New perspectives on the classical theory of motion, interaction and geometry of space-time

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Abstract

By examining the theory of relativity, as originally proposed by Lorentz, Poincare and Einstein, a fundamental theory of general motion is developed. From this, the relationship between space-time and matter is discovered. As a result, the geometrical theory of interaction is introduced. The corresponding geometrical theory of electrodynamics resolves the origin of electromagnetic interaction, as a vortex-like field, and clarifies some of the existing ambiguities.

1. Introduction

Poincare's theory of relativity explains the physical meaning of the Lorentz transformation among inertial systems by unification of space-time. Although it shows a relationship between pure Lorentz transformation and hyperbolic rotation, it does not specify what is rotating. This is the origin of most troubles in the theory of relativity and electrodynamics. For example, although the Maxwellian theory of electrodynamics is the most understood among the theories of fundamental forces, the electromagnetic interaction, called the Lorentz force, is not a direct consequence of Maxwell's equations. It has to be postulated in an independent manner, which is the manifest of incompleteness of the theory. Although it has been noted that the electromagnetic field strength tensor and Lorentz force are both a natural consequence of the geometric structure of Minkowskian space-time, its fundamental meaning has not been discovered.

Another trouble is the magnetic monopole whose existence is apparently compatible with fully symmetrized Maxwell's equations. It seems only modification of Maxwell's equations suffice to allow magnetic charges in electrodynamics. However, no magnetic monopole has been found to this date.

To resolve these and other difficulties, we develop a fundamental geometrical theory of motion and interaction, which shows that the Lorentz force and Maxwell's equations are simple geometrical relations based on four-dimensional rotation. It is seen that this geometry is non-Euclidean with interesting consequences. This theory clarifies the relativity of space-time and its relationship with matter. It also revives the idea of the electromagnetic field as vortex motion in a universal entity.

In the following section, we first present the theory of relative inertial systems and kinematics of particles in the framework of Poincare's relativity. Subsequently, in Section 3, we develop the consistent theory of moving particles by exploring the relation between mass and space-time. This resolves the troubles in Poincare's relativity by clarifying the origin of the governing non-Euclidean geometry.

Afterwards, in Section 4, we develop the geometrical theory of fundamental interaction, which shows that a Lorentz-like force as a rotational effect is an essential character of every fundamental interaction. Therefore, every fundamental interaction is specified by a four-dimensional vortex-like field. Interestingly, this means a unification of all forces based on the geometrical theory of motion and interaction. In section 5, we demonstrate all the details of this vortex theory for electromagnetic interaction. Therefore, electrodynamics is complete with electric charges and magnetic monopoles do not exist. The geometrical view also clarifies the spin dynamics of charged elementary particles. At the end, it is seen that the corresponding consistent theory of gravity is a generalized

Newtonian gravity. This analogous Maxwellian theory of gravity is also developed in detail in Section 6. A summary and general conclusion is presented in Section 7.

2. Poincare's theory of relative inertial systems

As an inertial reference frame in the Minkowskian space-time, a four-dimensional coordinate system $x_1x_2x_3x_4$ is considered such that $x_1x_2x_3$ is the usual space and x_4 the axis measuring time with imaginary values, such that $x_4 = ict$. By considering the unit four-vector bases

$$
\begin{aligned}
\mathbb{e}_1 &= (1,0,0,0) \\
\mathbb{e}_2 &= (0,1,0,0) \\
\mathbb{e}_3 &= (0,0,1,0) \\
\mathbb{e}_4 &= (0,0,0,1)
\end{aligned}
\tag{2.1}
$$

the space-time position four-vector can be represented by

$$\mathbb{X} = x_\mu \mathbb{e}_\mu \tag{2.2}$$

However, for simplicity we sometimes write

$$\mathbb{X} = (\mathbf{x}, x_4) = (\mathbf{x}, ict) = (x, y, z, ict) \tag{2.3}$$

or even

$$x_\mu = (\mathbf{x}, x_4) \tag{2.4}$$

and also often use x in place of $\mathbb{X}$.

With this convenient elementary notation, we do not need to use covariant and contravariant forms of four-tensors in metric notations. Importantly, it is seen that the non-Euclidean geometry governing motion and interaction is much clearer in this complex number notation. However, all developed theory can be easily presented in any other notation.

The square length of position four-vector is

$$\mathbb{X} \bullet \mathbb{X} = \mathbb{X}^{\mathrm{T}}\mathbb{X} = x_\mu x_\mu = \mathbf{x}^2 - c^2t^2 = x^2 + y^2 + z^2 - c^2t^2 \tag{2.5}$$

where we notice that the same symbol $\mathbb{X}$ also represents the matrix form of $\mathbb{X}$.

A homogeneous Lorentz transformation

$$x'_\mu = \Lambda_{\mu\nu} x_\nu \tag{2.6}$$

is any transformation which leaves the length of the four-vectors invariant

$$x'_\mu x'_\mu = x_\mu x_\mu \tag{2.7}$$

This requires

$$x'_\mu x'_\mu = \Lambda_{\mu\alpha} \Lambda_{\mu\beta} x_\alpha x_\beta \tag{2.8}$$

which leads to the following orthogonality condition on $\mathbf{\Lambda}$

$$\Lambda_{\mu\alpha} \Lambda_{\mu\beta} = \delta_{\alpha\beta} \tag{2.9}$$

As will be seen, we use only first and second order three and four-dimensional tensors. Therefore, for convenience we use the matrix representation for these tensors with the same symbol. Based on this convention, (2.9) can be written in more compact form

$$\mathbf{\Lambda}^T \mathbf{\Lambda} = \mathbb{1} \tag{2.10}$$

This shows that the Lorentz transformation is an orthogonal transformation in the specified four dimensional space-time. Conversely, any transformation, which satisfies this orthogonal condition, is a Lorentz transformation. All of these transformations form a group in the mathematical sense.

What we have is the relation between coordinates of a point or event in two different four-dimensional coordinate systems $x_1 x_2 x_3 x_4$ and $x'_1 x'_2 x'_3 x'_4$. One expects that understanding the meaning of this relation is crucial in developing a theory of space-time and motion.

2.1. Space rotation

A familiar example of a Lorentz transformation is the relative space orientation of two coordinate systems with common origin, which is spatial rotation. In general, for this transformation, we have

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{2.11}$$

where $\mathbf{Q}$ is a constant proper real orthogonal matrix specifying the space rotation of the new reference system relative to the original coordinate system. In this case, the transformation decomposes to

$$\mathbf{x}' = \mathbf{Q}\mathbf{x} \tag{2.12}$$

$$t' = t$$

As an example, for rotation about the z-axis with angle ϕ, we have

$$\mathbf{Q} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

In general, for rotation about an arbitrary axis denoted by unit vector $\mathbf{n}$ with angle ϕ, where

$$\boldsymbol{\varphi} = \phi\mathbf{n} \tag{2.14}$$

we have[1)]

$$Q_{ij} = \delta_{ij} - \varepsilon_{imj} n_m \sin\phi + (1-\cos\phi)(n_i n_j - \delta_{ij}) \tag{2.15}$$

It is convenient to associate an anti-symmetric matrix $\mathbf{R}_\mathbf{w}$ defined by

$$\mathbf{R}_\mathbf{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{2.16}$$

to some axial vector $\mathbf{w} = (w_1, w_2, w_3)$. If $\mathbf{G}$ is an arbitrary vector, then

$$\mathbf{w} \times \mathbf{G} = \mathbf{R}_\mathbf{w}\mathbf{G} \tag{2.17}$$

which is a relation frequently used in this article. Therefore, (2.15) can be written as

$$\mathbf{Q} = \mathbf{1} - \sin\phi \mathbf{R}_\mathbf{n} + (1-\cos\phi)(\mathbf{n}\mathbf{n}^T - \mathbf{1}) \tag{2.18}$$

In terms of elements

$$\mathbf{Q} = \begin{bmatrix} \cos\phi + (1-\cos\phi)n_1^2 & n_1 n_2(1-\cos\phi) + n_3\sin\phi & n_1 n_3(1-\cos\phi) - n_2\sin\phi \\ n_1 n_2(1-\cos\phi) - n_3\sin\phi & \cos\phi + (1-\cos\phi)n_2^2 & n_2 n_3(1-\cos\phi) + n_1\sin\phi \\ n_1 n_3(1-\cos\phi) + n_2\sin\phi & n_2 n_3(1-\cos\phi) - n_1\sin\phi & \cos\phi + (1-\cos\phi)n_3^2 \end{bmatrix} \tag{2.19}$$

By using Cayley-Hamilton theorem, it can be shown that

$$\begin{aligned} \mathbf{Q} &= \exp(\mathbf{R}_\boldsymbol{\varphi}) \\ &= \exp\begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} = \exp\begin{bmatrix} 0 & -n_3\phi & n_2\phi \\ n_3\phi & 0 & -n_1\phi \\ -n_2\phi & n_1\phi & 0 \end{bmatrix} \end{aligned} \tag{2.20}$$

Based on the Euler theorem for the three-dimensional motion of a rigid body, every proper orthogonal matrix $\mathbf{Q}$ is equivalent to a rotation about an axis [1]. This means that the form given here for $\mathbf{Q}$ is a general form. In practice, the Euler angles are widely used to represent the rotation matrix $\mathbf{Q}$ [1].

It should be noticed that the relations for base unit space three-vectors are

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j \tag{2.21}$$

$$\mathbf{e}_i = Q_{ji}\mathbf{e}'_j \tag{2.22}$$

with

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{1} \tag{2.23}$$

It is obvious that for the four-dimensional base vectors we have

$$\mathbb{e}'_\mu = \Lambda_{\mu\nu}\mathbb{e}_\nu \tag{2.24}$$

$$\mathbb{e}_\mu = \Lambda_{\nu\mu}\mathbb{e}'_\nu \tag{2.25}$$

It should be noticed that

$$\mathbb{e}'_4 = \mathbb{e}_4 = (0,0,0,1) \tag{2.26}$$

which means the new time coordinate is the same as the old one.

Let the real orthogonal matrix transforming from $\mathbf{x}$ to $\mathbf{x}'$ be designated by $\mathbf{Q}_1$

$$\mathbf{x}' = \mathbf{Q}_1\mathbf{x} \tag{2.27}$$

and the second orthogonal matrix from $\mathbf{x}'$ to $\mathbf{x}''$ be $\mathbf{Q}_2$

$$\mathbf{x}'' = \mathbf{Q}_2\mathbf{x}' \tag{2.28}$$

Hence the matrix of complete transformation $\mathbf{Q}$ from $\mathbf{x}$ to $\mathbf{x}''$

$$\mathbf{x}'' = \mathbf{Q}\mathbf{x} \tag{2.29}$$

is

$$\mathbf{Q} = \mathbf{Q}_2\mathbf{Q}_1 \tag{2.30}$$

In general the rotations are not commutative. In other words, the rotation vectors $\boldsymbol{\varphi}_1 = \phi_1\mathbf{n}_1$ and $\boldsymbol{\varphi}_2 = \phi_2\mathbf{n}_2$ do not follow the Euclidean vector summations. It can be shown that

$$\cos\frac{\phi}{2} = \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\mathbf{n_1} \bullet \mathbf{n_2} \tag{2.31}$$

$$\sin\frac{\phi}{2}\mathbf{n} = \sin\frac{\phi_1}{2}\cos\frac{\phi_2}{2}\mathbf{n_1} + \sin\frac{\phi_2}{2}\cos\frac{\phi_1}{2}\mathbf{n}_2 + \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\mathbf{n_1} \times \mathbf{n_2} \tag{2.32}$$

These relations are more conveniently derived, if a quaternion representation of rotations or unimodular representation with Cayley-Klein parameters is used [1]. It is seen that the summation of half vector of rotations $\frac{1}{2}\boldsymbol{\varphi}_1$ and $\frac{1}{2}\boldsymbol{\varphi}_2$ obey the rules of spherical geometry. The triangle representing these vectors can be considered as a spherical triangle on a unit sphere, with the angle opposite to elliptic vector $\frac{1}{2}\boldsymbol{\varphi}$ given by the angle between the two axes of rotation. Therefore, the vectorial representation of spatial rotation is governed by an elliptic type of non-Euclidean geometry. However, for infinitesimal rotations, this geometry reduces to Euclidean geometry, where the infinitesimal rotation vectors commute.

2.2. Boost

The other important form of Lorentz transformation is a pure Lorentz transformation or boost specified with relative velocity $\mathbf{v}$. The boost parameter or rapidity ξ is defined by

$$\tanh\xi = \frac{v}{c} \tag{2.33}$$

The inversion of this relation gives

$$\xi = \tanh^{-1}\left(\frac{v}{c}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right) = \frac{v}{c} + \frac{1}{3}\left(\frac{v}{c}\right)^3 + \frac{1}{5}\left(\frac{v}{c}\right)^5 + \frac{1}{7}\left(\frac{v}{c}\right)^7 + \cdots \tag{2.34}$$

The vector rapidity also can be considered as

$$\boldsymbol{\xi} = \xi\mathbf{e}_t \tag{2.35}$$

where $\mathbf{e}_t$ is the unit vector in the direction of $\mathbf{v}$. Here, we emphasize the use of rapidity ξ as an essential parameter.

A simple example of a boost is the boost along the x-axis, for which

$$\mathbf{\Lambda} = \begin{pmatrix} \cosh\xi & 0 & 0 & i\sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \tag{2.36}$$

which is usually written as

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{2.37}$$

with $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$, where

$$\gamma = \cosh\xi \tag{2.38}$$

$$\beta\gamma = \sinh\xi \tag{2.39}$$

The structure of this transformation tensor is reminiscent of a rotation tensor, but with hyperbolic functions instead of circular. Interestingly, we can define

$$\psi = i\xi \tag{2.40}$$

where

$$\sin\psi = i\sinh\xi \tag{2.41}$$

$$\cos\psi = \cosh\xi \tag{2.42}$$

Therefore, the transformation matrix can be written as

$$\mathbf{\Lambda} = \begin{pmatrix} \cos\psi & 0 & 0 & \sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\psi & 0 & 0 & \cos\psi \end{pmatrix} \tag{2.43}$$

One can realize this is a rotation with imaginary angle $\psi = i\xi$ representing the deviation of plane $x_1'x_4'$ relative to plane x_1x_4. Analogous to the spatial rotation, the base four-vectors of the new system are

$$\begin{aligned} \mathbb{e}_1' &= (\cosh\xi, 0, 0, i\sinh\xi) \\ \mathbb{e}_2' &= (0,1,0,0) \\ \mathbb{e}_3' &= (0,0,1,0) \\ \mathbb{e}_4' &= (-i\sinh\xi, 0, 0, \cosh\xi) \end{aligned} \tag{2.44}$$

Therefore, we have

$$\begin{aligned} \cos(\mathbb{e}_1', \mathbb{e}_1) &= \cosh\xi = \cos\psi \\ \cos(\mathbb{e}_1', \mathbb{e}_4) &= i\sinh\xi = \sin\psi \\ \cos(\mathbb{e}_4', \mathbb{e}_1) &= -i\sinh\xi = -\sin\psi \\ \cos(\mathbb{e}_4', \mathbb{e}_4) &= \cosh\xi = \cos\psi \end{aligned} \tag{2.45}$$

Relations (2.45) show that these imaginary and complex angles are

$$(\mathbb{e}_1', \mathbb{e}_1) = i\xi = \psi$$

$$
\begin{gathered}
\left(\mathbb{e}_1', \mathbb{e}_4\right) = -i\xi + \frac{\pi}{2} = -\psi + \frac{\pi}{2} \\
\left(\mathbb{e}_4', \mathbb{e}_1\right) = i\xi + \frac{\pi}{2} = \psi + \frac{\pi}{2} \\
\left(\mathbb{e}_4', \mathbb{e}_4\right) = i\xi
\end{gathered}
\tag{2.46}
$$

For a general boost, which is not parallel to any of coordinate axes, we have

$$
\mathbf{\Lambda} = \begin{bmatrix} \mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix}
\tag{2.47}
$$

By using the Cayley-Hamilton theorem, we can show

$$
\begin{aligned}
\mathbf{\Lambda} &= \exp\begin{bmatrix} \mathbf{0} & i\boldsymbol{\xi} \\ -i\boldsymbol{\xi}^T & 0 \end{bmatrix} = \exp\begin{bmatrix} \mathbf{0} & i\xi\mathbf{e}_t \\ -i\xi\mathbf{e}_t^T & 0 \end{bmatrix} = \\
&= \mathbf{1} + \sinh\xi\begin{bmatrix} \mathbf{0} & i\mathbf{e}_t \\ -i\mathbf{e}_t^T & 0 \end{bmatrix} + \left(\cosh\xi - 1\right)\begin{bmatrix} \mathbf{e}_t\mathbf{e}_t^T & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}
\end{aligned}
\tag{2.48}
$$

In terms of the elements we have

$$
\mathbf{\Lambda} = \begin{bmatrix}
1 + \left(\cosh\xi - 1\right)e_{t1}^{\ 2} & \left(\cosh\xi - 1\right)e_{t1}e_{t2} & \left(\cosh\xi - 1\right)e_{t1}e_{t3} & ie_{t1}\sinh\xi \\
\left(\cosh\xi - 1\right)e_{t2}e_{t1} & 1 + \left(\cosh\xi - 1\right)e_{t2}^{\ 2} & \left(\cosh\xi - 1\right)e_{t2}e_{t3} & ie_{t2}\sinh\xi \\
\left(\cosh\xi - 1\right)e_{t3}e_{t1} & \left(\cosh\xi - 1\right)e_{t3}e_{t2} & 1 + \left(\cosh\xi - 1\right)e_{t3}^{\ 2} & ie_{t3}\sinh\xi \\
-ie_{t1}\sinh\xi & -ie_{t2}\sinh\xi & -ie_{t3}\sinh\xi & \cosh\xi
\end{bmatrix}
\tag{2.49}
$$

Therefore, we expect the base four-vectors of the new system in terms of old ones to be

$$
\begin{aligned}
\mathbb{e}_1' &= \left(1 + (\cosh\xi - 1)e_{t1}^{\ 2},\ (\cosh\xi - 1)e_{t1}e_{t2},\ (\cosh\xi - 1)e_{t1}e_{t3},\ i\sinh\xi e_{t1}\right) \\
\mathbb{e}_2' &= \left((\cosh\xi - 1)e_{t2}e_{t1},\ 1 + (\cosh\xi - 1)e_{t2}^{\ 2},\ (\cosh\xi - 1)e_{t2}e_{t3},\ i\sinh\xi e_{t2}\right) \\
\mathbb{e}_3' &= \left((\cosh\xi - 1)e_{t3}e_{t1},\ 1 + (\cosh\xi - 1)e_{t3}e_{t2},\ 1 + (\cosh\xi - 1)e_{t3}^{\ 2},\ i\sinh\xi e_{t3}\right) \\
\mathbb{e}_4' &= \left(-i\sinh\xi e_{t1}, -i\sinh\xi e_{t2}, -i\sinh\xi e_{t3}, \cosh\xi\right) \\
&= \left(-i\sinh\xi\mathbf{e}_t,\ \cosh\xi\right)
\end{aligned}
\tag{2.50}
$$

It is seen that the angles among new and old axes can be obtained easily. For example, from

$$\cos\left(\mathbb{e}_4', \mathbb{e}_4\right) = \cosh\xi \tag{2.51}$$

we obtain

$$\left(\mathbb{e}_4', \mathbb{e}_4\right) = i\xi = \psi \tag{2.52}$$

This shows the angle between the time axes is specified by rapidity, which is expected.

2.3. General Lorentz transformations

Every homogeneous Lorentz transformation in general can be decomposed into a pure Lorentz transformation $\mathbf{\Lambda}_B$ (boost) and a spatial rotation $\mathbf{\Lambda}_R$ (in either order) [1]. For the case where a Lorentz transformation is represented as the product of a boost

$$\mathbf{\Lambda}_B = \begin{bmatrix} \mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \tag{2.53}$$

from the old system $x_1x_2x_3x_4$ to the intermediate system $y_1y_2y_3y_4$, where

$$y_\mu = \Lambda_{B\mu\nu}x_\nu \tag{2.54}$$

followed by a spatial rotation

$$\mathbf{\Lambda}_R = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.55}$$

from $y_1y_2y_3y_4$ to the new system $x_1'x_2'x_3'x_4'$

$$x_\mu' = \Lambda_{R\mu\nu}y_\nu \tag{2.56}$$

we have the total homogeneous Lorentz transformation

$$\mathbf{\Lambda} = \mathbf{\Lambda}_R\mathbf{\Lambda}_B \tag{2.57}$$

which is

$$\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q} + (\cosh\xi - 1)\mathbf{Q}\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{Q}\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \end{aligned} \tag{2.58}$$

It is obvious the transformations $\mathbf{\Lambda}_R$ and $\mathbf{\Lambda}_B$ are not generally commutative. This is because the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\varphi}$ are non-Euclidean and therefore their addition does not follow the rules of Euclidean geometry.

Now we demonstrate the important property of a pure Lorentz transformation or boost which follows the hyperbolic type of non-Euclidean geometry. Let the pure Lorentz transformation from x_μ to x'_μ be designated by $\mathbf{\Lambda}_1$

$$x'_\mu = \Lambda_{1\,\mu\nu} x_\nu \tag{2.59}$$

and a second Lorentz transformation from x'_μ to x''_μ be $\mathbf{\Lambda}_2$

$$x''_\mu = \Lambda_{2\,\mu\nu} x'_\nu \tag{2.60}$$

Hence the matrix of complete transformation $\mathbf{\Lambda}$ from x_μ to x''_μ

$$x''_\mu = \Lambda_{\mu\nu} x_\nu \tag{2.61}$$

is

$$\mathbf{\Lambda} = \mathbf{\Lambda}_2 \mathbf{\Lambda}_1 \tag{2.62}$$

where

$$\mathbf{\Lambda} = \\ = \begin{bmatrix} \mathbf{1} + (\cosh\xi_2 - 1)\mathbf{e}_{t2}\mathbf{e}_{t2}^T & i\sinh\xi_2 \mathbf{e}_{t2} \\ -i\sinh\xi_2 \mathbf{e}_{t2}^T & \cosh\xi_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh\xi_1 - 1)\mathbf{e}_{t1}\mathbf{e}_{t1}^T & i\sinh\xi_1 \mathbf{e}_{t1} \\ -i\sinh\xi_1 \mathbf{e}_{t1}^T & \cosh\xi_1 \end{bmatrix} \tag{2.63}$$

It is seen that the complete transformation is not in general a pure Lorentz transformation. This transformation is in general form (2.58), where

$$\cosh\xi = \cosh\xi_1 \cosh\xi_2 + \sinh\xi_1 \sinh\xi_2 \mathbf{e}_{t1} \bullet \mathbf{e}_{t2} \tag{2.64}$$

This result is the indication of hyperbolic geometry governing the velocity addition law. This has been noticed and developed extensively by early investigators of relativity such as Varičak [2-4]. It is seen that this non-Euclidean geometry is the origin of the famous Thomas-Wigner rotation, which has been explained by Borel [5]. An account of these investigations can be found in the article by Walter [6]. One can realize that for infinitesimal rapidity vectors, the hyperbolic geometry reduces to Euclidean geometry, where the rapidity or velocity vectors commute.

If the transformations $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are general Lorentz transformations, it is seen that

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{Q}_2 + (\cosh\xi_2 - 1)\mathbf{Q}_2\mathbf{e}_{t2}\mathbf{e}_{t2}{}^T & i\sinh\xi_2\mathbf{Q}_2\mathbf{e}_{t2} \\ -i\sinh\xi_2\mathbf{e}_{t2}^T & \cosh\xi_2 \end{bmatrix}\begin{bmatrix} \mathbf{Q}_1 + (\cosh\xi_1 - 1)\mathbf{Q}_1\mathbf{e}_{t1}\mathbf{e}_{t1}{}^T & i\sinh\xi_1\mathbf{Q}_1\mathbf{e}_{t1} \\ -i\sinh\xi_1\mathbf{e}_{t1}^T & \cosh\xi_1 \end{bmatrix} \tag{2.65}$$

where

$$\cosh\xi = \cosh\xi_1\cosh\xi_2 + \sinh\xi_1\sinh\xi_2\mathbf{e}_{t2}^T\mathbf{Q}_1\mathbf{e}_{t1} \tag{2.66}$$

It will be shown that this relation can be further generalized to accelerating systems.

We can see that the inertial systems are oriented from each other by a four-dimensional rotation. The homogeneous Lorentz transformation just specifies this rotation relative to a fixed inertial system as reference frame. This transformation in general can be decomposed into a pure Lorentz transformation (boost) and a spatial rotation. In a geometrical view, the Lorentz transformation can be specified by a hyperbolic vector $\boldsymbol{\xi}$ representing the hyperbolic angle associated with the boost and an elliptic vector $\boldsymbol{\varphi}$ representing the space angle rotation. The geometry governing these vectors is non-Euclidean as was demonstrated.

In general, the base unit four-vectors of two inertial systems are related by

$$\mathbb{e}'_\mu = \Lambda_{\mu\nu}\mathbb{e}_\nu \tag{2.67}$$

or

$$\mathbb{e}_\mu = \Lambda_{\nu\mu}\mathbb{e}'_\nu \tag{2.68}$$

Therefore, the angles among these directions are such that

$$\cos\left(\mathbb{e}'_\mu, \mathbb{e}_\nu\right) = \Lambda_{\mu\nu} \tag{2.69}$$

$$\cos\left(\mathbb{e}_{\mu}, \mathbb{e}'_{\nu}\right) = \Lambda_{\nu\mu} \tag{2.70}$$

It would be interesting to present a simple general Lorentz transformation. Let this transformation be the product of a boost in the x-direction followed by a spatial rotation around a z-axis

$$\mathbf{\Lambda} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\xi & 0 & 0 & i\sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\xi & 0 & 0 & \cosh\xi \end{bmatrix} \tag{2.71}$$

which can be written as

$$\mathbf{\Lambda} = \begin{bmatrix} \cos\phi\cosh\xi & \sin\phi & 0 & i\cos\phi\sinh\xi \\ -\sin\phi\cosh\xi & \cos\phi & 0 & -i\sin\phi\sinh\xi \\ 0 & 0 & 1 & 0 \\ -i\sinh\xi & 0 & 0 & \cosh\xi \end{bmatrix} \tag{2.72}$$

The base unit four-vectors of the new system are

$$\begin{aligned} \mathbb{e}'_1 &= \left(\cos\phi\cosh\xi,\ \sin\phi,\ 0,\ i\cos\phi\sinh\xi\right) \\ \mathbb{e}'_2 &= \left(-\sin\phi\cosh\xi,\ \cos\phi,\ 0,\ \text{-}\,i\sin\phi\sinh\xi\right) \\ \mathbb{e}'_3 &= \left(0,\ 0,\ 1,\ 0\right) \\ \mathbb{e}'_4 &= \left(-i\sinh\xi,\ 0,\ 0,\ \cosh\xi\right) \end{aligned} \tag{2.73}$$

It is noticed that

$$\begin{aligned} \cos\left(\mathbb{e}'_1, \mathbb{e}_1\right) &= \mathbb{e}'_1 \bullet \mathbb{e}_1 = \cos\phi\cosh\xi \\ \cos\left(\mathbb{e}'_1, \mathbb{e}_2\right) &= \sin\phi \\ \cos\left(\mathbb{e}'_1, \mathbb{e}_3\right) &= 0 \\ \cos\left(\mathbb{e}'_1, \mathbb{e}_4\right) &= i\cos\phi\sinh\xi \end{aligned} \tag{2.74}$$

It is seen that these relations are the result of the addition of non-commutative non-Euclidean vectors $\boldsymbol{\xi}$ and $\boldsymbol{\varphi}$.

What we have demonstrated is the very important character of the set of four-dimensional systems with three real coordinates and one imaginary coordinate. It is seen that these systems are oriented from each other, in a manner which can be represented by a combination of circular and hyperbolic angles. It can be realized that this set is the set of all inertial systems in Poincare's relativity. In this theory, space and time are no longer separated as in Galilean relativity and motion is nothing but rotation. However, Poincare's relativity does not specify what is rotating. Our aim in the following sections is to resolve this fundamental question.

Although we have been using the concept of four-vector and four-tensor repeatedly, we have not given their rigorous definition. Therefore, for future reference, the definition of a four-tensor is provided here. A four-tensor $\mathbb{G}$ of order n is defined as a mathematical object with n indices which has 4^n components $G_{\mu_1\mu_2\cdots\mu_n}$ in a given inertial system and transforms via

$$G'_{\mu_1\mu_2\cdots\mu_n} = \Lambda_{\mu_1\nu_1}\Lambda_{\mu_2\nu_2}\cdots\Lambda_{\mu_n\nu_n}G_{\nu_1\nu_2\cdots\nu_n} \tag{2.75}$$

to a new inertial system. The most important four-tensors are those involved in the theory of electrodynamics, which will be discussed later. For simplicity, we have been using the same symbols such as $\mathbf{x}$, $\mathbf{e}$, $\mathbf{Q}$, $\mathbb{X}$, $\mathbb{e}$ and $\mathbf{\Lambda}$ to represent the matrix form of their corresponding three and four tensors.

3. Fundamental theory of motion

In this section, we develop the theory of accelerating particles, which shows the fundamental relation between space-time and matter. It clarifies the relativity of space-time and shows how an inertial system transforms to other inertial systems. This is nothing but the geometrical theory of interaction. It is seen that the relative motion is the result of the four-dimensional rotation of these systems relative to each other. We start with classical particle kinematics and develop the fundamental theory of motion. The theory of interaction will be discussed in the next section.

3.1. Kinematics of a particle

Let us specify the inertial system $x_1x_2x_3x_4$ as an inertial reference frame. Consider a particle with mass m moving relative to this inertial frame. At any position, the motion may be considered as taking place in the plane that contains the path at this position. This plane is often called the osculating plane. The velocity vector $\mathbf{v}$ is tangent to the path curve in this plane. The acceleration of the particle

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{3.1}$$

lies also in this plane. We can consider a local coordinate system by defining the unit vector $\mathbf{e}_t$ tangent to the curve at this position, the unit vector $\mathbf{e}_n$ in the direction of principal normal to the curve in the osculating plane, and the bi-normal unit vector $\mathbf{e}_b$, which is normal to the osculating plane at the point. The relation

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \tag{3.2}$$

among these vectors holds. In this local (tangential, normal, bi-normal) coordinate system we have

$$\mathbf{v} = v\mathbf{e}_t \tag{3.3}$$

with

$$v = \frac{ds_s}{dt} \tag{3.4}$$

where ds_s is the length of the infinitesimal displacement of the particle on the space curve in time interval dt. For acceleration, we have

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt} \tag{3.5}$$

For the second term, we apply the concept of curvature in the form

$$\frac{d\mathbf{e}_t}{ds_s} = \frac{1}{R_s}\mathbf{e}_n \tag{3.6}$$

where R_s is the radius of curvature at the particle position point. Therefore, the acceleration in terms of tangential and normal components $\mathbf{a}_t$ and $\mathbf{a}_n$ is

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n \tag{3.7}$$

where

$$\mathbf{a}_t = a_t \mathbf{e}_t = \frac{dv}{dt} \mathbf{e}_t \tag{3.8}$$

$$\mathbf{a}_n = a_n \mathbf{e}_n = \frac{v^2}{R_s} \mathbf{e}_n \tag{3.9}$$

It seems the differential geometry governing the kinematics of the particle is more complete if we introduce the concept of torsion of the curve defined by

$$\frac{d\mathbf{e}_b}{ds_s} = -\frac{1}{R_{tor}} \mathbf{e}_n \tag{3.10}$$

where R_{tor} is the radius of torsion of the curve. It can be easily shown that

$$\frac{d\mathbf{e}_n}{ds_s} = -\frac{1}{R_s} \mathbf{e}_n + \frac{1}{R_{tor}} \mathbf{e}_b \tag{3.11}$$

Therefore, the equations for curvature of curve might be written as

$$\begin{bmatrix} \dfrac{d\mathbf{e}_t}{ds_S} \\ \dfrac{d\mathbf{e}_n}{ds_S} \\ \dfrac{d\mathbf{e}_b}{ds_S} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{R_s} & 0 \\ -\dfrac{1}{R_s} & 0 & \dfrac{1}{R_{tor}} \\ 0 & -\dfrac{1}{R_{tor}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_n \\ \mathbf{e}_b \end{bmatrix} \tag{3.12}$$

This relation is called the Frenet-Serret formula in differential geometry. The anti-symmetric tensor on the right hand side possesses the whole information about the curvature and twist of the curve at the point under consideration. However, an interesting interpretation of this relation can be given as follows. The principal directions $\mathbf{e}_t - \mathbf{e}_n - \mathbf{e}_b$ specify a local orthogonal reference system attached to the particle. This reference system rotates as the particle moves on the curve path. It is obvious this relation shows the gradual rotation of this local system with respect to any inertial system. If we write the relation as

$$\begin{bmatrix} \dfrac{d\mathbf{e}_t}{dt} \\ \dfrac{d\mathbf{e}_n}{dt} \\ \dfrac{d\mathbf{e}_b}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{v}{R_s} & 0 \\ -\dfrac{v}{R_s} & 0 & \dfrac{v}{R_{tor}} \\ 0 & -\dfrac{v}{R_{tor}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_n \\ \mathbf{e}_b \end{bmatrix} \tag{3.13}$$

the anti-symmetric tensor is the angular velocity tensor of the rotating local system $\mathbf{e}_t - \mathbf{e}_n - \mathbf{e}_b$. By considering the angular velocity vector

$$\boldsymbol{\omega} = \begin{bmatrix} -\dfrac{v}{R_{tor}} \\ 0 \\ -\dfrac{v}{R_s} \end{bmatrix} \tag{3.14}$$

we obtain the relation

$$\begin{bmatrix} \dfrac{d\mathbf{e}_t}{dt} \\ \dfrac{d\mathbf{e}_n}{dt} \\ \dfrac{d\mathbf{e}_b}{dt} \end{bmatrix} = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_n \\ \mathbf{e}_b \end{bmatrix} \tag{3.15}$$

It is interesting to note that there is no angular velocity component in the $\mathbf{e}_n$ direction.

The Frenet–Serret formulas can be generalized to higher dimensional Euclidean spaces by defining generalized curvatures. It can be shown that in the principal local coordinate system, which is called the Frenet–Serret frame, the anti-symmetric curvature tensor is tri-diagonal [7]. An important analogy will be seen in developing the relativistic theory of motion.

3.2. Relativistic kinematics of a particle

In a relativistic study, the velocity and acceleration of the particle must be defined as four-vectors. However, it is seen that the vectors **v** and **a** are still useful in this development.

The position of a particle in the inertial reference frame describes a path known as the world line. By considering two neighboring events on the world line of the particle with coordinates x_μ and $x_\mu + dx_\mu$, we have

$$dx_\mu = (d\mathbf{x}, icdt) = (\mathbf{v}, ic)dt \tag{3.16}$$

The square length of this infinitesimal four-vector

$$ds^2 = d\mathbb{X} \bullet d\mathbb{X} = dx_\mu dx_\mu = d\mathbf{x}^2 - c^2 dt^2 = -c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \tag{3.17}$$

is the scalar invariant under all Lorentz transformation. It is seen that the imaginary length on the world line is

$$ds = icdt\sqrt{1 - \frac{v^2}{c^2}} \tag{3.18}$$

The proper time between the events $d\tau$ is defined by

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = dt/\gamma \tag{3.19}$$

Therefore,

$$ds = icd\tau \tag{3.20}$$

By using the concept of rapidity

$$\tanh \xi = \frac{v}{c} \tag{2.33}$$

we notice

$$dt = \gamma d\tau = d\tau \cosh \xi \tag{3.21}$$

The unit four-vector tangent to the world line $\mathbb{e}_t$ is defined as

$$\mathbb{e}_t = \frac{d\mathbb{X}}{ds} = \frac{dx_\mu}{ds}\mathbb{e}_\mu \tag{3.22}$$

where

$$\mathbb{e}_t \bullet \mathbb{e}_t = 1 \tag{3.23}$$

The four-vector velocity $\mathbb{U} = u_\mu \mathbb{e}_\mu$ is defined as the rate of change of the position vector of particle $\mathbb{X}$ with respect to its proper time

$$\mathbb{U} = \frac{d\mathbb{X}}{d\tau} \tag{3.24}$$

The space and time components of $\mathbb{U}$

$$u_\mu = (\mathbf{u}, u_4) \tag{3.25}$$

are

$$\mathbf{u} = \gamma\mathbf{v} = \frac{\mathbf{v}}{\sqrt{1-v^2/c^2}} = c\sinh\xi\mathbf{e}_t \tag{3.26}$$

and

$$u_4 = i\gamma c = \frac{ic}{\sqrt{1-v^2/c^2}} = ic\cosh\xi \tag{3.27}$$

Therefore

$$\mathbb{U} = c(\sinh\xi\mathbf{e}_t,\ \mathrm{i}\cosh\xi) \tag{3.28}$$

which can be written as

$$\mathbb{U} = ic\mathbb{e}_t \tag{3.29}$$

with

$$\mathbb{e}_t = (-i\sinh\xi\mathbf{e}_t,\ \cosh\xi) \tag{3.30}$$

The length of the four-vector velocity is a constant since

$$\mathbb{U} \bullet \mathbb{U} = u_\mu u_\mu = \mathbf{u}^2 + u_4{}^2 = -c^2 \tag{3.31}$$

and it is thus time-like. It is seen

$$\mathbb{e}_t \bullet \frac{d\mathbb{e}_t}{ds} = 0 \tag{3.32}$$

which means $\frac{d\mathbb{e}_t}{ds}$ is normal to the world line. By considering the unit four-vector $\mathbb{e}_n$ in this normal direction called the first normal and using the concept of curvature, we have

$$\frac{d\mathbb{e}_t}{ds} = -\frac{1}{R}\mathbb{e}_n \tag{3.33}$$

where R is the world line radius of curvature at the point under consideration. The minus sign is for convenience and it will be justified shortly. It is seen that

$$\frac{d\mathbb{e}_t}{ds} \bullet \frac{d\mathbb{e}_t}{ds} = \frac{1}{R^2} \tag{3.34}$$

The four-acceleration $\mathbb{b} = b_\mu \mathbb{e}_\mu$ is defined as

$$\mathbb{b} = \frac{d\mathbb{u}}{d\tau} = \frac{d^2\mathbb{x}}{d\tau^2} \tag{3.35}$$

which is always perpendicular to the four-vector velocity where,

$$\mathbb{u} \bullet \frac{d\mathbb{u}}{d\tau} = 0 \tag{3.36}$$

It can be easily shown that

$$\mathbb{b} = \left[\gamma^2 \mathbf{a} + \gamma^4 \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c^2}\right)\mathbf{v}, i\gamma^4 \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c}\right)\right] \tag{3.37}$$

The length of four-vector acceleration can be found to be

$$|\mathbb{b}|^2 = b_\mu b_\mu = \gamma^4 a^2 + \gamma^6 \left(\frac{\mathbf{v} \bullet \mathbf{a}}{c}\right)^2 \tag{3.38}$$

Since $b_\mu b_\mu$ is positive, the four-acceleration is space-like. However, it is more appealing to consider the four-acceleration relative to the world line. By using (3.33), we obtain

$$\mathbb{b} = ic\frac{d\mathbb{e}_t}{d\tau} = \frac{c^2}{R}\mathbb{e}_n \tag{3.39}$$

which shows the four acceleration is in direction of first-normal of world line and its value is

$$|\mathbb{b}| = \frac{c^2}{R} \tag{3.40}$$

Now we study the relative motion of a particle A in different inertial systems. Let us consider the inertial systems $x_1x_2x_3x_4$ and $x_1'x_2'x_3'x_4'$, which are related by the Lorentz transformation

$$x'_\mu = \Lambda_{\mu\nu} x_\nu \tag{2.6}$$

where

$$\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q} + (\cosh\xi - 1)\mathbf{Q}\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{Q}\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \end{aligned} \tag{2.58}$$

Assume particle A moves in the first inertial system with $\mathbf{x}_A = \mathbf{x}_A(t)$ and its four-vector velocity is

$$u_{A\mu} = (\mathbf{u}_A, u_{A4}) = c(\sinh\xi_A\mathbf{e}_{tA}, i\cosh\xi_A) \tag{3.41}$$

where

$$\tanh\xi_A = \frac{v_A}{c} \tag{3.42}$$

This particle also moves in the second inertial system with $\mathbf{x}'_A = \mathbf{x}'_A(t')$, such that

$$u'_{A\mu} = (\mathbf{u}'_A, u'_{A4}) = c(\sinh\xi'_A\mathbf{e}'_{tA}, i\cosh\xi'_A) \tag{3.43}$$

where

$$\tanh\xi'_A = \frac{v'_A}{c} \tag{3.44}$$

These four-vector velocities are also related by the tensor transformation

$$u'_\mu = \Lambda_{\mu\nu} u_\nu \tag{3.45}$$

Therefore,

$$\mathbb{U}'_A = \begin{bmatrix} \mathbf{Q} + (\cosh\xi - 1)\mathbf{Q}\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{Q}\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \begin{bmatrix} c\sinh\xi_A\mathbf{e}_{tA} \\ ic\cosh\xi_A \end{bmatrix} \tag{3.46}$$

which gives the relations

$$\sinh\xi'_A\mathbf{e}'_{tA} = \mathbf{Q}[\mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T]\sinh\xi_A\mathbf{e}_{tA} - \sinh\xi\cosh\xi_A\mathbf{Q}\mathbf{e}_t \tag{3.47}$$

$$\cosh\xi'_A = \cosh\xi\cosh\xi_A - \sinh\xi\sinh\xi_A\mathbf{e}_t \bullet \mathbf{e}_{tA} \tag{3.48}$$

The relation (3.48) shows that the velocity addition law is valid even when one of the velocities is not constant. We investigate shortly the validity of this law when all particles are accelerating.

3.3. Motion of particle as a four-dimensional rotation

After reviewing kinematics of a particle, we develop the important character of its motion as a four-dimensional rotation. To show this we consider the motion of the particle as the transformation of its four-velocity vector $\mathbb{U}$ in the inertial reference frame system. Let $\mathbb{U}_0$ be the initial four-vector velocity at position $\mathbb{X}_0$, such that $\mathbb{U}(\mathbb{X}_0) = \mathbb{U}_0$. We can consider the transformation

$$u_\mu(x) = L_{\mu\nu}(x, x_0)u_\nu(x_0) \tag{3.49}$$

where the transformation tensor $L_{\mu\nu}(x, x_0)$ depends on the current position of the particle. This relation can be written as

$$\mathbb{U}(x) = \mathbb{L}(x, x_0)\mathbb{U}_0 \tag{3.50}$$

Since the length of the four-vector velocity is constant, we have

$$u_\mu(x)u_\mu(x) = u_\mu(x_0)u_\mu(x_0) = -c^2 \tag{3.51}$$

Therefore,

$$[L_{\mu\alpha}(x, x_0)L_{\mu\beta}(x, x_0) - \delta_{\alpha\beta}]u_\alpha(x_0)u_\beta(x_0) = 0 \tag{3.52}$$

This requires the orthogonality condition

$$L_{\mu\alpha}(x, x_0)L_{\mu\beta}(x, x_0) = \delta_{\alpha\beta} \tag{3.53}$$

It is seen that although $\mathbb{L}(x, x_0)$ looks similar to a Lorentz transformation among inertial systems, it varies with motion of the particle.

Meanwhile, the inverse relation is

$$\mathbb{u}(x_0) = \mathbb{L}^T(x, x_0)\mathbb{u}(x) \tag{3.54}$$

which in terms of components is

$$u_\mu(\mathbb{X}_0) = L_{\nu\mu}(x, x_0)u_\nu(x) \tag{3.55}$$

This gives the orthogonality condition in the form

$$\mathbb{L}(x, x_0)\mathbb{L}^T(x, x_0) = \mathbb{1} \tag{3.56}$$

or

$$L_{\mu\alpha}(x, x_0)L_{\nu\alpha}(x, x_0) = \delta_{\mu\nu} \tag{3.57}$$

By taking the derivative of (3.57) with respect to the proper time of the particle, we obtain

$$\frac{dL_{\mu\alpha}(x, x_0)}{d\tau}L_{\nu\alpha}(x, x_0) + L_{\mu\alpha}(x, x_0)\frac{dL_{\nu\alpha}(x, x_0)}{d\tau} = 0 \tag{3.58}$$

Now by defining the four-tensor

$$\Omega_{\mu\nu}(x) = \frac{dL_{\mu\alpha}(x, x_0)}{d\tau}L_{\nu\alpha}(x, x_0) \tag{3.59}$$

we can see that the relation (3.58) becomes

$$\Omega_{\mu\nu}(x) + \Omega_{\nu\mu}(x) = 0 \tag{3.60}$$

or

$$\mathbf{\Omega}(x) + \mathbf{\Omega}^T(x) = \mathbf{0} \tag{3.61}$$

which shows $\Omega_{\mu\nu}(x)$ is an anti-symmetric four-tensor. In compact form, we have

$$\mathbf{\Omega}(x) = \frac{d\mathbb{L}(x, x_0)}{d\tau}\mathbb{L}^T(x, x_0) \tag{3.62}$$

By multipling the relation (3.59) with $L_{\nu\beta}(x,x_0)$ and using the orthogonality condition, we obtain

$$\frac{dL_{\mu\nu}(x,x_0)}{d\tau}=\Omega_{\mu\alpha}(x)L_{\alpha\nu}(x,x_0) \tag{3.63}$$

This may readily be written as

$$\frac{d\mathbb{L}(x,x_0)}{d\tau}=\Omega(x)\mathbb{L}(x,x_0) \tag{3.64}$$

Now the acceleration from the original transformation relation

$$u_\mu(x)=L_{\mu\nu}(x,x_0)u_\nu(x_0) \tag{3.49}$$

is

$$\frac{du_\mu(x)}{d\tau}=\frac{dL_{\mu\nu}(x,x_0)}{d\tau}u_\nu(x_0) \tag{3.65}$$

By substituting from (3.63), we have

$$\frac{du_\mu(x)}{d\tau}=\Omega_{\mu\alpha}(x)L_{\alpha\nu}(x,x_0)u_\nu(x_0) \tag{3.66}$$

which reduces to the relation

$$\frac{du_\mu(x)}{d\tau}=\Omega_{\mu\alpha}(x)u_\alpha(x) \tag{3.67}$$

This is the relation between four acceleration $\frac{du_\mu(x)}{d\tau}$ and four-velocity at each point on the world line. It should be noticed that the relation (3.67) is actually (3.35) and (3.37) written as a transformation.

It is also noticed that the relation (3.67) is similar to the non-relativistic relation for rate of change of a constant length vector $\mathbf{G}$ attached to a rotating system

$$\frac{d\mathbf{G}}{dt}=\boldsymbol{\omega}\times\mathbf{G} \tag{3.68}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of that rotating system. It is remembered that the components ω_1, ω_2 and ω_3 are the angular velocities of the body system in the yz, zx and xy planes of an inertial frame. Because of the importance of (3.68), it is advantageous to demonstrate the mathematical details of its derivation. Let the prime system to be the body system. Then the components of this vector $\mathbf{G}'$ are constant in this system. Therefore, we have

$$\mathbf{G}' = \mathbf{Q}(t)\mathbf{G}(t) \tag{3.69}$$

where $\mathbf{Q}(t)$ is the orthogonal rotation matrix. This relation can be written as

$$\mathbf{G}(t) = \mathbf{Q}^T(t)\mathbf{G}' \tag{3.70}$$

The rate of change of the vector $\mathbf{G}(t)$ relative to the fixed reference frame is

$$\frac{d\mathbf{G}}{dt} = \frac{d\mathbf{Q}^T(t)}{dt}\mathbf{G}' \tag{3.71}$$

After eliminating $\mathbf{G}'$ by using (3.69), we obtain

$$\frac{d\mathbf{G}}{dt} = \frac{d\mathbf{Q}^T}{dt}\mathbf{Q}\mathbf{G} \tag{3.72}$$

Now by defining the tensor

$$\mathbf{W} = \frac{d\mathbf{Q}^T}{dt}\mathbf{Q} \tag{3.73}$$

we have

$$\frac{d\mathbf{G}}{dt} = \mathbf{W}\mathbf{G} \tag{3.74}$$

which in the index notation can be written as

$$\frac{dG_i}{dt} = W_{ij}G_j \tag{3.75}$$

Now by differentiating the orthogonality condition

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{1} \tag{3.76}$$

with respect to the time, we obtain

$$\frac{d\mathbf{Q}^T}{dt}\mathbf{Q}+\mathbf{Q}^T\frac{d\mathbf{Q}}{dt}=\mathbf{0} \tag{3.77}$$

which may readily be written as

$$\mathbf{W}+\mathbf{W}^T=\mathbf{0} \tag{3.78}$$

This relation shows that the tensor $\mathbf{W}$ is anti-symmetric. This tensor is the known angular velocity tensor of the rotating system relative to the inertial system. In terms of elements, this tensor is

$$\mathbf{W}=\mathbf{R}_{\boldsymbol{\omega}}=\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{3.79}$$

Therefore, the relation (3.74) is the other form of (3.68)

$$\frac{d\mathbf{G}}{dt}=\boldsymbol{\omega}\times\mathbf{G} \tag{3.68}$$

It should be noticed that the Frenet-Serret formula (3.15) is the application of this equation for fundamental base unit vectors.

Now, we have a remarkable analogy for

$$\frac{du_\mu(x)}{d\tau}=\Omega_{\mu\alpha}(x)u_\alpha(x) \tag{3.67}$$

with

$$\frac{dG_i}{dt}=W_{ij}G_j \tag{3.75}$$

It is seen that the four-vector acceleration $\dfrac{du_\mu}{d\tau}$ is the result of continuous rotation of the four-vector velocity u_μ in a four-dimensional sense. Therefore, it seems u_μ is attached to a four-dimensional system $x_1'x_2'x_3'x_4'$ in the $x_4'=ict'$ direction, where

$$u_\mu'=(0,0,0,ic) \tag{3.80}$$

and this system is rotating with four-dimensional angular velocity $\Omega_{\mu\nu}$ relative to the inertial system, such that

$$\frac{dL_{\mu\nu}(x, x_0)}{d\tau} = \Omega_{\mu\alpha}(x) L_{\alpha\nu}(x, x_0) \tag{3.63}$$

Therefore, we have discovered that there is a fundamental relation between space-time and matter. A massive particle specifies a local four-dimensional orthogonal system with three real axes and one imaginary axis. Within this local $x_1' x_2' x_3' x_4'$ space-time system, the particle has attached four-velocity with magnitude c in the time direction. The rotation of this space-time or four-dimensional system generates motion of the particle relative to the inertial system. This rotation is represented by the four-tensor angular velocity $\mathbf{\Omega} = \Omega_{\mu\nu} \mathbf{e}_\mu \mathbf{e}_\nu$ in the inertial reference frame. The nature of this four-dimensional angular velocity is explored very shortly.

As was mentioned above, at any point on the world line, we have the transformation

$$u'_\mu = \Lambda_{\mu\nu}(x) u_\nu(x) \tag{3.81}$$

where the varying tensor transformation $\Lambda_{\mu\nu}(x)$ looks like a Lorentz transformation. Therefore, there must be a relation between tensors $\Lambda_{\mu\nu}(x)$ and $L_{\mu\nu}(x)$. For a particle at an initial point x_0, we have

$$u'_\mu = \Lambda_{\mu\nu}(x_0) u_\nu(x_0) \tag{3.82}$$

Therefore

$$\Lambda_{\mu\nu}(x) u_\nu(x) = \Lambda_{\mu\nu}(x_0) u_\nu(x_0) \tag{3.83}$$

By substituting for $u_\nu(x)$ from (3.49), we obtain

$$\Lambda_{\mu\nu}(x) L_{\nu\alpha}(x, x_0) u_\alpha(x_0) = \Lambda_{\mu\alpha}(x_0) u_\alpha(x_0) \tag{3.84}$$

This shows the relation

$$\Lambda_{\mu\nu}(x) L_{\nu\alpha}(x, x_0) = \Lambda_{\mu\alpha}(x_0) \tag{3.85}$$

which can be written as

$$\mathbf{\Lambda}(x) \mathbb{L}(x, x_0) = \mathbf{\Lambda}(x_0) \tag{3.86}$$

Therefore

$$\mathbb{L}(x, x_0) = \mathbf{\Lambda}^T(x) \mathbf{\Lambda}(x_0) \tag{3.87}$$

It should be noticed that although $\mathbf{\Lambda}(x)$ is not constant any more, it follows the general form of Lorentz transformation (2.58)

$$\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \mathbf{Q}_P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh \xi - 1) \mathbf{e}_{tP} \mathbf{e}_{tP}^T & i \sinh \xi_P \mathbf{e}_{tP} \\ -i \sinh \xi_P \mathbf{e}_{tP}^T & \cosh \xi_P \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_P + (\cosh \xi_P - 1) \mathbf{Q}_P \mathbf{e}_{tP} \mathbf{e}_{tP}^T & i \sinh \xi_P \mathbf{Q}_P \mathbf{e}_{tP} \\ -i \sinh \xi_P \mathbf{e}_{tP}^T & \cosh \xi_P \end{bmatrix} \end{aligned} \tag{3.88}$$

where the physical meaning of the parameters in $\boldsymbol{\xi}_P$ and $\mathbf{Q}_P$ has not been specified. However, we can explore their relation with the motion of the particle in the course of our development. By using the relation (3.81), we obtain

$$u_\mu = (c \sinh \xi_P \mathbf{e}_{tP}, ic \cosh \xi_P) \tag{3.89}$$

This shows the vector $\boldsymbol{\xi}_P$ is actually the rapidity vector $\boldsymbol{\xi}$ of the particle. Therefore,

$$\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \mathbf{Q}_P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh \xi - 1) \mathbf{e}_t \mathbf{e}_t^T & i \sinh \xi \mathbf{e}_t \\ -i \sinh \xi \mathbf{e}_t^T & \cosh \xi \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_P + (\cosh \xi - 1) \mathbf{Q}_P \mathbf{e}_t \mathbf{e}_t^T & i \sinh \xi \mathbf{Q}_P \mathbf{e}_t \\ -i \sinh \xi \mathbf{e}_t^T & \cosh \xi \end{bmatrix} \end{aligned} \tag{3.90}$$

It is seen that the position vector $\mathbf{x} = \mathbf{x}(t)$ of the particle does not specify its relative position in the reference inertial frame completely. It is also necessary to specify its body frame orientation $\mathbf{\Lambda} = \mathbf{\Lambda}(t)$ relative to this frame. However, the rapidity vector $\boldsymbol{\xi}(t) = \xi \mathbf{e}_t$ is obtained from the velocity vector $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$. Therefore, the position vector $\mathbf{x} = \mathbf{x}(t)$ and orthogonal matrix $\mathbf{Q}_P = \mathbf{Q}_P(t)$ completely specify the particle position.

Now, we investigate the character of the anti-symmetric tensor $\mathbf{\Omega} = \Omega_{\mu\nu}\mathbb{e}_{\mu}\mathbb{e}_{\nu}$. The initial four-vector velocity is

$$\mathbb{U}_0 = c\left(\sinh\xi_0\mathbf{e}_{t0},\ \mathrm{i}\cosh\xi_0\right) \tag{3.91}$$

For simplicity we take the space coordinates of the initial body frame to be parallel to the stationary inertial frame, where $\mathbf{Q}_P(t_0) = \mathbf{1}$. Therefore,

$$\mathbf{\Lambda}_0 = \begin{bmatrix} \mathbf{1} + (\cosh\xi_0 - 1)\mathbf{e}_{t0}\mathbf{e}_{t0}^T & i\sinh\xi_0\mathbf{e}_{t0} \\ -i\sinh\xi_0\mathbf{e}_{t0}^T & \cosh\xi_0 \end{bmatrix} \tag{3.92}$$

By taking the derivative with respect to the proper time τ in the equation

$$\mathbb{L}(x, x_0) = \mathbf{\Lambda}^T(x)\mathbf{\Lambda}(x_0) \tag{3.87}$$

we obtain

$$\frac{d\mathbb{L}(x, x_0)}{d\tau} = \frac{d\mathbf{\Lambda}^T(x)}{d\tau}\mathbf{\Lambda}(x_0) \tag{3.93}$$

Therefore, the relation

$$\mathbf{\Omega}(x) = \frac{d\mathbb{L}(x, x_0)}{d\tau}\mathbb{L}^T(x, x_0) \tag{3.94}$$

becomes

$$\mathbf{\Omega}(x) = \frac{d\mathbf{\Lambda}^T(x)}{d\tau}\mathbf{\Lambda}(x_0)\mathbf{\Lambda}^T(x_0)\mathbf{\Lambda}(x) \tag{3.95}$$

and finally we have

$$\mathbf{\Omega}(x) = \frac{d\mathbf{\Lambda}^T(x)}{d\tau}\mathbf{\Lambda}(x) \tag{3.96}$$

For $\mathbf{\Lambda}^T(x)$ from (3.90)

$$\mathbf{\Lambda}^T = \begin{bmatrix} \mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & -i\sinh\xi\mathbf{e}_t \\ i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix}\begin{bmatrix} \mathbf{Q}_P{}^T & 0 \\ 0 & 1 \end{bmatrix} \tag{3.97}$$

By taking the derivative with respect to the proper time, we obtain

$$\begin{aligned} \frac{d\mathbf{\Lambda}^T}{d\tau} = & \begin{bmatrix} \sinh\xi\mathbf{e}_t\mathbf{e}_t^T\dfrac{d\xi}{d\tau}+(\cosh\xi-1)\left(\dfrac{d\mathbf{e}_t}{dt}\mathbf{e}_t^T+\mathbf{e}_t\dfrac{d\mathbf{e}_t^T}{dt}\right) & -i\cosh\xi\mathbf{e}_t\dfrac{d\xi}{d\tau}-i\sinh\xi\dfrac{d\mathbf{e}_t}{d\tau} \\ i\cosh\xi\mathbf{e}_t^T\dfrac{d\xi}{d\tau}+i\sinh\xi\dfrac{d\mathbf{e}_t^T}{d\tau} & \sinh\xi\dfrac{d\xi}{d\tau} \end{bmatrix}\begin{bmatrix} \mathbf{Q}_p{}^T & 0 \\ 0 & 1 \end{bmatrix} \\ & +\begin{bmatrix} \mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & -i\sinh\xi\mathbf{e}_t \\ i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix}\begin{bmatrix} \dfrac{d\mathbf{Q}_P{}^T}{d\tau} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{3.98}$$

Therefore, for $\mathbf{\Omega}(x)$ in (3.96), we have

$$\begin{aligned} \mathbf{\Omega} = & \\ & \begin{bmatrix} \sinh\xi\dfrac{d\xi}{d\tau}\mathbf{e}_t\mathbf{e}_t^T+(\cosh\xi-1)\left(\dfrac{d\mathbf{e}_t}{dt}\mathbf{e}_t^T+\mathbf{e}_t\dfrac{d\mathbf{e}_t^T}{dt}\right) & -i\left(\cosh\xi\mathbf{e}_t\dfrac{d\xi}{d\tau}+\sinh\xi\dfrac{d\mathbf{e}_t}{d\tau}\right) \\ i\left(\cosh\xi\mathbf{e}_t^T\dfrac{d\xi}{d\tau}+\sinh\xi\dfrac{d\mathbf{e}_t^T}{d\tau}\right) & \sinh\xi\dfrac{d\xi}{d\tau} \end{bmatrix}\times \\ & \times\begin{bmatrix} \mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \\ & +\begin{bmatrix} \mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & -i\sinh\xi\mathbf{e}_t \\ i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix}\begin{bmatrix} \dfrac{d\mathbf{Q}_P{}^T}{d\tau} & 0 \\ 0 & 0 \end{bmatrix}\times \\ & \times\begin{bmatrix} \mathbf{Q}_P & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{1}+(\cosh\xi-1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \end{aligned} \tag{3.99}$$

Now by using the relation

$$\mathbf{R}_{\boldsymbol{\omega}_P} = \frac{d\mathbf{Q}_P{}^T}{d\tau}\mathbf{Q}_P \tag{3.100}$$

where $\boldsymbol{\omega}_P$ is an angular velocity vector in a mathematical sense, we obtain

$$
\mathbf{\Omega} = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega}_P} + (\cosh\xi - 1)\left(\mathbf{e}_t \dfrac{d\mathbf{e}_t^T}{dt} - \dfrac{d\mathbf{e}_t}{dt}\mathbf{e}_t^T\right) & -i\left(\dfrac{d\xi}{d\tau}\mathbf{e}_t + \sinh\xi \dfrac{d\mathbf{e}_t}{dt} - \sinh\xi \mathbf{R}_{\boldsymbol{\omega}_P}\mathbf{e}_t\right) \\ i\left(\dfrac{d\xi}{d\tau}\mathbf{e}_t^T + \sinh\xi \dfrac{d\mathbf{e}_t^T}{dt} - \sinh\xi \mathbf{e}_t^T \mathbf{R}_{\boldsymbol{\omega}_P}^T\right) & 0 \end{bmatrix} \tag{3.101}
$$

This relation can be written in the form

$$
\mathbf{\Omega} = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega}} & -i\dfrac{1}{c}\boldsymbol{\eta} \\ i\dfrac{1}{c}\boldsymbol{\eta}^T & 0 \end{bmatrix} \tag{3.102}
$$

where

$$
\mathbf{R}_{\boldsymbol{\omega}} = \mathbf{R}_{\boldsymbol{\omega}_P} + (\cosh\xi - 1)\left(\mathbf{e}_t \frac{d\mathbf{e}_t^T}{dt} - \frac{d\mathbf{e}_t}{dt}\mathbf{e}_t^T\right) \tag{3.103}
$$

$$
\frac{1}{c}\boldsymbol{\eta} = \frac{d\xi}{d\tau}\mathbf{e}_t + \sinh\xi\left(\frac{d\mathbf{e}_t}{d\tau} - \mathbf{R}_{\boldsymbol{\omega}_P}\mathbf{e}_t\right) \tag{3.104}
$$

One can see these relations can also be written as

$$
\boldsymbol{\omega} = \boldsymbol{\omega}_P + (\cosh\xi - 1)\frac{d\mathbf{e}_t}{d\tau} \times \mathbf{e}_t \tag{3.105}
$$

$$
\frac{1}{c}\boldsymbol{\eta} = \frac{d\xi}{d\tau}\mathbf{e}_t + \sinh\xi\left(\frac{d\mathbf{e}_t}{d\tau} - \boldsymbol{\omega}_P \times \mathbf{e}_t\right) \tag{3.106}
$$

These relations can be simplified further by using the relations

$$
\frac{d\mathbf{e}_t}{d\tau} = \frac{v}{R_s}\cosh\xi \mathbf{e}_n \tag{3.107}
$$

and

$$
\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \tag{3.2}
$$

The four-tensor $\mathbf{\Omega}(x)$ in terms of elements in the inertial reference frame is

$$
\mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & -i\eta_1/c \\ \omega_3 & 0 & -\omega_1 & -i\eta_2/c \\ -\omega_2 & \omega_1 & 0 & -i\eta_3/c \\ i\eta_1/c & i\eta_2/c & i\eta_3/c & 0 \end{bmatrix} \tag{3.108}
$$

This is the general form of an anti-symmetric four-tensor angular velocity $\Omega_{\mu\nu}$. It should be noticed that the elements $\Omega_{4i} = -\Omega_{i4} = i\eta_i / c$ are imaginary. It is observed that the angular velocities ω_1, ω_2 and ω_3 in xy, yz and zx planes generate space rotation of the body frame; the imaginary angular velocities $\frac{i}{c}\eta_1$, $\frac{i}{c}\eta_2$ and $\frac{i}{c}\eta_3$ in xt , yt and zt planes generate boost of the body frame. Therefore, the space-time body frame system rotates relative to the inertial system with angular velocity tensor $\Omega_{\mu\nu}$, which is a combination of elliptic and hyperbolic angular velocities $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$.

Returning to the equation for four-acceleration

$$\frac{du_\mu(x)}{d\tau} = \Omega_{\mu\alpha}(x)u_\alpha(x) \tag{3.67}$$

we have the space and time components of four-vector acceleration as

$$\frac{d\mathbf{u}}{d\tau} = \boldsymbol{\omega} \times \mathbf{u} - \frac{i}{c}\boldsymbol{\eta} u_4 \tag{3.109}$$

$$\frac{du_4}{d\tau} = i\frac{1}{c}\boldsymbol{\eta} \bullet \mathbf{u} \tag{3.110}$$

These relations can also be written in the form

$$\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = \frac{d\mathbf{u}}{dt} = \boldsymbol{\eta} + \boldsymbol{\omega} \times \mathbf{v} \tag{3.111}$$

$$\frac{d}{dt}\left(\frac{c}{\sqrt{1-v^2/c^2}}\right) = \frac{1}{i}\frac{du_4}{dt} = \frac{1}{c}\boldsymbol{\eta} \bullet \mathbf{u} \tag{3.112}$$

To demonstrate the physical meaning of the four-tensor angular velocity $\Omega_{\mu\nu}$, we consider the case where the particle starts moving from rest at $t = 0$. This requires $\xi = 0$ in (3.106) and (3.107). Therefore, at this moment,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_P \tag{3.113}$$

$$\frac{1}{c}\boldsymbol{\eta} = \frac{d\xi}{d\tau}\mathbf{e}_t = \frac{1}{c}a_t\mathbf{e}_t \tag{3.114}$$

These mean $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta} = \frac{1}{c}\mathbf{a}$ are the circular and hyperbolic angular velocity of the body frame relative to the inertial frame. We notice that at this instant $d\tau = dt$ and

$$\boldsymbol{\eta} = \mathbf{a}_t = \mathbf{a} \tag{3.115}$$

and therefore

$$d\boldsymbol{\varphi} = \boldsymbol{\omega}d\tau \tag{3.116}$$

$$d\mathbf{v} = \mathbf{a}_t d\tau = \mathbf{a}d\tau \tag{3.117}$$

The infinitesimal anti-symmetric four-dimensional rotation tensor $d\boldsymbol{\Phi}$ is defined

$$d\boldsymbol{\Phi} = \boldsymbol{\Omega}d\tau \tag{3.118}$$

which can be written as

$$d\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{R}_{d\boldsymbol{\varphi}} & -i\frac{1}{c}d\mathbf{v} \\ i\frac{1}{c}d\mathbf{v}^T & 0 \end{bmatrix} \tag{3.119}$$

This tensor in terms of elements is

$$d\boldsymbol{\Phi} = \begin{bmatrix} 0 & -d\phi_3 & d\phi_2 & -idv_1/c \\ d\phi_3 & 0 & -d\phi_1 & -idv_2/c \\ -d\phi_2 & d\phi_1 & 0 & -idv_3/c \\ idv_1/c & idv_2/c & idv_3/c & 0 \end{bmatrix} \tag{3.120}$$

This explanation can be used for the special case where the inertial system is coincident with the body frame instantly, which is often called a commoving inertial frame system. For this case, we have $\mathbf{Q}'_P = \mathbf{1}$ and the relation

$$\frac{dL'_{\mu\nu}(x)}{d\tau'} = \Omega'_{\mu\alpha}(x)L'_{\alpha\nu}(x) \tag{3.121}$$

becomes

$$\frac{dL'_{\mu\nu}(x)}{d\tau'} = \Omega'_{\mu\nu}(x) \tag{3.122}$$

where

$$\boldsymbol{\Omega}' = \begin{bmatrix} \mathbf{R}_{\omega'} & -i\frac{1}{c}\boldsymbol{\eta}' \\ i\frac{1}{c}\boldsymbol{\eta}'^T & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{\omega'} & -i\frac{1}{c}\mathbf{a}'_t \\ i\frac{1}{c}\mathbf{a}'^T_t & 0 \end{bmatrix} \tag{3.123}$$

It is seen that

$$\boldsymbol{\omega}' = \boldsymbol{\omega}'_P \tag{3.124}$$

$$\frac{1}{c}\boldsymbol{\eta}' = \frac{1}{c}\mathbf{a}'_t = \frac{1}{c}\mathbf{a}' \tag{3.125}$$

and at this instant $d\tau' = dt'$ and we have

$$d\boldsymbol{\varphi}' = \boldsymbol{\omega}' d\tau \tag{3.126}$$

$$d\mathbf{v}' = \mathbf{a}'_t d\tau = \mathbf{a}' d\tau \tag{3.127}$$

It is seen that $\boldsymbol{\omega}'$ and $\frac{1}{c}\boldsymbol{\eta}' = \frac{1}{c}\mathbf{a}'$ are the circular and hyperbolic angular velocity of the body frame relative to the commoving inertial frame system. The infinitesimal four-dimensional rotation $d\boldsymbol{\Phi}'$ of the body frame relative to the commoving inertial frame system is

$$d\boldsymbol{\Phi}' = \begin{bmatrix} \mathbf{R}_{d\varphi'} & -i\frac{1}{c}d\mathbf{v}' \\ i\frac{1}{c}d\mathbf{v}'^T & 0 \end{bmatrix} \tag{3.128}$$

It is obvious the four-tensor angular velocity tensor $\Omega'_{\mu\nu}$ is the representation of $\boldsymbol{\Omega}$ on the commoving frame and we have

$$\Omega'_{\mu\nu}\mathbb{e}'_\mu\mathbb{e}'_\nu = \Omega_{\mu\nu}\mathbb{e}_\mu\mathbb{e}_\nu \tag{3.129}$$

From this, it is expected that

$$\Omega'_{\mu\nu} = \Lambda_{\mu\alpha}\Lambda_{\mu\beta}\Omega_{\alpha\beta} \tag{3.130}$$

This tensor transformation can also be written as

$$\Omega_{\mu\nu} = L_{\mu\alpha}L_{\mu\beta}\Omega'_{\alpha\beta} \tag{3.131}$$

Although we still use the notations $\boldsymbol{\omega}$ and $\frac{i}{c}\boldsymbol{\eta}$ and call them angular velocities, these vectors cannot be taken as a proper angular velocity vectors like vectors $\boldsymbol{\omega}'$ and $\frac{i}{c}\boldsymbol{\eta}'$. This is the result of the non-Euclidean geometry governing the four-dimensional rotations. A combination of the circular and hyperbolic angular velocities $\boldsymbol{\omega}'$ and $\frac{1}{c}\boldsymbol{\eta}'$ in the relation (3.131) gives the vectors $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$. The famous Thomas precession for accelerating particles is manifest of the governing hyperbolic geometry. Now it is clear why we denoted subscript P in the orthogonal tensor $\mathbf{Q}_P$, which specifies $\boldsymbol{\omega}_P$. The orthogonal tensor $\mathbf{Q}$ specifies $\boldsymbol{\omega}$ through the relation

$$\mathbf{R}_{\boldsymbol{\omega}} = \frac{d\mathbf{Q}^T}{dt}\mathbf{Q} \tag{3.132}$$

Although $\mathbf{Q}$ and $\boldsymbol{\omega}$ are essential mathematical entities, they cannot be demonstrated geometrically as directly as $\mathbf{Q}_P$ or $\boldsymbol{\omega}_P$. However, we must be careful when we consider $\boldsymbol{\omega}$ as a circular angular velocity. We might drop the subscript P cautiously. Therefore, we have learned that the motion of a particle in the classical sense is the result of the hyperbolic part of rotation of its body frame. The space rotation is also part of the motion, which is the origin of spin precession of an electron in a magnetic field. This will be discussed in more detail shortly.

It is realized that the non-Euclidean geometry is the result of transforming four-tensors and four-vectors among different space-time body frames. Through this important physical reality, one appreciates the work of those who considered the possibility of non-Euclidean geometry. The non-Euclidean aspect of the velocity addition law for uniform motion has been studied by Robb, Varičak, Lewis, Wilson and Borel [6]. However, these discoveries have not been appreciated enough by later investigators. Fortunately, there have been some advocates of reviving this important issue recently [8]. Now we appreciate that this path resolves inconsistencies and paradoxes in relativity. It also explains the geometrical mechanism behind motion and interaction, which will be developed in the next section.

We have also noticed an important issue regarding the four-vector velocity of a particle. It has been shown that the four-vector velocity is attached to its body frame such that

$$u'_\mu = \Lambda_{\mu\nu}(x)u_\nu(x) \tag{3.81}$$

This has been shown symbolically in Fig. 1 by considering a two dimensional space and one time direction. It should be noticed that the inertial reference frame and body frame of the particle both have attached four vector-velocities $\mathbb{u}^R$ and $\mathbb{u}^P$ in their space-time frames, respectively. However, the Lorentz transformation (3.81) relates the components of four-vector velocity u'_μ of particle P in its frame and its components of four-vector velocity $u_\mu(x)$ in the inertial reference frame of particle R. It should be noticed that the four-vector velocity components $u'_\mu = (0,0,0,ic)$ and $u_\mu(x) = c(\sinh\xi\mathbf{e}_t, i\cosh\xi)$ are representations of $\mathbb{u}^P$ in body frame of particle and inertial reference frame, respectively.

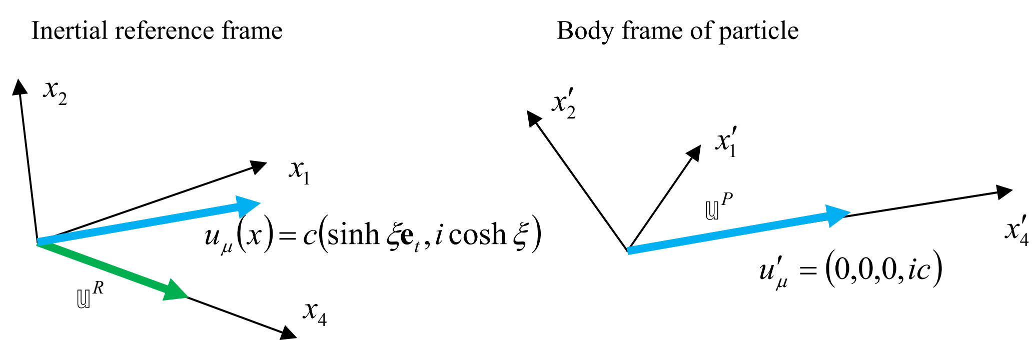

Fig. 1. Inertial reference frame and body frame.

Therefore, we can consider a new type of four-vector $\mathbb{G}$ called an attached four-vector and defined as a four-vector attached to the body frame of a particle, such that

$$G'_\mu = \Lambda_{\mu\nu}G_\nu \tag{3.133}$$

no matter whether the body frame is inertial or accelerating. For this four-vector

$$\frac{dG_\mu}{d\tau} = \Omega_{\mu\nu} G_\nu \tag{3.134}$$

This can be written as

$$\frac{dG_\mu}{ds} = -\frac{i}{c} \Omega_{\mu\nu} G_\nu \tag{3.135}$$

or in compact form

$$\frac{d\mathbb{G}}{ds} = -\frac{i}{c} \boldsymbol{\Omega} \mathbb{G} \tag{3.136}$$

Using this relation for unit base tangential four-vector $\mathbb{e}_t$, we have

$$\frac{d\mathbb{e}_t}{ds} = -\frac{i}{c} \boldsymbol{\Omega} \mathbb{e}_t \tag{3.137}$$

By comparing this relation with the relation (3.33) for the world line radius of curvature, we obtain

$$\frac{1}{R} \mathbb{e}_n = \frac{i}{c} \boldsymbol{\Omega} \mathbb{e}_t \tag{3.138}$$

It is seen that the world line radius of curvature satisfies

$$\frac{1}{R^2} = -\frac{1}{c^2} \mathrm{e}_t^T \boldsymbol{\Omega}^2 \mathrm{e}_t \tag{3.139}$$

where the symmetric tensor $\boldsymbol{\Omega}^2$ is

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} \boldsymbol{\omega}\boldsymbol{\omega}^T - \omega^2 \mathbf{1} + \frac{1}{c^2} \boldsymbol{\eta}\boldsymbol{\eta}^T & -\frac{i}{c} \boldsymbol{\omega} \times \boldsymbol{\eta} \\ -\frac{i}{c} (\boldsymbol{\omega} \times \boldsymbol{\eta})^T & \frac{1}{c^2} \eta^2 \end{bmatrix} \tag{3.140}$$

It is also seen that the fundamental equation (3.63) can be written as

$$\frac{dL_{\mu\nu}(x)}{ds} = -\frac{i}{c} \Omega_{\mu\alpha}(x) L_{\alpha\nu}(x) \tag{3.141}$$

or in the compact form

$$\frac{d\mathbb{L}(x)}{ds} = -\frac{i}{c} \boldsymbol{\Omega}(x) \mathbb{L}(x) \tag{3.142}$$

For the base four-vectors of body frame $\mathbb{e}'_\mu$, we have

$$\frac{d}{ds}\begin{bmatrix}\mathbb{e}'_1\\ \mathbb{e}'_2\\ \mathbb{e}'_3\\ \mathbb{e}'_4\end{bmatrix} = -\frac{i}{c}\mathbf{\Omega}(x)\bullet\begin{bmatrix}\mathbb{e}'_1\\ \mathbb{e}'_2\\ \mathbb{e}'_3\\ \mathbb{e}'_4\end{bmatrix} \tag{3.143}$$

which can be written as

$$\begin{bmatrix}\frac{d\mathbb{e}'_1}{ds}\\ \frac{d\mathbb{e}'_2}{ds}\\ \frac{d\mathbb{e}'_3}{ds}\\ \frac{d\mathbb{e}'_4}{ds}\end{bmatrix} = -\frac{i}{c}\begin{bmatrix}0 & -\omega_3 & \omega_2 & -i\eta_1/c\\ \omega_3 & 0 & -\omega_1 & -i\eta_2/c\\ -\omega_2 & \omega_1 & 0 & -i\eta_3/c\\ i\eta_1/c & i\eta_2/c & i\eta_3/c & 0\end{bmatrix}\begin{bmatrix}\mathbb{e}'_1\\ \mathbb{e}'_2\\ \mathbb{e}'_3\\ \mathbb{e}'_4\end{bmatrix} \tag{3.144}$$

One realizes that this equation is actually a Frenet-Serret-like formula for orientation of the local body frame relative to the inertial system. However, it should be noticed that this orientation is in terms of generalized curvatures of the world line but not generally in principal directions. The tangent to the world line specified by $\mathbb{e}_T = \mathbb{e}'_4$ is a principal direction, but the perpendicular directions to the tangent are not usually principal directions. It should be also mentioned that Synge has already studied the Minkowskian Frenet-Serret moving frame [9]. What we have shown is that this frame is a representation of the fundamental body frame of a particle.

In this section, it has been demonstrated that there is a relationship between Minkowskian space-time and massive particles. The particle specifies its space-time body frame relative to the inertial reference frame. Now the natural question concerns the very existence of these space-time systems. It is seen that we are compelled to admit the existence of a universal entity, which has nothing to do with any special space-time. It is in this universal entity in which particles and their corresponding space-time body frame exist. Later we will investigate more about this universal entity.

3.4. General relative motion and velocity addition law

Now we develop the theory of relative motion for general accelerating particles. It is seen that the governing relations and velocity addition law in Poincare's relativity are still valid for this general case.

Consider two particles A and B moving with velocities $\mathbf{v}_A = \mathbf{v}_A(t)$ and $\mathbf{v}_B = \mathbf{v}_B(t)$ relative to an inertial system. The four-vector velocities $\mathbb{U}_A$ and $\mathbb{U}_B$ are attached four-vectors, where we have

$$(\mathbb{U}_A)_A = \mathbf{\Lambda}_A \mathbb{U}_A \tag{3.145}$$

$$(\mathbb{U}_B)_B = \mathbf{\Lambda}_B \mathbb{U}_B \tag{3.146}$$

$(\mathbb{U}_A)_A$ and $(\mathbb{U}_B)_B$ are representing these four-vectors on their corresponding body frame where

$$(\mathbb{U}_A)_A = (\mathbb{U}_B)_B = (0,0,0,ic) \tag{3.147}$$

The transformations $\mathbf{\Lambda}_A = \mathbf{\Lambda}_A(t)$ and $\mathbf{\Lambda}_B = \mathbf{\Lambda}_B(t)$ represent the orientation of these body frames relative to the inertial frame. For these transformations, we explicitly have

$$\mathbf{\Lambda}_A(t) = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A \mathbf{e}_{tA}\mathbf{e}_{tA}^T & i\sinh\xi_A \mathbf{Q}_A \mathbf{e}_{tA} \\ -i\sinh\xi_A \mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix} \tag{3.148}$$

and

$$\mathbf{\Lambda}_B(t) = \begin{bmatrix} \mathbf{Q}_B + (\cosh\xi_B - 1)\mathbf{Q}_B \mathbf{e}_{tB}\mathbf{e}_{tB}^T & i\sinh\xi_B \mathbf{Q}_B \mathbf{e}_{tB} \\ -i\sinh\xi_B \mathbf{e}_{tB}^T & \cosh\xi_B \end{bmatrix} \tag{3.149}$$

By using (3.147) and combining (3.145) and (3.146), we obtain

$$\mathbb{U}_A = \mathbf{\Lambda}_A^T \mathbf{\Lambda}_B \mathbb{U}_B \tag{3.150}$$

Relative orientation of the body frame B relative to A at time t is denoted by $\mathbf{\Lambda}_{B/A}$ and is defined such that

$$\mathbf{\Lambda}_B = \mathbf{\Lambda}_A \mathbf{\Lambda}_{B/A} \tag{3.151}$$

This relation shows

$$\mathbf{\Lambda}_{B/A} = \mathbf{\Lambda}_A^T \mathbf{\Lambda}_B \tag{3.152}$$

Therefore, (3.150) becomes

$$\mathbb{U}_A = \mathbf{\Lambda}_{B/A} \mathbb{U}_B \tag{3.153}$$

which can also be written as

$$\mathbb{U}_B = \mathbf{\Lambda}_{B/A}^T \mathbb{U}_A \tag{3.154}$$

It should be noticed that $\mathbf{\Lambda}_{B/A}$ is the relative Lorentz transformation from body frame A to body frame B measured by our inertial reference frame at time *t*. Therefore all the relations are relative to this observer at time *t*. However, we should derive similar relations relative to the observer attached to the body frame *A*. For this we notice that the velocity of *B* relative to *A* measured by an observer in the body frame of *A* is

$$(\mathbb{U}_B)_A = (\mathbb{U}_{B/A})_A = \mathbf{\Lambda}_A \mathbb{U}_B \tag{3.155}$$

By substituting for $\mathbb{U}_B$ from (3.146), we obtain

$$(\mathbb{U}_B)_A = (\mathbb{U}_{B/A})_A = \mathbf{\Lambda}_A \mathbf{\Lambda}_B^T (\mathbb{U}_B)_B \tag{3.156}$$

We also have the obvious relation

$$(\mathbb{U}_A)_A = (\mathbf{\Lambda}_{B/A})_A (\mathbb{U}_{B/A})_A \tag{3.157}$$

which can be written as

$$(\mathbb{U}_{B/A})_A = (\mathbf{\Lambda}_{B/A})_A^T (\mathbb{U}_A)_A \tag{3.158}$$

By comparing (3.156) and (3.157) and using (3.155) we obtain the relation

$$(\mathbf{\Lambda}_{B/A})_A^T = \mathbf{\Lambda}_A \mathbf{\Lambda}_B^T \tag{3.159}$$

which can be written as

$$(\mathbf{\Lambda}_{B/A})_A = \mathbf{\Lambda}_B \mathbf{\Lambda}_A^T \tag{3.160}$$

Interestingly, it is seen that

$$\left(\mathbf{\Lambda}_{B/A}\right)_A = \mathbf{\Lambda}_A \mathbf{\Lambda}_{B/A} \mathbf{\Lambda}_A^T \tag{3.161}$$

which looks like the transformation for tensor $\mathbf{\Lambda}_{B/A}$ from inertial reference frame to the body frame A. What we have is the development of the general theory of relative motion.

Explicitly from (3.155), we have

$$\left(\mathbb{U}_{B/A}\right)_A = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A \mathbf{e}_{tA} \mathbf{e}_{tA}^T & i\sinh\xi_A \mathbf{Q}_A \mathbf{e}_{tA} \\ -i\sinh\xi_A \mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix} \begin{bmatrix} c\sinh\xi_B \mathbf{e}_{tB} \\ ic\cosh\xi_B \end{bmatrix} \tag{3.162}$$

From this, we obtain the relations

$$\left(\sinh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A = -\sinh\xi_A \cosh\xi_B \mathbf{Q}_A \mathbf{e}_{tA} + \mathbf{Q}_A[\mathbf{1} + (\cosh\xi_A - 1)\mathbf{e}_{tA}\mathbf{e}_{tA}^T]\sinh\xi_B \mathbf{e}_{tB} \tag{3.163}$$

$$\left(\cosh\xi_{B/A}\right)_A = \cosh\xi_A \cosh\xi_B - \sinh\xi_A \sinh\xi_B \mathbf{e}_{tA} \bullet \mathbf{e}_{tB} \tag{3.164}$$

These relations are the manifest of hyperbolic geometry governing the velocity addition law even for accelerating particles. This property holds for all attached four-vectors and four tensors. Inertial observers relate components of attached four-vectors and four-tensors by Lorentz transformations. This is the origin of non-Euclidean geometry governing the three vector and three tensors. As we saw the addition of three vector velocities follow hyperbolic geometry.

It should be noticed that these relations hold despite the fact that the transformation

$$x'_\mu = \Lambda_{\mu\nu} x_\nu \tag{3.165}$$

is not valid among accelerating systems. What we have here is the completion of the Poincare's relativity for accelerating systems.

4. Fundamental interaction

After developing the theory of accelerating motion, we are ready to develop the theory of fundamental interaction. The equation of motion for a particle in an inertial reference frame system is given by

$$m\frac{du_\mu}{d\tau} = F_\mu \tag{4.1}$$

where F_μ is the four-vector Minkowski force. This force is the result of interaction of the particle with a field, such as an electromagnetic field. We are looking to explore the geometrical character of this field. By substituting for four-acceleration from (3.67) in the relation (4.1), we obtain

$$F_\mu = m\Omega_{\mu\nu}u_\nu \tag{4.2}$$

for the Minkowski force. Since $\Omega_{\mu\nu}$ is anti-symmetric, we have

$$F_\mu u_\mu = m\Omega_{\mu\nu}u_\mu u_\nu = 0 \tag{4.3}$$

which means the four-vector Minkowski force F_μ is perpendicular to the four-vector velocity u_μ. The relation (4.2) shows that this force depends on four-vector velocity u_μ and four-tensor angular velocity $\Omega_{\mu\nu}$ at the position of the particle $\tilde{x}$. As a result, the field strength must depend on the four-tensor angular velocity $\Omega_{\mu\nu}$. It is seen that the simplest admissible field is characterized by a field strength four-tensor $\Theta_{\mu\nu}(x)$ such that at the position of the particle

$$m\Omega_{\mu\nu} = \alpha\Theta_{\mu\nu}(\tilde{x}) \tag{4.4}$$

Scalar α is a property of the particle and depends on the type of interaction. This quantity can be recognized as electric charge in electromagnetic interaction. Therefore, we can consider a fundamental interaction to be an interaction characterized by an anti-symmetric strength tensor field $\Theta_{\mu\nu}(x)$, such that at the position of the particle $\tilde{x}$

$$\Omega_{\mu\nu} = \frac{\alpha}{m}\Theta_{\mu\nu}(\tilde{x}) \tag{4.5}$$

Although $\Theta_{\mu\nu}(\tilde{x})$ is independent of the particle, the Minkowski force depends on the particle through α and four vector velocity u_μ, such that

$$F_\mu = \alpha \Theta_{\mu\nu} u_\nu \tag{4.2}$$

Therefore, the equation of motion becomes

$$m \frac{du_\mu}{d\tau} = \alpha \Theta_{\mu\nu}(\tilde{x}) u_\nu \tag{4.6}$$

One can see that the anti-symmetric strength tensor $\Theta_{\mu\nu}(x)$ looks like a four-dimensional vorticity field analogous to the three-dimensional vorticity in rotational fluid flow. Therefore, we can consider a four-vector velocity-like field $\mathbb{V} = V_\mu \mathbb{e}_\mu$ induced to the space-time of the inertial reference frame, such that its four dimensional curl is the vorticity-like strength tensor

$$\Theta_{\mu\nu}(x) = \partial_\nu V_\mu - \partial_\mu V_\nu \tag{4.7}$$

From our familiarity with electrodynamics, it is obvious that electromagnetic interaction is completely compatible with this geometrical theory of interaction. Therefore, in the next section, we present the covariant theory of electromagnetics and explore its geometrical aspects based on the four-dimensional vorticity theory. It is seen that this geometrical theory resolves some ambiguities in the traditional theory of electromagnetics. More importantly, one realizes that this theory is a model for any other fundamental interaction. Therefore, the corresponding gravitational theory is also developed in detail in Section 6.

We should remember that the theory of relativity has its origin in the theory of electrodynamics. Now we can see that the theory of interaction also has its origin in this theory.

5. Geometrical theory of electromagnetic interaction

In the theory of electrodynamics [10], in an inertial reference frame, the force on a charged particle can be expressed in terms of two vector fields, an electric field **E(x**, t) and a magnetic field **B(x**, t). In terms of these fields, the force on a particle with charge q moving with velocity **v** is given by

$$\begin{aligned} \mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &= q(\mathbf{E} - \mathbf{B} \times \mathbf{v}) \end{aligned} \tag{5.1}$$

This is known as the Lorentz force in SI units. It is noticed that the vector **B** is actually an axial or pseudo-vector. Therefore, there is a corresponding anti-symmetric tensor

$$\mathbf{R}_{\mathbf{B}} = \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix} \tag{5.2}$$

such that the Lorentz force in matrix form is

$$\mathbf{F} = q(\mathbf{E} - \mathbf{R}_{\mathbf{B}} \mathbf{v}) \tag{5.3}$$

In the covariant theory of electrodynamics, the corresponding four-vector Minkowski force is

$$F_\mu = qF_{\mu\nu}(\tilde{x})u_\nu \tag{5.4}$$

where the electromagnetic strength field $F_{\mu\nu}$ is

$$\mathbb{F} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_B & -i\dfrac{1}{c}\mathbf{E}^T \\ i\dfrac{1}{c}\mathbf{E} & 0 \end{bmatrix} \tag{5.5}$$

Therefore, the equation of motion of this particle is given by

$$m\frac{du_\mu}{d\tau} = qF_{\mu\nu}(\tilde{x})u_\nu \tag{5.6}$$

It is obvious that the equation (5.6) has the form of the equation (4.6), which was obtained based on the kinematical considerations. It is seen that

$$\alpha \to q \tag{5.7}$$

$$\Theta_{\mu\nu} \to F_{\mu\nu} \tag{5.8}$$

Therefore, the space-time body frame of the particle rotates with four-tensor angular velocity

$$\Omega_{\mu\nu} = \frac{q}{m} F_{\mu\nu}(\tilde{x}) \tag{5.9}$$

relative to the inertial frame. It is seen that the hyperbolic and circular angular velocities of the body frame are

$$\frac{\boldsymbol{\eta}}{c} = \frac{q}{mc} \mathbf{E}(\tilde{x}) \tag{5.10}$$

and

$$\boldsymbol{\omega} = -\frac{q}{m} \mathbf{B}(\tilde{x}) \tag{5.11}$$

respectively. Now we realize that the electromagnetic strength field tensor and Lorentz force vector are both a natural consequence of the geometric structure of relative space time. Based on our experience with continuum mechanics, as we mentioned before, the strength tensor $F_{\mu\nu}$ field seems like a four-dimensional vorticity field. This electromagnetic vorticity four-tensor field is a combination of hyperbolic electromagnetic vorticity $\frac{1}{c}\mathbf{E}$ and circular electromagnetic vorticity $-\mathbf{B}$. It is seen that the scalar $\frac{q}{m}$ maps the vorticity field $F_{\mu\nu}$ at the position of the particle to the four-tensor angular velocity $\Omega_{\mu\nu}$ of its body frame. Therefore, the effect of electromagnetic interaction on a charged particle is nothing but the instantaneous four-dimensional rotation of its body frame. The equations (3.111) and (3.112) for the particle can be written as

$$\frac{d\mathbf{u}}{d\tau} = \boldsymbol{\omega} \times \mathbf{u} + \frac{1}{\sqrt{1 - v^2 / c^2}} \boldsymbol{\eta} \tag{5.12}$$

$$\frac{d}{d\tau}\left(\frac{c}{\sqrt{1-v^2/c^2}}\right)=\frac{1}{c}\boldsymbol{\eta}\bullet\mathbf{u} \tag{5.13}$$

These equations are equivalent to the space and time components of equation (5.6) for electromagnetic interaction as

$$m\frac{d\mathbf{u}}{dt}=\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{5.14}$$

$$\frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}}=q\mathbf{E}\bullet\mathbf{v} \tag{5.15}$$

As we know, the first equation is the equation of motion, where its right hand side is the familiar Lorentz force. The second equation is the rate at which the electromagnetic field does work on the particle and changes its energy.

In covariant electromagnetic theory, the four-vector electric current density

$$J_{E\mu}=(\mathbf{J}_E,J_{E4})=(\mathbf{J}_E,i\rho_E c)=\rho_E(\mathbf{v},ic) \tag{5.16}$$

satisfying the continuity equation

$$J_{E\mu,\mu}=\nabla\bullet\mathbf{J}_E+\frac{\partial\rho_E}{\partial t}=0 \tag{5.17}$$

generates the electromagnetic four-vector potential $\mathbb{A}$, where

$$\mathbb{A}=A_\mu \mathbb{e}_\mu=(\mathbf{A},A_4) \tag{5.18}$$

in space-time corresponding to the inertial reference frame. The space component $\mathbf{A}$ is the magnetic vector potential and the time component A_4 is related to the electric scalar potential ϕ as

$$A_4=i\frac{1}{c}\phi \tag{5.19}$$

The four-dimensional curl of A_μ gives the electromagnetic field strength tensor $F_{\mu\nu}$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{5.20}$$

Therefore, the fields $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of these potentials as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{5.21}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5.22}$$

It should be noticed that the four-vector $\mathbb{V}$ corresponds to the negative of $\mathbb{A}$

$$V_\mu \to -A_\mu \tag{5.23}$$

and can be considered as an electromagnetic velocity field induced in four-dimensional space-time relative to the inertial frame. As was mentioned previously, its four-dimensional curl is the electromagnetic vorticity four-tensor $F_{\mu\nu}$

$$\Theta_{\mu\nu} \to F_{\mu\nu} \tag{5.24}$$

The covariant form of the governing equation for strength or vorticity tensor $F_{\mu\nu}$ due to the electric current density is

$$\partial_\nu F_{\mu\nu} = \frac{4\pi K}{c^2} J_{E\mu} \tag{5.25}$$

which is the compact form of Maxwell's inhomogeneous equations

$$\nabla \bullet \mathbf{E} = 4\pi K \rho_E \tag{5.26}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi K}{c^2}\mathbf{J}_E \tag{5.27}$$

Equation (5.26) is Gauss's law and equation (5.27) is Ampere's law with Maxwell's correction. In these equations, the constant K is the electrostatic or Coulomb constant that usually is written as $K = \frac{1}{4\pi\varepsilon_0}$, where ε_0 is the permittivity of free space. There is also the relation $c^2 = \frac{1}{\mu_0 \varepsilon_0}$, where constant μ_0 is called the permeability of free space and the relation $\frac{4\pi K}{c^2} = \mu_0$ holds. Therefore, the equation (5.25) can be written as

$$\partial_\nu F_{\mu\nu} = \mu_0 J_{E\mu} \tag{5.28}$$

and also the Gauss and Ampere's laws (4.26) and (4.27) become

$$\nabla \bullet \mathbf{E} = \rho_E / \varepsilon_0 \tag{5.29}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}_E \tag{5.30}$$

The compatibility equation for $F_{\mu\nu}$ is

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{5.31}$$

This is the necessary condition to obtain the electromagnetic velocity A_μ from vorticity field $F_{\mu\nu}$. It simply checks if a given electromagnetic vorticity field is acceptable or not. This equation is the covariant form of Maxwell's homogeneous equations

$$\nabla \bullet \mathbf{B} = 0 \tag{5.32}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{5.33}$$

As we know, the equation (5.32) is Gauss's law for magnetism and the equation (5.33) is Faraday's law of induction. The set of equations (5.29)-(5.30) and (5.32)-(5.33) are Maxwell's equations in SI units. They simply show the relations governing the electromagnetic vorticity induced to space-time. It is seen that the geometrical theory of electromagnetic interaction is very clear in SI units. Interestingly, it is realized that the electromagnetic theory would have been much more compatible with the geometrical theory if the scalar and vector potentials ϕ and $\mathbf{A}$, and magnetic field $\mathbf{B}$ had been defined as the negative of their present form.

The four-vector potential field A_μ is not uniquely determined from compatible strength four-tensor $F_{\mu\nu}$ due to the gauge freedom. Indeed, the new field

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda \tag{5.34}$$

does not change the field strength tensor $F_{\mu\nu}$. Such transformation is called a gauge transformation in which the function λ is a function of coordinate x. This gauge freedom allows us to have the Lorentz gauge constraint

$$\partial_\mu A_\mu = \nabla \bullet \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{5.35}$$

Therefore, λ is not that arbitrary. It must satisfy the wave equation

$$\partial_\mu \partial_\mu \lambda = \nabla^2 \lambda - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} = 0 \tag{5.36}$$

This wave equation can be considered as representing the inertial electromagnetic waves. Using the Lorentz gauge in (5.28) produces the manifestly covariant wave equation

$$\partial_\alpha \partial_\alpha A_\mu = -\mu_0 J_{E\mu} \tag{5.37}$$

What we have shown is that Maxwell's equations are equations governing the hyperbolic and circular angular electromagnetic vorticities $\frac{1}{c}\mathbf{E}$ and $-\mathbf{B}$. The equation

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{5.31}$$

is nothing but a kinematic compatibility for these electromagnetic vorticities. The non-homogeneous equation

$$\partial_\nu F_{\mu\nu} = \mu_0 J_{E\mu} \tag{5.28}$$

is the relation among these vorticities and electric four-vector density current. An analogy with continuum mechanics suggests this relation is the equation of motion for electromagnetic vorticities.

Maxwell's equations are covariant, which means they are invariant under Lorentz transformations among inertial systems. Therefore, the four-vector $\mathbb{A}$, and four-tensor $\mathbb{F}$ are fundamental fields independent of any specific space-time induced in the universal entity mentioned before. It is the inertial observer who specifies a space-time in this universal entity and measures components for these four-vector and four-tensor, for example $F_{\mu\nu}$ for $\mathbb{F}$. The components of this four-tensor transform under Lorentz transformation among inertial systems as

$$F'_{\mu\nu} = \Lambda_{\mu\alpha}\Lambda_{\nu\beta}F_{\alpha\beta} \tag{5.38}$$

The non-Euclidean character of electromagnetic field tensors is obvious from these transformations. Interestingly, the scalars

$$F_{\mu\nu}F_{\mu\nu} = 2(B^2 - \frac{1}{c^2}E^2) \tag{5.39}$$

$$\det\mathbb{F} = -\frac{1}{c^2}(\mathbf{E}\bullet\mathbf{B})^2 \tag{5.40}$$

are the invariants of the four-tensor $F_{\mu\nu}$ under the Lorentz transformations. They show that the scalar $\left(\mathbf{B} + \frac{i}{c}\mathbf{E}\right)^2$ is invariant.

It is obvious that the non-inertial observers are not qualified to use (5.38), because the transformation

$$x'_\mu = \Lambda_{\mu\nu}x_\nu \tag{2.6}$$

does not hold among them. We demonstrate this fact by a simple example. Consider the electromagnetic vorticity field generated by a free charged particle. Its body frame is an inertial frame and has a uniform motion relative to other inertial observers. The particle generates the electric field in its inertial body frame, such that

$$\mathbf{E}' = \frac{q}{4\pi\varepsilon_0 r'^2}\hat{\mathbf{r}}' \tag{5.41}$$

Therefore, it is seen that

$$A'_\mu = (0,0,0,\frac{i}{c}\phi') \tag{5.42}$$

where

$$\phi' = \frac{q}{4\pi\varepsilon_0 r'} \tag{5.43}$$

is parallel to the four-vector velocity $u'_\mu = (0,0,0,ic)$. Relative to the reference inertial frame, we have

$$x_\mu = \Lambda_{\nu\mu} x'_\nu \tag{5.44}$$

and

$$A_\mu = \Lambda_{\nu\mu} A'_\nu \tag{5.45}$$

It seems as if the four-vector field A'_μ were attached to the body frame rigidly in the time direction, such that it looks rotated relative to the fixed inertial system. However, this rigid like character and covariant relations are not valid when the particle is accelerating. It seems the space-time body frame of an accelerating particle does not look rigid in this sense to any observer. Therefore, the position four-vector $\mathbb{X}$, four-vector potential $\mathbb{A}$ and four-tensor $\mathbb{F}$ field do not transform under a uniform hyperbolic rotation. This non-rigid character can be considered as the geometrical origin of electromagnetic radiation. The radiation of an accelerating particle can be analyzed by using the general equation

$$\partial_\alpha \partial_\alpha A_\mu = -\mu_0 J_{E\mu} \tag{5.37}$$

in the context of Liénard-Wiechert potential [10].

What is the consequence of the non-rigidity of the space-time body frame of an accelerating particle? As we saw, the four-acceleration of a charged particle

$$\frac{du_\mu}{d\tau} = \frac{q}{m} F_{\mu\nu}(\tilde{x}) u_\nu \tag{5.46}$$

is the result of rigid-like instantaneous rotation of its body frame. However, we now realize that the global rigid-like character of the body frame is not a requirement for this geometrical derivation. It is only necessary to consider the instantaneous rotation of the body frame in the neighborhood of the particle. Therefore, we define $\Omega_{\mu\nu}(\tilde{x})$ as the four-dimensional angular velocity of the body frame at the position of the particle

$$\Omega_{\mu\nu}(\tilde{x}) = \frac{q}{m} F_{\mu\nu}(\tilde{x}) \tag{5.47}$$

Interestingly, with this development we can examine the character of particles in quantum theory to explain the wave-particle duality of matter. It is in classical mechanics where we specify position of a particle, for example, at the origin of its space body frame. In quantum mechanics, a free elementary particle with specified momentum does not have a specified position in its space-time and can be anywhere in its body or inertial reference frame. One can suggest that the wave function of the particle represents the trace of its space-time body frame on the inertial reference frame. Therefore, it is necessary to understand the Dirac spinor wave function in the framework of the present space-time theory. This new view looks very promising if we remember that the wave function of an interacting particle is localized and is different from the wave function of a free particle. It is seen that this is nothing but the manifestation of a deformation-like character of the space-time body frame of an interacting particle. Interestingly, we realize that the creation and annihilation of particles can be explained as the result of constraints in the time direction. It is clear that we may expect to resolve ambiguities in the quantum world and other branches of modern physics with our new view of space-time. Here, we should mention that the affinity of the Lorentz transformation with electromagnetic strength field tensor and Lorentz force has been realized before. For example, Buitrago has stated that the electromagnetic strength field tensor and Lorentz force are both a natural consequence of the geometric structure of Minkowskian space time, which indicates a fundamental meaning in physics [11]. Obviously, what we have here is development of this fundamental meaning.

Now it is time to explore more about the universal fundamental entity in which particles create their space-time and interact through vorticity fields. It turns out that the review of electromagnetic energy-momentum tensor and Maxwell stress tensor is useful.

5.1. Electromagnetic energy-momentum tensor

Relative to the space-time inertial reference frame, the Lorentz force per unit volume on a medium with a charge density ρ_E and current density $\mathbf{J}_E$ is given by

$$\mathbf{f} = \rho_E \mathbf{E} + \mathbf{J}_E \times \mathbf{B} \tag{5.48}$$

The generalization of this force in covariant electrodynamics is

$$f_\mu = F_{\mu\nu} J_{E\nu} \tag{5.49}$$

where $f_\mu = (\mathbf{f}, f_4)$ is the force-density four vector with

$$f_4 = \frac{i}{c} \mathbf{J}_E \bullet \mathbf{E} \tag{5.50}$$

We note that

$$\frac{\partial w}{\partial t} = \mathbf{J}_E \bullet \mathbf{E} \tag{5.51}$$

is the work done per unit time per unit volume by the electric field on moving charges. Therefore

$$f_4 = \frac{i}{c} \frac{\partial w}{\partial t} \tag{5.52}$$

By substituting $J_{E\mu}$ from the equations of motion of the electromagnetic field

$$\partial_\nu F_{\mu\nu} = \mu_0 J_{E\mu} \tag{5.30}$$

and some tensor algebra, we obtain

$$f_\mu = \partial_\nu T_{\mu\nu} \tag{5.53}$$

where $T_{\mu\nu}$ is the electromagnetic energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{1}{\mu_0} \left(F_{\mu\sigma} F_{\sigma\nu} + \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} \right) \tag{5.54}$$

The explicit form of the components of this four-tensor in terms of $\mathbf{E}$ and $\mathbf{B}$ are

$$T_{ij} = \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} B_k B_k \delta_{ij}) \tag{5.55}$$

called the Maxwell stress tensor, and

$$T_{44} = u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{5.56}$$

the electromagnetic energy density, and

$$T_{4i} = T_{i4} = -\frac{i}{c}\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})_i = -\frac{i}{c} S_i \tag{5.57}$$

where the Poynting vector $\mathbf{S}$ is defined by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{5.58}$$

Therefore, the symmetric four-tensor $T_{\mu\nu}$ can be written in schematic matrix form as

$$\mathbb{T} = \begin{bmatrix} T_{ij} & -\frac{i}{c}\mathbf{S} \\ -\frac{i}{c}\mathbf{S} & u \end{bmatrix} \tag{5.59}$$

The traction or force exerted by this field on a unit area of a surface in space with unit normal vector n_i is

$$T_{ij} n_j = T_i^{(n)} \tag{5.60}$$

Through this similarity with continuum mechanics, we can take $T_{\mu\nu}$ as a four-stress tensor. The time-space components of the equation (5.53) are

$$f_i = \frac{\partial T_{ij}}{\partial x_j} - \frac{1}{c^2}\frac{\partial S_i}{\partial t} \tag{5.61}$$

$$-icf_4 = -\frac{\partial S_j}{\partial x_j} - \frac{\partial u}{\partial t} \tag{5.62}$$

Integrating these relations over a volume V bounded by surface A, and using the divergence theorem, we obtain

$$\int_V f_i dV + \frac{1}{c^2}\frac{\partial}{\partial t}\int_V S_i dV = \int_A T_{ij} n_j dA \tag{5.63}$$

$$-ic\int_V f_4 dV + \frac{\partial}{\partial t}\int_V u dV + \int_A S_i n_i dA = 0 \tag{5.64}$$

These equations show that the electromagnetic field has energy and carries momentum. The Poynting vector **S** represents the energy per unit time, per unit area, transported by the fields in space. It is also seen that the electromagnetic field carries momentum, such that

$$\mathbf{G} = \frac{1}{c^2}\mathbf{S} = \varepsilon_0 \mathbf{E} \times \mathbf{B} \tag{5.65}$$

is the electromagnetic momentum density vector. By noticing

$$F_i = \int_V f_i dV = \text{total force acting on volume } V \tag{5.66}$$

and

$$\frac{\partial W}{\partial t} = -ic\int_V f_4 dV = \text{work done per unit time by the electric field on moving charges in } V \tag{5.67}$$

we obtain the equations (5.63) and (5.64) as

$$F_i + \frac{\partial}{\partial t}\int_V G_i dV = \int_A T_i^{(n)} dA \tag{5.68}$$

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}\int_V u dV + \int_A S_i n_i dA = 0 \tag{5.69}$$

It is seen that by considering

$$F_i = \frac{\partial P_{i \text{ mech}}}{\partial t} \tag{5.70}$$

where

$$P_{i \text{ mech}} = \text{mechanical momentum of charges in volume } V \tag{5.71}$$

and

$$P_{i \text{ field}} = \int_V G_i dV = \text{electromagnetic momentum in volume } V \tag{5.72}$$

and

$$U_{\text{field}} = \int_V u dV = \text{electromagnetic energy in volume } V \tag{5.73}$$

we obtain the momentum and energy conservation laws

$$\frac{\partial}{\partial t}\left(P_{i\ \text{mech}} + P_{i\ \text{field}}\right) = \int_A T_i^{(n)} dA \tag{5.74}$$

$$\frac{\partial}{\partial t}\left(W + U_{\text{field}}\right) + \int_A S_i n_i dA = 0 \tag{5.75}$$

These relations in vectorial form are

$$\frac{\partial}{\partial t}\left(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}}\right) = \int_A \mathbf{T}^{(n)} dA \tag{5.76}$$

$$\frac{\partial}{\partial t}\left(W + U_{\text{field}}\right) + \int_A \mathbf{S} \bullet \mathbf{n} dA = 0 \tag{5.77}$$

In addition, note that the relation

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\sigma}F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right) \tag{5.54}$$

looks like a constitutive relation for four-stress tensor $T_{\mu\nu}$ in term of the four-tensor electromagnetic vorticity $F_{\mu\nu}$ in the universal entity. In linear continuum mechanics, the constitutve equations relate the stress tensor linearly to strain or strain rate, but the energy density is a quadratic function of strain or strain rate tensor. However, what we have here is four-dimensional analogous case in which the stress four-tensor $\mathbb{T}$ is a quadratic function of vorticity four-tensor in the universal entity. Therefore, it is seen that the universal entity behaves like a continuum in which charged particles create stresses and electromagnetic vorticities. Interestingly, the point charged particles are singularites of these vorticities and four stress tensors. Therefore, the Minkowski forces exerted on these point particles are the Lorentz forces, which can be considered also as four-dimensional lift forces. Although this conclusion looks very interesting, historical accounts show it is not completely new. This development is similar to the efforts of investigators of ether theory. Ether was the term used to describe a medium for the propagation of electromagnetic waves. For example, it is very interesting to note that McCullaugh [12] considered ether to be a new kind of medium in which the energy density depends only on the rotation of the volume element of ether. The work of McCullough has been a base for work of other proponents of ether theory such as Lord Kelvin, Maxwell, Kirchhoff, Lorentz and Larmor. Whitaker [13] gives a detailed account of these investigations in which we learn that Maxwell agreed to a rotational character for magnetic field and a translational character for electric field. We also learn that Larmor [14] considered that the ether was separate from matter and that particles, such as electrons, serve as source of vortices in ether.

What is surprising is that we have used similar ideas about stress and vorticity, but in a four-dimensional context. In our development, the magnetic field has the same character as circular rotation, but the electric field has the character of hyperbolic rotation. It is seen that it is well justified to call our fundamental universal entity the historical ether out of respect, which now is represented by four-dimensional space-time systems. Therefore, in the new view, particles specify their space-time body frames in the ether and interact with each other through four-vorticity and four-stress that they create in the ether. As we mentioned, the Lorentz force

$$F_{\mu} = qF_{\mu\nu}(\tilde{x})u_{\nu} \tag{5.4}$$

is analogous to the lift force in fluid dynamics. The lift on an airfoil is perpendicular to the velocity of flow past the surface. This is the mechanical explanation of four-vector electromagnetic Lorentz force.

It is obvious that understanding more about ether and space-time is an important step toward understanding more about modern physics. However, the geometrical theory of electromagnetic interaction resolves some difficulties even in this classical state. We address two important cases.

5.2. Magnetic monopole does not exist

With the new view, the magnetic field $\mathbf{B}$ is the space electromagnetic vorticity induced to the ether relative to the reference inertial frame. This is analogous to the vorticity field in a rotational fluid flow. From non-relativistic fluid mechanics, we know that the vorticity is the curl of the velocity field of the fluid and it is twice the angular velocity of the fluid element. Therefore, we see the same for the electromagnetic vorticity. The magnetic field $\mathbf{B}$ is the curl of the electromagnetic velocity vector field $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5.22}$$

This definition requires

$$\nabla \bullet \mathbf{B} = 0 \tag{5.32}$$

which is the kinematical compatibility equation. This is the necessary condition for the existence of vector potential $\mathbf{A}$ for a given magnetic field $\mathbf{B}$. Existence of a magnetic monopole violates this trivial kinematical compatibility equation. We demonstrate this further by contradiction as follows.

Let us assume, at the origin, there is a point magnetic monopole of strength q_m. Therefore, in SI units

$$\nabla \bullet \mathbf{B} = \mu_0 q_m \delta^{(3)}(\mathbf{x}) \tag{5.78}$$

and the static magnetic field is then given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}} \tag{5.79}$$

However, the relation (5.78) contradicts the kinematical compatibility (5.32). Interestingly, based on the Helmholtz decomposition theorem, this field can only be represented by a scalar potential [15]

$$\phi_m(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{q_m}{r} \tag{5.80}$$

where the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = -\nabla \phi_m \tag{5.81}$$

But this is absurd because the electromagnetic vorticity vector field $\mathbf{B}$ has to be always represented by curl of the electromagnetic velocity vector $\mathbf{A}$. Therefore, magnetic monopoles cannot exist. It is concluded that the magnetic field $\mathbf{B}$ is only generated by moving electric charges.

It has been long speculated that magnetic monopoles might not exist because there is no complete symmetry between **B** and **E**. This is due to the fact that **B** is a pseudo-vector, but **E** is a polar vector. What we have here is the confirmation of this correct speculation that there is no duality between **E** and **B** in electrodynamics. We have shown that the magnetic field **B** has the character of a circular vorticity field and is divergence free. However, the electric field **E** has the character of a hyperbolic vorticity with electric charges as its sources, where

$$\nabla \bullet \mathbf{E} = \rho_E / \varepsilon_0 \tag{5.29}$$

It is seen that this explanation is actually clarification of Larmor's ether theory.

As mentioned previously, the electric charge q of a particle has the property of a kinematical coupling, which maps the four-dimensional electromagnetic vorticity at the position of the particle to the angular velocity of its body frame. We have shown that electric charge is the only coupling present. Furthermore, there is no need for any other coupling. It is naïve to assume that a simplistic modification of Maxwell's equations suffice to allow the existence of magnetic charges in electrodynamics.

5.3. Spin dynamics and magnetic moment

It is known that every elementary particle, such as an electron, has an intrinsic angular momentum called spin. The spin can be considered as a constant length four-vector $s_\mu = (\mathbf{s}, s_4)$ such that relative to the particle body frame, the spin four-vector has only space components. This means that it is normal to the particle's four-vector velocity relative to its frame and also the inertial reference frame

$$\mathbb{U} \bullet \mathbb{S} = u_\mu s_\mu = 0 \tag{5.82}$$

If the electromagnetic fields are uniform, the equation for spin is given by the BMT equation[10)]

$$\frac{ds_\mu}{d\tau} = \frac{q}{m}\left[\frac{g}{2}F_{\mu\nu}s_\nu + \frac{1}{c^2}\left(\frac{g}{2}-1\right)u_\mu s_\nu F_{\nu\lambda}u_\lambda\right] \tag{5.83}$$

where g is called the gyro-magnetic ratio. By using an analogy with orbital angular momentum of systems of charged particles and the concept of magnetic dipole moment, we can show $g = 1$. However, experiments show it is a number very near 2. The Dirac relativistic wave equation for an electron shows $g = 2$ [10]. Therefore, the BMT equation becomes

$$\frac{ds_\mu}{d\tau} = \frac{q}{m}F_{\mu\nu}s_\nu \tag{5.84}$$

This is fantastic! It is seen that the value $g = 2$ is compatible with the developed geometrical-kinematical theory of electrodynamics. The spin four-vector is an attached four-vector, which is rotating with

$$\Omega_{\mu\nu} = \frac{q}{m}F_{\mu\nu}(\tilde{x}) \tag{5.9}$$

Therefore, the constant length spin rotates with the body frame, such that

$$\frac{ds_\mu}{d\tau} = \Omega_{\mu\nu}s_\nu = \frac{q}{m}F_{\mu\nu}s_\nu \tag{5.85}$$

It should be noticed that the spin four-vector has only space components in its body frame, which is consistent with (5.82).

Interestingly, now we realize that the analogy to orbital angular momentum and using the concept of magnetic dipole moment, which leads to $g = 1$, is misleading.

6. Maxwellian theory of gravity

The Maxwellian theory of gravity generalizes the Newtonian theory of gravity to moving masses. It is clear that this is the compatible theory with our geometrical theory of interaction. The peculiarity of this theory, although classical theory offers no compelling reason behind it, is that the gravitational charge m_G is proportional to the inertial mass m, as far as we know. This is called the equivalence principle, which means in a proper system of units, such as the SI system, these two masses are equal

$$m_G = m \tag{6.1}$$

However, it should be noticed that in the developed geometrical interaction theory, the equivalence principle is not a fundamental necessity at all. If, in future, this principle is invalidated in some range of masses, this theory will still remain valid.

In this theory, the gravitational mass (charge) induces the four-momentum per unit gravitational mass or gravitational four-velocity $\mathbb{U}$, where

$$\mathbb{U} = U_\mu \mathbb{e}_\mu = (\mathbf{U}, U_4) \tag{6.2}$$

to the ether relative to the space-time inertial observer. Because of the equivalence principle, the gravitational four-velocity field $\mathbb{U}$ looks like the four-velocity $\mathbb{u}$ of the particle. This explains why we use the symbol $\mathbb{U}$ to represent this velocity-like field.

By analogy to the electromagnetic theory, U_4 should be related to the scalar Newtonian potential $\varPhi$. It will be shortly shown that

$$U_4 = -i\frac{\varPhi}{c} \tag{6.3}$$

The anti-symmetric four-tensor gravitational intensity field is characterized by the curl

$$\varOmega_{G\mu\nu} = \left(\partial_\nu U_\mu - \partial_\mu U_\nu\right) \tag{6.4}$$

which is the gravitational four-vorticity induced in the ether measured by an inertial observer analogous to $F_{\mu\nu}$ in electrodynamics. We have chosen the symbol $\varOmega_{G\mu\nu}$ to emphasize the analogy of the space gravitational vorticity to vorticity in classical fluid mechanics. In terms of components

$$\mathbf{\Omega}_G(x) = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega} G} & -i\frac{1}{c}\boldsymbol{\eta}_G \\ i\frac{1}{c}\boldsymbol{\eta}_G{}^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\omega_{G3} & \omega_{G2} & -i\eta_{G1}/c \\ \omega_{G3} & 0 & -\omega_{G1} & -i\eta_{G2}/c \\ -\omega_{G2} & \omega_{G1} & 0 & -i\eta_{G3}/c \\ i\eta_{G1}/c & i\eta_{G2}/c & i\eta_{G3}/c & 0 \end{bmatrix} \tag{6.5}$$

where $\boldsymbol{\omega}_G$ and $\boldsymbol{\eta}_G/c$ are gravitational circular and hyperbolic angular vorticitiy fields, respectively. By decomposition of the vorticity tensor defined by (6.4), we obtain

$$\frac{1}{c}\boldsymbol{\eta}_G = \frac{1}{c}\frac{\partial \mathbf{U}}{\partial t} - i\nabla U_4 \tag{6.6}$$

$$\boldsymbol{\omega}_G = \nabla \times \mathbf{U} \tag{6.7}$$

The space vorticity $\boldsymbol{\omega}_G$ is called the co-gravitational, magnetic gravity or gyro-gravitation vector [16]. From the Newtonian theory of gravity, we see

$$\boldsymbol{\eta}_G = \mathbf{g} \tag{6.8}$$

and therefore

$$U_4 = -i\frac{\Phi}{c} \tag{6.3}$$

and the relation (6.6) can be written as

$$\mathbf{g} = \boldsymbol{\eta}_G = \frac{\partial \mathbf{U}}{\partial t} - \nabla \Phi \tag{6.9}$$

Equation (4.6) represents the equation of motion for a particle in an arbitrary field, such as the electromagnetic field. The corresponding equation of motion for a particle in a general gravitational field becomes

$$m\frac{du_\mu}{d\tau} = m_G \Omega_{G\mu\nu}(\tilde{x})u_\nu \tag{6.10}$$

By using the equivalence principle (6.1), we obtain the geometrical equation of motion

$$\frac{du_\mu}{d\tau} = \Omega_{G\mu\nu}(\tilde{x})u_\nu \tag{6.11}$$

Therefore, the four-tensor angular velocity of the body frame is

$$\Omega_{\mu\nu} = \Omega_{G\mu\nu}(\tilde{x}) \tag{6.12}$$

This equation for components gives

$$\boldsymbol{\eta} = \boldsymbol{\eta}_G(\tilde{x}) = \mathbf{g}(\tilde{x}) \tag{6.13}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_G(\tilde{x}) \tag{6.14}$$

Note that $\boldsymbol{\eta}_G(\tilde{x})$ and $\boldsymbol{\omega}_G(\tilde{x})$ are the gravitational vorticities of the field at the position of the particle, while $\boldsymbol{\eta}$ and $\boldsymbol{\omega}$ represent the angular velocities of the body frame of the particle.

It is seen that the time and space components of the equations of motion are

$$\frac{d\mathbf{u}}{d\tau} = \gamma\mathbf{g} + \boldsymbol{\omega}_G \times \mathbf{u} \tag{6.15}$$

$$\frac{du_4}{d\tau} = \frac{i}{c}\mathbf{g} \bullet \mathbf{u} \tag{6.16}$$

which can also be written as

$$m\frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right) = m(\mathbf{g} + \boldsymbol{\omega}_G \times \mathbf{v}) \tag{6.17}$$

$$\frac{d}{dt}\frac{mc^2}{\sqrt{1 - v^2/c^2}} = m\mathbf{g} \bullet \mathbf{v} \tag{6.18}$$

similar to the electromagnetic interaction. The first of these is the equation of motion and its right hand side is the gravitational Lorentz force. The second equation defines the rate at which the gravitational field does work on the particle.

In analogy with electromagnetic theory, the field equation is postulated to be

$$\partial_\nu \Omega_{G\mu\nu} = -\frac{4\pi G}{c^2} J_\mu \tag{6.19}$$

where G is the gravitational constant in Newton's theory of gravitation and

$$J_\mu = (\mathbf{J}, J_4) = (\mathbf{J}, i\rho c) = \rho(\mathbf{v}, ic) \tag{6.20}$$

is the four-vector mass current density. It should be noticed that the negative sign in the right hand side of (6.19) is in agreement with the Newtonian gravity, where masses attract each other. Equation (6.19) is the compact form of the inhomogeneous equations

$$\nabla \bullet \mathbf{g} = -4\pi G\rho \tag{6.21}$$

$$\nabla \times \boldsymbol{\omega}_G = -\frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t} + \frac{4\pi G}{c^2}\mathbf{J} \tag{6.22}$$

where equation (6.21) is Gauss's law for the gravitation and equation (6.22) is the general Ampere's law for this Maxwellian theory of gravity.

Based on the definition of the vorticity field $\Omega_{G\alpha\beta}$ in (6.4), we have the compatibility equation

$$\partial_\alpha \Omega_{G\beta\gamma} + \partial_\beta \Omega_{G\gamma\alpha} + \partial_\gamma \Omega_{G\alpha\beta} = 0 \tag{6.23}$$

which gives the homogeneous equations

$$\nabla \bullet \boldsymbol{\omega}_G = 0 \tag{6.24}$$

$$\nabla \times \mathbf{g} - \frac{\partial \boldsymbol{\omega}_G}{\partial t} = 0 \tag{6.25}$$

The first equation (6.24) is Gauss's law for the gyro-gravitation vector, while the second equation (6.25) is Faraday's law of induction for this Maxwellian theory of gravity.

In analogy with electrodynamics, the induced four-vector velocity field U_μ is not uniquely determined from $\Omega_{G\mu\nu}$. The new field

$$U_\mu \to U'_\mu = U_\mu + \partial_\mu \chi \tag{6.26}$$

does not change the gravitational vorticity field $\Omega_{G\mu\nu}$. Here χ is a scalar function of x. This gauge freedom allows us to impose the constraint

$$\partial_\mu U_\mu = \frac{\partial U_\mu}{\partial x_\mu} = 0 \tag{6.27}$$

Therefore, the scalar χ is not that arbitrary. It must satisfy the wave equation

$$\partial_\mu \partial_\mu \chi = \nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = 0 \tag{6.28}$$

which can be considered as representing gravitational inertial waves. By considering the gauge invariance, we obtain the covariant wave equation

$$\partial_\alpha \partial_\alpha U_\mu = -\frac{4\pi G}{c^2} J_\mu \tag{6.29}$$

This equation relates the mass current density to the gravitational velocity field U_μ induced to the ether relative to the inertial reference frame. Here, the right hand side coefficient has been adjusted such that the Newtonian theory can be recovered for stationary masses. For this case

$$\nabla^2 U_4 = -i\frac{4\pi G}{c}\rho \tag{6.30}$$

Then by using U_4 from equation (6.3), we obtain

$$\nabla^2 \Phi = 4\pi G \rho \tag{6.31}$$

which is the well known Poisson equation in Newtonian theory. In this theory, the gravitational field $\mathbf{g}$ is obtained from the relation

$$\mathbf{g} = \boldsymbol{\eta}_G = -\nabla \Phi \tag{6.32}$$

which shows the gravitational hyperbolic vorticity $\frac{1}{c}\mathbf{g}$ is the result of only the time component of the gravitational velocity $-\frac{1}{c}\Phi$.

It should be noticed that a vortex theory describing gravity is not new. Descartes devised a theory of vortices which postulated that the space was entirely filled with a subtle matter, some kind of effluvium, not much different from the ether of later authors. He postulated that the sun by its rotation causes this effluvium to be concentrated in space vortices that carry the planets around the sun on their orbits [17]. It is seen that this form of vortex theory resembles the co-gravitational part of the Maxwellian gravity. As we know now, this co-gravitational part does not have that much affect on planetary motions around the sun. Actually, this part can be considered as a small perturbation to the dominant Newtonian gravitation.

Now we understand why Newton refuted a vortex theory to explain gravity. This is because he could not relate the vortex theory to his theory of gravitation. How could he have imagined his theory as a hyperbolic vortex theory with hyperbolic rotation instead of familiar circular rotation? He could not believe that his theory could be completed by adding circular vorticity as gyro-gravity part of gravity. Despite the extensive geometrical analysis in his work, Newton did not have any geometrical explanation for his theory of gravitation [18]. However, the vortex theory of Descartes was so appealing that it had many proponents such as Bernoulli who proposed that space is permeated with tiny whirlpools.[13)] It is this theory which Maxwell and other investigators used to explain the electromagnetic phenomenon as we discussed in Section 5. Now, we clearly know that this vortex theory only explains the magnetic part of the electromagnetic phenomenon.

Based on historical records, the developed Maxwellian theory of gravity should be called the Newton-Heaviside theory of gravity [19]. Jefimenko [16] provides a collection of solved problems regarding moving and stationary bodies of different shapes, sizes and configurations.

6.1. Gravitational four-stress tensor and mechanical view of Lorentz force

In the Maxwellian theory of gravity, massive particles are the source of the gravitational vorticity field four-tensor $\Omega_{G\mu\nu}$, where

$$\partial_\nu \Omega_{G\mu\nu} = -\frac{4\pi G}{c^2} J_\mu \tag{6.19}$$

They are also the source of stress four-tensor $T_{\mu\nu}$ fields in the ether. To obtain the constitutive relation, we notice that the gravitational four-vector Lorentz force density relative to the space-time inertial reference frame is given by

$$f_\mu = \Omega_{G\mu\nu} J_\nu \tag{6.33}$$

By substituting J_μ from the equation of gravitational vorticity (6.19) and some tensor algebra, we obtain

$$f_\mu = \partial_\nu T_{\mu\nu} \tag{6.34}$$

where the four-stress tensor is

$$T_{\mu\nu} = -\frac{c^2}{4\pi G}\left(\Omega_{G\mu\sigma}\Omega_{G\sigma\nu} + \frac{1}{4}\delta_{\mu\nu}\Omega_{G\alpha\beta}\Omega_{G\alpha\beta}\right) \tag{6.35}$$

The explicit form of the components of this four-tensor in terms of $\mathbf{g}$ and $\boldsymbol{\omega}_G$ are the gravitational Maxwell stress tensor

$$T_{ij} = -\frac{1}{4\pi G}(g_i g_j - \frac{1}{2}g_k g_k \delta_{ij}) - \frac{c^2}{4\pi G}(\omega_{Gi}\omega_{Gj} - \frac{1}{2}\omega_{Gk}\omega_{Gk}\delta_{ij}) \tag{6.36}$$

the gravitational energy density

$$T_{44} = u_G = -\frac{1}{8\pi G}\left(g^2 + c^2\omega_G^2\right) \tag{6.37}$$

and

$$T_{4i} = T_{i4} = i\frac{c}{4\pi G}\left(\mathbf{g}\times\boldsymbol{\omega}_G\right)_i = -\frac{i}{c}S_{Gi} \tag{6.38}$$

where $\mathbf{S}_G$ is the gravitational Poynting vector

$$\mathbf{S}_G = -\frac{c}{4\pi G}\mathbf{g}\times\boldsymbol{\omega}_G \tag{6.39}$$

Therefore, the schematic matrix form of the symmetric four-tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = \begin{bmatrix} T_{ij} & -\frac{i}{c}\mathbf{S}_G \\ -\frac{i}{c}\mathbf{S}_G & u_G \end{bmatrix} \tag{6.40}$$

What we notice is that the components of the four-stress tensor are the negative of their corresponding electromagnetic ones. This is the character of the gravitational interaction in Newtonian theory. As in electromagnetic interaction, massive particles interact via inducing four-stress and four-vorticity tensors in ether. Similarly, the point massive particles are singularities of these vorticities and four stress tensors. Therefore, the gravitational Lorentz forces exerting on these particles can also be considered as four-dimensional lift forces. This is also the mechanical explanation of four-vector gravitational Lorentz force.

7. Conclusion and discussion

We have seen that every massive particle specifies a Minkowskian space-time body frame in a universal entity, here referred to as ether. This aspect of space-time clarifies Poincare's theory of relativity. Inertial observers relate components of four-vectors and four-tensors by Lorentz transformation. This is the origin of non-Euclidean geometry governing the three vector and three tensor components. The hyperbolic geometry of the velocity addition law is the manifest of this fact. The space components of the four-vector velocity in the particle body frame are zero. However, its component in the space of a reference frame is the result of a hyperbolic angle deviation. Therefore, the three-velocity vector in the reference frame is a hyperbolic vector and geometry governing the three-velocity addition law in the reference frame is hyperbolic.

The acceleration of a particle is the result of the instantaneous rotation of its body frame in the ether. This instantaneous rotation is specified by the four-dimensional angular velocity tensor in the inertial reference frame. The hyperbolic part of this rotation is actually what is known as accelerating motion. However, there is also circular space rotation, which is essential in understanding some phenomena, such as the spin precession of a stationary charged particle in a magnetic field.

Based on the theory of motion, the geometrical character of fundamental interaction has been developed. This theory shows that every fundamental interaction is represented by an anti-symmetric strength four-tensor field with characteristics of a vorticity field. Charged particles interact with each other through four-vorticity and four-stress that they induce in the ether. The four-vorticity tensor is a combination of three-vector circular and hyperbolic vorticities. It is seen that a Lorentz-like Minkowski force is an essential feature of every fundamental interaction. This vortex theory gives a clear geometrical explanation of electrodynamics, which is a model for any other interaction. Through this theory, we realize that the homogeneous Maxwell's equations are actually necessary compatibility equations for electromagnetic vorticity vectors, and the inhomogeneous Maxwell's equations are equations governing motion of these vorticities. It is seen that the energy-momentum four-tensor has the character of a four-stress tensor and its expression in terms of electromagnetic vorticities is a constitutive relation. This reveals the mechanical character of Lorentz force as a four-dimensional lift force perpendicular to four-vector velocity. This vortex theory shows why a magnetic monopole cannot exist. It also clarifies the spin dynamics of charged elementary particles in a classical view.

In addition, the geometrical theory of interaction shows that a Maxwellian theory of gravity is inevitable. Interestingly, this is the reconciliation of the vortex theory of Descartes and Bernoulli with Newton's theory of gravitation. This is more compelling when we notice that the other fundamental forces such as weak and strong forces are generalizations of the electromagnetic theory in non-Abelian gauge theory based on local symmetry groups SU(2) and SU(3). This is completely compatible with our unification of fundamental interactions based on the vortex theory. Therefore, it is necessary to develop the geometrical aspect of these quantum mechanical generalizations.

Interestingly, the new theory of space-time has the potential to clarify the wave-particle duality of matter. In this regard, the quantum mechanical wave function of a particle seems to be the trace of its space-time body frame on the observer's reference frame. This is more promising when we realize the deformability of space-time of an interacting particle. Therefore, quantum theory has the same fate as electrodynamics theory and must be presented based on the new theory of relative space-time in the ether. However, we should notice the ambiguity in introducing ether. We used the term ether for the universal entity in which a massive observer specifies a space-time. How can we visualize ether when the concepts of where and when cannot be applied? It should be noticed that this ether is different from the ether conceived by earlier proponents of space vortex theory. They considered the ether some sort of matter filling the space. But our ether is something in which a particle specifies a four-dimensional orthogonal system with three real and one imaginary axis, which we call space-time. In other words, the space-time body frames of particles are different representations of ether.

Although Lorentz, Poincare, Minkowski, Varičak, Borel and others have developed important aspects of the theory of relativity, the fundamental meaning of space-time and its relation with the ether have not been appreciated. It is realized that these are the origin of many inconsistencies in modern physics. Through the developed theory of motion and interaction, one appreciates the work of those who questioned the fifth postulate in Euclidean geometry about parallel lines and considered the possibility of non-Euclidean geometry by modifying this postulate. It is stunning to see that the rules of motion and interaction are governed by non-Euclidean geometry, because all motion is a four-dimensional rotation. We realize that the theory of motion is a model for hyperbolic geometry.

One can see that continuum mechanics has played an essential role in developing the present theory of motion and interaction. Interestingly, Maxwell also used continuum mechanics in his development of electrodynamics. It is known that he generalized Ampere's law (5.30) by adding displacement current to have a consistency with the electric charge continuity equation (5.17). Had Maxwell, Lord Kelvin or their contemporaries known about a four-dimensional space-time, Lorentz transformation and covariance of electrodynamics, could they not have developed this four-dimensional vortex theory? Answering is not difficult when one learns that they were already talking about vortices in ether, which they inherited from Descartes.

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