A complete graph model of the Schwarzschild black hole in $\mathbf{R}3$

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November 19, 2010

Where $\hbar = c = G = 1$, the following components will be used to model a Schwarzschild black hole of rest mass-energy E in $\mathbf{R}3$:

1. A 2-sphere (event horizon) S_0 at coordinate distance $R_0 = 2E$, upon which lies $N_0 = E$ uniformly distributed vertices V_0.
2. A complete graph's worth of edges E_0 generated by V_0.
3. An exterior volume at $R > R_0$, upon which lies a countable number of vertices V_{ext}.
4. A countable number of (non-complete) graph edges E_{ext} generated by the Delaunay tetrahedralization of V_0 and V_{ext}.

The following presumptions are made:

1. Neither the event horizon nor the black hole centre at $R = 0$ are singular in any way.
2. The complete graph edges E_0 define a universal edge coordinate length of

$$L = \frac{1}{\sqrt{1 - R_0/R}}, \tag{1}$$

   where R is the coordinate distance between an edge's midpoint and the black hole centre. Accordingly, edge proper length is L^2 (e.g., light travels across an edge at coordinate speed $1/L$).
3. The complete graph edges E_0 define a universal minimum edge coordinate length of $L = 1$ (e.g., the Planck length).

The following steps are used to construct the model's components:

1. With regard to the 2-sphere S_0, numerically solve for the coordinate radial distance $R_1 > R_0$ of a second 2-sphere S_1. Using the formula for the height of a regular tetrahedron

$$H_{\text{tet}} = L\sqrt{2/3} \tag{2}$$

   as a guide:

$$R_1 = R_0 + H_0, \tag{3}$$

$$\frac{H_0}{\sqrt{2/3}} \approx \frac{1}{\sqrt{1 - \frac{R_0}{R_0 + H_0/2}}}. \tag{4}$$

2. Calculate the number of vertices N_1 that lie upon S_1. Using the formulas for the area of a regular triangle

$$A_{\text{tri}} = L^2(1/4)\sqrt{3} \tag{5}$$

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and the Euler characteristic of a closed convex polyhedron

$$N_1 + F_1 - E_1 = 2, \quad F_1 = 2N_1 - 4 \tag{6}$$

as a guide:

$$L_1 = \frac{1}{\sqrt{1 - R_0/R_1}}, \tag{7}$$

$$F_1 = \frac{4\pi R_1^2}{L_1^2} \frac{4}{\sqrt{3}}, \tag{8}$$

$$N_1 = \frac{4 + F_1}{2}. \tag{9}$$

3. Numerically solve for H_1:

$$\frac{H_1}{\sqrt{2/3}} \approx \frac{1}{\sqrt{1 - \frac{R_0}{R_1 + H_1/2}}}. \tag{10}$$

4. Repeat steps 2 and 3 for each subsequent 2-sphere $S_{\geq 2}$:

$$R_{\geq 2} = R_{\geq 1} + H_{\geq 1}, \tag{11}$$

$$N_{\geq 2} = 2 + \frac{8\pi R_{\geq 2}(R_{\geq 2} - R_0)}{\sqrt{3}}, \tag{12}$$

$$H_{\geq 2} \approx \frac{1}{\sqrt{\frac{3}{2} - \frac{3R_0}{2R_{\geq 2} + H_{\geq 2}}}}. \tag{13}$$

5. Generate the vertices V_0 that lie upon S_0. Use Coulomb repulsion on S_0 to make the vertex distribution roughly uniform.

6. Obtain the complete graph edges E_0 generated by V_0.

7. Generate the vertices $V_{\geq 1}$ that lie along each 2-sphere $S_{\geq 1}$. Use Coulomb repulsion on each 2-sphere to make its vertex distribution roughly uniform, if desired.

8. Obtain the (non-complete) graph edges generated by the Delaunay tetrahedralization of all vertices $V_{\geq 0}$.

Depending on how well the vertices V_{ext} are uniformly distributed along their respective shells, one will have to multiply $H_{\geq 0}$ and $N_{\geq 1}$ by some small constant values (e.g., roughly on the order of 1) in order to meet the edge coordinate length requirement given in Eq. 1 with accuracy.

See Ref. [1] for a public domain C++ code that generates this model's vertices and edges. Edge analysis code is included. The default configuration produces an edge coordinate length accuracy of ~ 0.99. See Fig. 1 for an example manifold. As with all discretization models [2], edge coordinate length accuracy is based on an average.

Unlike most other discretization models, this model does not allow one to arbitrarily choose the scale of the tetrahedra (e.g., $dx^\mu dx^\nu \equiv 1$ here). As such, the manifold is geodesically complete by definition, not by choice.

Thank you to P. Gibbs for his work on complete graphs [3].

References

[1] Google Code. (2010) `http://code.google.com/p/cgmetric/downloads/list`

[2] McDonald JR, Miller WA. A Discrete Representation of Einsteins Geometric Theory of Gravitation: The Fundamental Role of Dual Tessellations in Regge Calculus. (2008) arXiv:0804.0279v1 [gr-qc]

[3] Gibbs PE. Event-Symmetric Space-Time. (1998) viXra:0911.0042

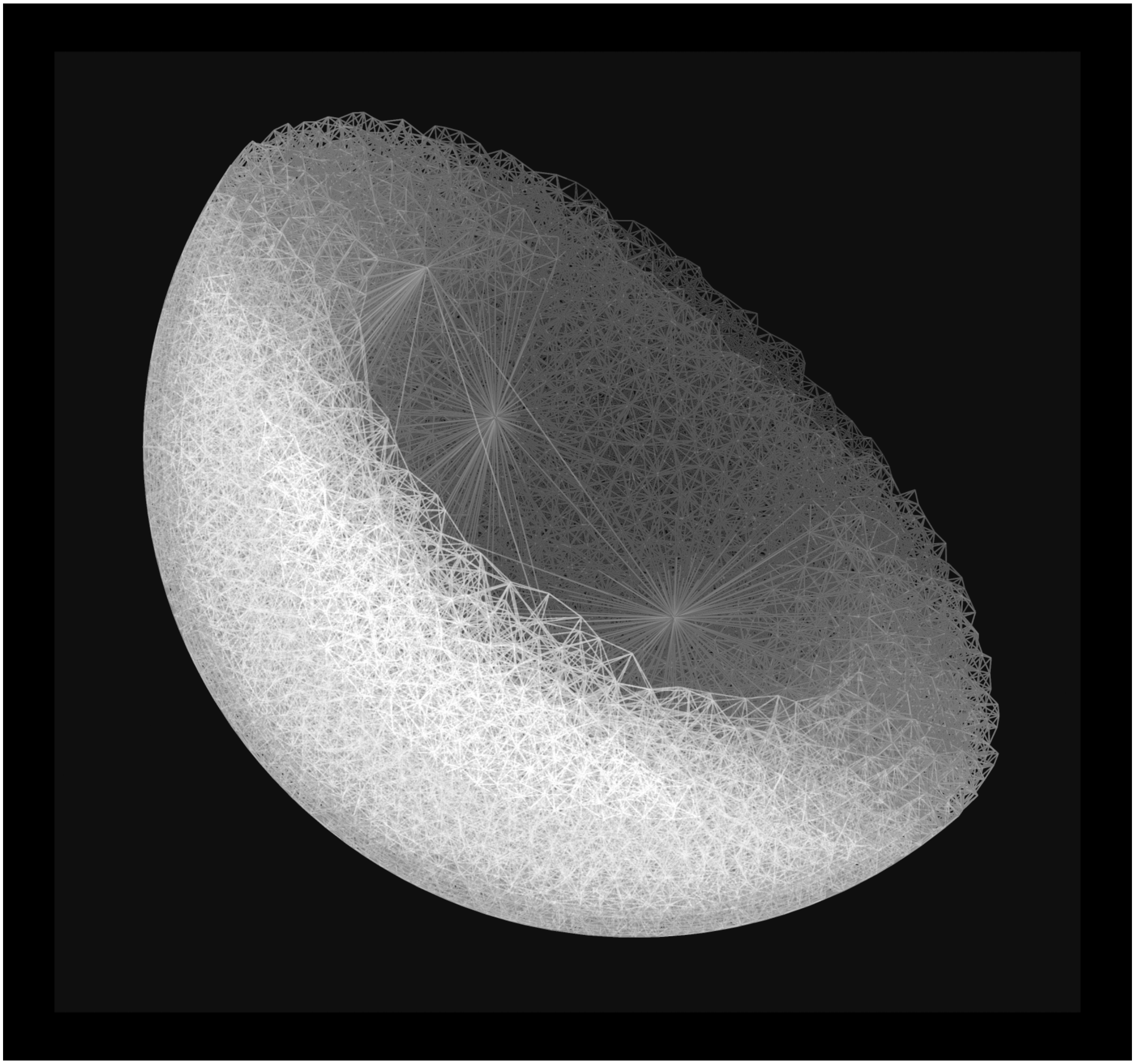

Figure 1: One half of the edges for 2-spheres S_0 through S_{10}, where $N_0 = 10$. The manifold is geodesically complete, since it does not contain any infinitely small or large edges, or "dead end" paths. Edge coordinate length decreases as r increases. The figure was rendered using OpenGL / Microsoft Visual C++ Express 2010, and was post-processed using Rick Brewster's Paint.NET. Space kitten is generally nonplussed by π.