 0 \quad (33)$$

$$eJ^\nu = \partial_\mu F^{\mu\nu} + m^2\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu, \quad (34)$$

and it turns obvious that the equations of motion of this gauge invariant version of massive vector model coincides with the Proca one if we fix the Lorentz gauge $\partial_\mu A^\mu = 0$, showing equivalence between both formulations. It can be seen that such gauge choice is equivalent to choose θ constant before integration over the scalar. We shall return to this point next sections.

This illustrative example is just a guideline to reach a rather more interesting and less *common sense* result presented in next section.

IV. EQUIVALENCE BETWEEN THE ORIGINAL AND ENHANCED VERSIONS OF ABELIAN ANOMALOUS MODELS

Now, we return to the anomalous generic gauge model defined in (1), where

$$I\left[\psi,\bar{\psi},A\right]=I_{M(Ano)}\left[\psi,\bar{\psi},A\right]+I_S\left[A\right], \tag{35}$$

with $I_{M(Ano)}\left[\psi,\bar{\psi},A\right]$ being the anomalous matter action and $I_S\left[A\right]$ is the gauge invariant free bosonic one.

The local gauge invariance breakdown of the effective action (6) is used to be referred with current nonconservation. To understand this, we see that since the effective action is not gauge invariant we may say that we do not have the *Noether* identity $\partial_\mu\left(-\frac{1}{e}\frac{\delta W[A]}{\delta A_\mu(x)}\right)\equiv 0$, *i. e.*, identically

$$\mathcal{A}\equiv\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right)\neq 0. \tag{36}$$

The quantity defined by (36) is used to be referred as an *anomaly*. To understand the relation between (36) and current divergence, we notice that

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right)\exp\left(iW\left[A\right]\right)=\int d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)}\right)\exp\left(iI\left[\psi,\bar{\psi},A\right]\right). \tag{37}$$

Since $I_S\left[A^\theta\right]=I_S\left[A\right]$, we have

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta I_S\left[A\right]}{\delta A_\mu(x)}\right)\equiv 0, \tag{38}$$

and, therefore,

$$\int d\psi d\bar{\psi}\partial_\mu J^\mu(x)\exp\left(iI\left[\psi,\bar{\psi},A\right]\right)=\mathcal{A}\exp\left(iW\left[A\right]\right), \tag{39}$$

where

$$J^\mu(x)\equiv-\frac{1}{e}\frac{\delta I_{(Ano)M}\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)} \tag{40}$$

is the classical conserved current that may be obtained by global invariance of the action. If $\mathcal{A}$ is considered non-null, then eq. (39) means current conservation breakdown at quantum level, representing one of the most intriguing problems in quantum field theory. In this sense, to be very precise in our purposes, we *define* the anomaly by (39), generalizing it to the mean expectation value of the classical current divergence over the remaining fields beside the gauge one,

$$\int d\varphi d\psi d\bar{\psi}\partial_\mu J^\mu(x)\exp\left(iI\left[\psi,\bar{\psi},A,\varphi\right]\right)=\mathcal{A}\exp\left(iW\left[A\right]\right), \tag{41}$$

where φ represents all other fields that may enter the theory beside the ones being considered, and an *anomalous model* as being the one whose anomaly defined in (41) is not *identically* null.

Although such theories may bring theoretical problems, we may alternatively face an anomalous model as a faithful one, take the gauge field equation of motion from the effective action

$$\frac{\delta W\left[A\right]}{\delta A_\mu(x)}=0, \tag{42}$$

and, in straight analogy with the Proca model, obtain the nullity of the anomaly as a *subsidiary* condition

$$\mathcal{A}\equiv\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right)=0. \tag{43}$$

However, this means constraints into the theory. It remains to be proved, though, the internal consistency of a theory leading with such constraints. In this sense, we shall analyze a concrete example, the anomalous chiral Schwinger model, whose action is

$$I_{Sch}\left[\psi,\bar{\psi},A\right]=\int d^2x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}i\gamma^\mu\left[\partial_\mu-ieA_\mu P_+\right]\psi\right\}, \tag{44}$$

where

$$P_+\equiv\frac{1}{2}\left(1+\gamma_5\right). \tag{45}$$

This action is gauge invariant and the classical conserved current obtained by its symmetry is given by

$$J^\mu(x)=\bar{\psi}\gamma^\mu P_+\psi. \tag{46}$$

The effective action is exactly soluble [12], and given by

$$W_{Sch}\left[A\right]=\int d^2x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{e^2}{8\pi}A_\mu\left[ag^{\mu\nu}-\left(g^{\mu\alpha}+\epsilon^{\mu\alpha}\right)\frac{\partial_\alpha\partial_\beta}{\Box}\left(g^{\beta\nu}-\epsilon^{\beta\nu}\right)\right]A_\nu\right\}, \tag{47}$$

where $g^{\mu\nu}$ is the $2-D$ Minkowski metric, $\epsilon^{\mu\alpha}$ is the Levi-Civita tensor and a is an arbitrary regularization parameter.

Now, it is easy to see that $W_{Sch}\left[A^{\theta}\right] \neq W_{Sch}\left[A\right]$ [5]. Indeed,

$$\begin{aligned}\alpha_1\left[A,\theta\right] &= W_{Sch}\left[A^{\theta}\right] - W_{Sch}\left[A\right] \\ &= \frac{1}{4\pi}\int d^2x \left\{\frac{1}{2}\left(a-1\right)\partial_\mu\theta\partial^\mu\theta - e\theta\left[\left(a-1\right)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}. \end{aligned} \tag{48}$$

Therefore, the chiral Schwinger model *is* anomalous, with the anomaly being

$$\mathcal{A} = -\frac{e}{4\pi}\left\{\left(a-1\right)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu\right\}. \tag{49}$$

On the other hand, by the alternative point-of-view above explained, we may impose the variational principle to the effective action (47), and we just find the equation of motion of the vector field

$$\partial_\mu F^{\mu\nu} + \frac{e^2}{4\pi}\left(aA^\nu - \frac{\partial^\nu\partial^\mu}{\Box}A_\mu + \epsilon^{\alpha\mu}\frac{\partial^\nu\partial_\alpha}{\Box}A_\mu - \epsilon^{\nu\alpha}\frac{\partial_\alpha\partial^\mu}{\Box}A_\mu + \epsilon^{\nu\alpha}\epsilon^{\beta\mu}\frac{\partial_\alpha\partial_\beta}{\Box}A_\mu\right) = 0. \tag{50}$$

Taking the divergence of (50) and using the fact that

$$\epsilon^{\mu\alpha}\epsilon^{\beta\nu} = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu}, \tag{51}$$

we just arrive with the subsidiary condition that cancels the anomaly

$$\left(a-1\right)\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu = 0. \tag{52}$$

Substituting it back to (50), it is straightforward to find the Proca gauge invariant version of the massive $2-D$ vector field's equation of motion

$$\partial_\mu F^{\mu\nu} + \frac{e^2}{4\pi}\frac{a^2}{\left(a-1\right)}\left(\eta^{\mu\nu} - \frac{\partial^\nu\partial^\mu}{\Box}\right)A_\mu = 0, \tag{53}$$

but with the vector field restricted to the condition (52).

We now turn back to the *general* case and proceed the enhanced mapping. Using (16), (15) and (11), we just obtain

$$\begin{aligned}&\int d\theta d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right) \\ &= \partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_\mu(x)}\right)\exp\left(iW_{eff}\left[A\right]\right). \end{aligned} \tag{54}$$

Since $W_{eff}\left[A^\theta\right] = W_{eff}\left[A\right]$, we have the Noether identity

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_\mu(x)}\right) \equiv 0, \tag{55}$$

and thus

$$\int d\theta d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right)\equiv 0. \tag{56}$$

Since in fermionic theories the gauge fields are used to be coupled linearly to the matter ones, and the difference between A_μ and A_μ^θ is just a translation, we may be sure that

$$\frac{\delta I_{M(Ano)}\left[\psi,\bar{\psi},A^\theta\right]}{\delta A_\mu(x)}=\frac{\delta I_{M(Ano)}\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)}. \tag{57}$$

By (38) we obtain, therefore

$$\int d\theta d\psi d\bar{\psi}\partial_\mu J^\mu\exp\left(iI_{en}\left[\psi,\bar{\psi},A,\theta\right]\right)\equiv 0, \tag{58}$$

which means that the abelian enhanced formulation is anomaly-free.

As already discussed, the enhanced formulation, before integration over the scalar, may be viewed as an anomalous analogue of the Stueckelberg mechanism [8], and it obviously reduces to the original one by the gauge choice where θ is set constant. We now return to the example of chiral Schwinger model and get its enhanced version. Then we have, after integrated the fermions,

$$W_{Sch}\left[A^\theta\right]=\alpha_1\left[A,\theta\right]+W_{Sch}\left[A\right]. \tag{59}$$

Therefore, one needs only to consider the Wess-Zumino term (48) in the integration over θ. Thus,

$$\begin{aligned}&\exp\left(iW_{eff}\left[A\right]\right)\\&=\exp\left(iW_{Sch}\left[A\right]\right)\int d\theta\exp\left(\frac{i}{4\pi}\int d^2x\left\{\frac{1}{2}\left(a-1\right)\partial_\mu\theta\partial^\mu\theta-e\theta\left[\left(a-1\right)\partial_\mu A^\mu+\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}\right)\end{aligned} \tag{60}$$

Using that

$$\begin{aligned}&\frac{i}{4\pi}\int d^2x\left\{\frac{1}{2}\left(a-1\right)\partial_\mu\theta\partial^\mu\theta-e\theta\left[\left(a-1\right)\partial_\mu A^\mu+\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}\\&=-\frac{i}{8\pi}\left(a-1\right)\int d^2x\left[\frac{1}{\Box}e\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right)\right.\\&\left.+\theta\right]\Box\left[\frac{1}{\Box}e\left(\partial_\alpha A^\alpha+\frac{1}{(a-1)}\epsilon^{\alpha\beta}\partial_\alpha A_\beta\right)+\theta\right]\\&-\frac{e^2}{\Box}\left(\partial_\mu A^\mu+\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right)\left(\partial_\alpha A^\alpha+\frac{1}{(a-1)}\epsilon^{\alpha\beta}\partial_\alpha A_\beta\right),\end{aligned} \tag{61}$$

performing the following translation over the $\theta - field$:

$$\theta' = \theta + \frac{1}{\Box} e \left(\partial_\mu A^\mu + \frac{1}{(a-1)} \epsilon^{\mu\nu} \partial_\mu A_\nu \right) ; d\theta' = d\theta, \tag{62}$$

and proceeding integration over the new parameter θ' in (70), it is straightforward to find

$$\exp\left(iW_{eff}[A]\right) = \exp\left(iW_{Sch}[A]\right) \int d\theta \exp\left\{ i\frac{e^2}{8\pi}(a-1) \int d^2x \left(\partial_\mu A^\mu + \frac{1}{(a-1)} \epsilon^{\mu\nu} \partial_\mu A_\nu \right) \right.$$
$$\left. \frac{1}{\Box} \left(\partial_\mu A^\mu + \frac{1}{(a-1)} \epsilon^{\mu\nu} \partial_\mu A_\nu \right) \right\}. \tag{63}$$

Using (47) and (51), we finally obtain

$$W_{eff}[A] = \int d^2x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \frac{e^2}{4\pi} \frac{a^2}{(a-1)} A_\mu \left[g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box} \right] A_\nu \right\}. \tag{64}$$

This result was also found in ref. [5]. We may observe that the effective action (64) is exactly the Proca $2-D$ gauge invariant action that gives the equation of the anomalous original model (53), but without the restriction (52) over the gauge field. Therefore, analogously to the *Proca/Stueckelberg* case, if we fix the gauge by (52) in the gauge invariant effective anomalous action, then the enhanced model reduces to the original anomalous one, showing equivalence between both formulations.

V. DISCUSSION

The examples mentioned above may lead us to the following statement: *A gauge theory is equivalent to a non-gauge one if the first is reducible to the second one by a gauge choice.* By the Pauli's conditions point-of-view, the original and enhanced formulations are obviously equivalent, since the second reduces to the first by the gauge choice where the Stueckelberg scalar θ is set constant.

By the canonical gauge theory point-of-view, on the other hand, our examples show us that the integrated effective models are reducible one to the other by the Lorentz gauge choice

$$\partial_\mu A^\mu = 0 \tag{65}$$

in the Proca case, and the rather distinct one

$$(a-1)\, \partial_\mu A^\mu + \epsilon^{\mu\nu} \partial_\mu A_\nu = 0 \tag{66}$$

in the chiral Schwinger model. We see that, to achieve these gauge conditions, we have to perform the following transformations over a not restricted generic gauge field A_μ:

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \Lambda \tag{67}$$

taking the divergence of A'_μ in the Proca case in (67), we have

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \frac{1}{e}\Box \Lambda_P = 0 \tag{68}$$

which means that

$$\Lambda_P = -\frac{e}{\Box}\partial_\mu A^\mu . \tag{69}$$

Doing the same for the chiral Schwinger model and adding $\frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu$, it is straightforward to find

$$\Lambda_{Sch} = -\frac{e}{\Box}\left(\partial_\mu A^\mu + \frac{1}{(a-1)}\epsilon^{\mu\nu}\partial_\mu A_\nu\right). \tag{70}$$

If we compare (69) and (70) with (30) and (62), respectively, we see that the translation over the $\theta - field$ to reach the pure gauge invariant action is just

$$\theta \rightarrow \theta' = \theta - \Lambda . \tag{71}$$

This suggests that the enhanced gauge condition $\theta\,(x) = k$ that ensures equivalence between both models is transferred to the gauge fields after integrated out the Stueckelberg, as manifested in (65) and (66), in such a way that it turns to be the subsidiary conditions of the original models.

We now turn our attention to the standard formulation. The work of ref. [11], in particularly analyzing the standard version of the chiral Schwinger model, shows that its gauge invariant correlation functions coincide with those of the original anomalous theory, but it also shows that it is not the case for gauge dependent Green's functions, and no choice of gauge conditions exists for which the generating functional of the standard formulation coincides with that of the original theory. The conclusion is, thus, that its physical contents are quite different. However, it was also shown that the action with the addition of the Wess-Zumino term is equivalent to the original anomalous model if the gauge condition (66) is *imposed* to both models. As it was shown, this condition may arise from the original model as a subsidiary condition. On the other hand, besides the final effective action is made gauge invariant, the starting one is not, since the Wess-Zumino term breaks it. To understand what it means, we consider a model with the standard action (13) and try to obtain the conserved quantity given by the gauge invariance of the effective action, we then find

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_\mu(x)}\right)\exp\left(iW_{eff}\left[A\right]\right)$$
$$=\int d\theta d\psi d\bar{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I_{st}\left[\psi,\bar{\psi},A,\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=0 \tag{72}$$

or, by (13) and (40)

$$\int d\theta d\psi d\bar{\psi}\partial_\mu J^\mu\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=\int d\theta d\psi d\bar{\psi}\partial_\mu\left(\frac{1}{e}\frac{\delta\alpha_1\left[A,\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right). \tag{73}$$

If we integrate the right hand side of (73) and use (55), we will just obtain

$$\int d\theta d\psi d\bar{\psi}\partial_\mu J^\mu\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=\mathcal{A}\exp\left(iW_{eff}\left[A\right]\right), \tag{74}$$

with $\mathcal{A}$ given by (36), which means, by our definition (41), that the standard formulation is still anomalous. We can notice that in this model, unlike the original anomalous one, no subsidiary condition arises in order to cancel the anomaly. To be more precise, a kind of subsidiary condition arises if we use the Dyson-Schwinger equation for the $\theta - field$. To see this, we writte

$$\int d\theta d\psi d\bar{\psi}\frac{\delta I_{st}}{\delta\theta}\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)$$
$$=\int d\theta d\psi d\bar{\psi}\frac{\delta\alpha_1}{\delta\theta}\left[A,\theta\right]\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=0, \tag{75}$$

but,

$$\frac{\delta\alpha_1}{\delta\theta}\left[A,\theta\right]=\frac{\delta W}{\delta\theta}\left[A^\theta\right]=\int d^n y\frac{\delta W\left[A^\theta\right]}{\delta A^\theta_\mu\left(y\right)}\frac{\delta A^\theta_\mu\left(y\right)}{\delta\theta\left(x\right)}$$
$$=\int d^n x\frac{1}{e}\frac{\delta W\left[A^\theta\right]}{\delta A^\theta_\mu}\partial_\mu\delta\left(x-y\right)=\partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A^\theta\right]}{\delta A^\theta_\mu}\right)=\mathcal{A}^\theta \tag{76}$$

and, therefore

$$\int d\theta d\psi d\bar{\psi}\mathcal{A}^\theta\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=0. \tag{77}$$

We see that, if the anomaly is made gauge invariant, which means to set $a=1$ in the chiral Schwinger model, then the left hand side of (77) reduces to (74) and the anomaly cancels out. However, in our simplest example, the choice $a = 1$ represents a gauge parameter (70), to be used in order to integrate the scalar by the translation in (62), which is infinite. It is easy to see, by eq. (60), that such a choice also represents a functional Dirac delta that has the anomaly as its parameter, that is, if $a = 1$, then

$$\begin{aligned}
&\exp\left(iW_{eff}\left[A\right]\right) \\
&= \exp\left(iW_{Sch}\left[A\right]\right)\int d\theta \exp\left(-\frac{i}{4\pi}\int d^2x\left\{e\theta\epsilon^{\mu\nu}\partial_\mu A_\nu\right\}\right) \\
&= \delta\left(\mathcal{A}\left[A\right]\right)\exp\left(iW_{Sch}\left[A\right]\right).
\end{aligned} \tag{78}$$

This is, indeed, trivially equivalent to the original anomalous model, since it has just the same non-gauge action with the redundant vanishing anomaly condition being imposed before the equation of motion of the vector field is taken from the effective action. The surviving of anomaly in (74) is, thus, caused by its gauge non-invariance. On the other hand, a distinct value of a clearly turns the anomaly cancellation impossible.

Yet, we see that if $a \neq 1$, the anomaly will not be invariant, but the final effective action will be. In this sense, if we were allowed to choose a gauge condition such as $\mathcal{A} = 0$, then the anomaly would cancel out, the current would become conserved and the standard formulation would turn to be the original one. We may also see that the standard formulation reduces to the original one if we set $\theta\left(x\right) = k$. However, obviously the situation imposed by such condition is not physically equivalent to leaving the anomaly non-null, since we have no current conservation in one situation, and have it conserved in another one. Therefore, we have a breaking of the physical equivalence between distinct gauge configurations in the final gauge invariant effective action of the standard formulation due to the fact that the standard action is not gauge invariant. These considerations may explain the results found in ref. [11].

On the other hand, we may try to adopt the point-of-view in which there may be a modified conserved current on the standard formulation due to the addition of the Wess-Zumino term. To calculate such current, we need to use the gauge invariance of the final effective action, which gives us, from (73)

$$\int d\theta d\psi d\bar{\psi}\partial_\mu\left(J^\mu - \frac{1}{e}\frac{\delta\alpha_1\left[A,\theta\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right) \equiv 0. \tag{79}$$

Therefore, the standard current must be taken from

$$J_{st}^{\mu}=J^{\mu}-\frac{1}{e}\frac{\delta\alpha_1\left[A,\theta\right]}{\delta A_{\mu}(x)}. \tag{80}$$

However, as it was seen (74)

$$\int d\theta d\psi d\bar{\psi}\partial_{\mu}J^{\mu}\exp\left(iI_{st}\left[\psi,\bar{\psi},A,\theta\right]\right)=\mathcal{A}\exp\left(iW_{eff}\left[A\right]\right).$$

If we choose our gauge choice where the anomaly is cancelled out $\mathcal{A}=0$, we arrive at the same classical conserved current

$$J_{st}^{\mu}=J^{\mu} \tag{81}$$

in one specific gauge, and such modified one (80) in all others, proving that such current cannot be physical. This may be explained, once again, by the non-invariance of the standard action.

It should be noticed, though, that such difference between the standard and enhanced models appears only before the integration over the matter fields. After that, the same intermediary effective action is found. In the sense of the effective theory, thus, this means that there is no net difference whether one is working initially with one or other formulation.

VI. CORRESPONDENCE BETWEEN THE GAUGE INVARIANT CHIRAL SCHWINGER AND STUECKELBERG MODELS

We saw that both gauge invariant formulations of the chiral Schwinger model give us, after integrated out the fermions,

$$W_{Sch}\left[A^{\theta}\right]=W_{Sch}\left[A\right]+\int d^2x\left\{\frac{1}{8\pi}\left(a-1\right)\partial_{\mu}\theta\partial^{\mu}\theta-e\theta\left[\left(a-1\right)\partial_{\mu}A^{\mu}+\epsilon^{\mu\nu}\partial_{\mu}A_{\nu}\right]\right\}, \tag{82}$$

while the Stueckelberg's gauge invariant model is described by

$$W_P\left[A^{\theta}\right]=-\frac{1}{4}\int d^4xF^{\mu\nu}F_{\mu\nu}+\frac{m^2}{2}\int d^4x\left(A^{\mu}+\frac{1}{e}\partial^{\mu}\theta\right)\left(A_{\mu}+\frac{1}{e}\partial_{\mu}\theta\right). \tag{83}$$

All these models, after integrated out the scalar, reach the same gauge invariant Proca action

$$W_{eff}\left[A\right]=\int d^2x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\frac{e^2}{4\pi}\frac{a^2}{(a-1)}A_{\mu}\left[g^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{\Box}\right]A_{\nu}\right\}. \tag{84}$$

The diference between them relies on the translation over the θ variable. While in the Stueckelberg model we change variables such as

$$\theta_P\rightarrow\theta_P+\frac{e}{\Box}\partial^{\mu}A_{\mu}, \tag{85}$$

in the chiral Schwinger one we have

$$\theta_{Sch} \to \theta_{Sch} + \frac{1}{\Box} e \left(\partial_\mu A^\mu + \frac{1}{(a-1)} \epsilon^{\mu\nu} \partial_\mu A_\nu \right). \tag{86}$$

Therefore, it is very convenient to find a mapping between these models. Indeed, it is straightforward to check that the relation

$$\theta_{Sch} = \frac{a}{(a-1)} \theta_P - \frac{e}{(a-1)} \frac{1}{\Box} \epsilon^{\mu\nu} \partial_\mu A_\nu + \frac{e}{(a-1)} \frac{1}{\Box} \partial_\mu A^\mu \tag{87}$$

turns one model in the other

$$W_{Sch(en)} [A, \theta_{Sch}] = W_{P(en)} [A, \theta_P] \tag{88}$$

$$\int d\theta_{Sch} \exp \left(i W_{Sch(en)} [A, \theta_{Sch}] \right) \sim \int d\theta_P \exp \left(i W_{P(en)} [A, \theta_P] \right), \tag{89}$$

which means that the gauge invariant chiral Schwinger model, after integrated out the fermions, may be identified with the original Stueckelberg's one which is known to be unitary and renormalizable in some gauge fixing condition [13], [14].

VII. CONCLUSION

This work has shown up a rather contra-intuitive idea: that a gauge invariant model may be equivalent to a non-invariant one. Besides being against the comon sense, it is perfectly possible, as it was shown, as long as one is reduced to the other by some gauge condition. The strangeness about this result, nevertheless, may be related to the idea that current conservation is due to gauge symmetry. However, at least in the classical case, Noether theorem ensures current conservation through *global* gauge invariance and the variational principle, instead of a rather stronger condition, which is local gauge symmetry, as it becomes evident in the Proca case. Work is in progress to clarify the relation between local gauge symmetry and current conservation in the context of quantum models.

On the other hand, this idea becomes manifest by an interesting procedure of recovering gauge symmetry from non-gauge theories, and the simpler example of the chiral Schwinger model shows us that, after integrated out the fermions, the effective theory may be identified with the original model proposed by Stueckelberg, showing a possible alternative mechanism of mass generation from chiral fermions that might be unitary and renormalizable.

Finally, we may suppose that such mass mechanism from chiral fermions is valid for higher dimensions. If one can prove that the final effective theory $W_{eff}[A]$ is the Proca invariant one (64) even in $4-D$, then it will be furnished more than the possibility of quantization of abelian anomalous models, but its conection with a quantizable mass generation mechanism able to substitute the *Higgs*, at least for the abelian case.

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