Non-Cartesian Systems : an Open Problem

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Dedicated to Marie-Louise Nykamp

Abstract

The following open problem is presented and motivated : Are there physical systems whose state spaces do not compose according to either the Cartesian product, as classical systems do, or the usual tensor product, as quantum systems do ?

"History is written with the feet ..."

Ex-Chairman Mao

1. Non-Cartesian Systems

The *state spaces* of systems within Classical Mechanics compose according to the Cartesian product. Namely, let X, Y be two such systems, each with the respective state spaces given by the sets E, F, then the state space of the composite system "$X \, and \, Y$" is the *Cartesian product* $E \times F$.

Here one can note that composing classically X and Y need not necessarily mean that the two systems shall interact, or that the states of the two systems may enter into a correlation. And if they do not, then the state space of "$X \, and \, Y$" will be the whole of $E \times F$. However, if they interact, and certain states of the component systems become correlated, then the state space of "$X \, and \, Y$" can in fact be a strict subset of $E \times F$.

Except for that classical way of composing systems, it appears that there is only one alternative known, namely, the quantum way of system composition. Namely, let X, Y be two quantum systems, each with the respective state space given by the Hilbert spaces E, F, then the state space of the composite quantum system "$X \, and \, Y$" is the *tensor product* $E \bigotimes F$.

Now, the considerable novelty in the non-Cartesian quantum way of composing systems is that the resulting quantum composite state space $E \bigotimes F$ is significantly *larger* than the classical composite state space $E \times F$. Indeed, if for instance E and F are finite dimensional Hilbert spaces, with the respective dimension n and m, then the dimension of $E \bigotimes F$ is no less than $n.m$, while the dimension of $E \times F$ is only $n + m$. Furthermore, one has the *injective* mapping

$$(1.1) \qquad E \times F \ni (x, y) \longmapsto x \otimes y \in E \bigotimes F$$

and for convenience of notation, we shall often identify the Cartesian product $E \times F$ with its image through the injective mapping (1.1) in the tensor product $E \bigotimes F$.

Now, the considerable number of composite states in

$$(1.2) \qquad (\, E \bigotimes F \,) \setminus (\, E \times F \,)$$

which therefore do *not* correspond to any pair of component states $x \in E$ and $y \in F$, and thus provide a typical quantum convenient feature that does not existent in the classical situation. This wealth of composite states

(1.3) $\quad z \in (\, E \bigotimes F \,) \setminus (\, E \times F \,)$

which are, therefore, *not* of the form

(1.4) $\quad z = x \otimes y$

for any $x \in E$ and $y \in F$, are called *entangled*. And as noted lately, they constitute a major and critically important resource in quantum computation and information.

In usual terms in quantum physics, *entanglement* is a property of *composite states*

(1.5) $\quad z = \sum_{1 \leq i \leq k} x_i \otimes y_i \in (\, E \bigotimes F \,) \setminus (\, E \times F \,)$

where $x_1, \ldots, x_k \in E, y_1, \ldots, y_k \in F$, of *not* being of the particular simple form (1.4), thus of not admitting $k = 1$ in (1.5), in other words, of not belonging to $E \times F$.

As for *correlations*, they are about the various x_i and y_j in (1.5) which are involved in a situation that is not like in (1.4).
In other words, correlation of states $x_i \in E$ and $y_j \in F$ can only occur *relative* to a composite state $z \in E \bigotimes F$ in which these states are components.

By the way, $x \in E$ and $y \in F$ in (1.4) are *not* correlated in the composite state $z \in E \bigotimes F$.

2. Is Tensor Product the Only Non-Cartesian Composition of Physical Systems ?

Tensor product may so far be the only known way non-Cartesian system compose, yet it has a rather strange history in quantum theory. The following citation from a recent paper, [2], of several physicists involved in state of the art research in quanta may be quite illustrative in this regard :

"In the beginning of modern quantum theory, the notion of entanglement was first noted by Einstein, et.al., [3], and by Schrödinger, [8]. While in those days quantum entanglement and its predicted physical consequences were - at least partially - considered as an un-physical property of the formalism - a 'paradox' - the modern perspective on this issue is very different. Now quantum entanglement is seen as an experimentally verified property of nature, that provides a resource for a vast variety of novel phenomena and concepts such as quantum computation, quantum cryptography, or quantum teleportation."

And the fact remains that, even at present, by far most of the 101 Quantum Mechanics courses do not mention entanglement ...

As for the possibility of the existence of physical systems which are non-Cartesian and at the same time non-quantum either, a surprising behaviour of a classical system presented in [1] may be worth attention.

In order to make a few first preparatory steps towards the study of the possible existence of physical systems which are non-Cartesian and at the same time non-quantum, in [4-7] a study was started on the deeper, simpler and more general roots of tensor products, and therefore, of entanglement as well. And as it turns out, mathematically there are indeed considerable generalizations of tensor products and entanglement, far beyond all algebraic structures. In particular, any finite or infinite family of sets of state spaces E_λ, with $\lambda \in \Lambda$, can be composed in infinitely many different rather natural ways, and according to one's choice, into a generalized tensor product $\bigotimes_{\lambda \in \Lambda} E_\lambda$, and the generalization of (1.1) holds, with an injective mapping

$$(2.1) \qquad \prod_{\lambda \in \Lambda} E_\lambda \ni (x_\lambda)_{\lambda \in \Lambda} \longmapsto \otimes_{\lambda \in \Lambda} x_\lambda \in \bigotimes_{\lambda \in \Lambda} E_\lambda$$

Furthermore, as in (1.2), there is in general a large amount of composite states in

$$(2.2) \qquad (\bigotimes_{\lambda \in \Lambda} E_\lambda) \setminus (\prod_{\lambda \in \Lambda} E_\lambda)$$

which thus do not correspond to any single family of component states $(x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} E_\lambda$. Therefore, the composite states in (2.2) can be seen as the generalization of the usual entangled states in the sense of quanta.

We note that in (2.1) and (2.2), and similar to section 1, we made the identification of the Cartesian product $\prod_{\lambda \in \Lambda} E_\lambda$ with its image in $\bigotimes_{\lambda \in \Lambda} E_\lambda$ through the injective mapping in (2.1).

3. Open Problem

The above lead to the following

Open Problem :

> Are there physical systems whose state spaces do not compose according to either the Cartesian product, as classical systems do, or the usual tensor product, as quantum systems do ?

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