Lattice Rationals

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Abstract

This paper redefines the addition of rational numbers, in a way that allows division by zero. This requires defining a "compensator" on the integers, plus extending least-common-multiple (LCM) to zero and negative numbers. "Compensated addition" defines ordinary addition on all ratios, including the 'infinities' n/0, and also 'zeroids' 0/n. The infinities and the zeroids form two 'double ringlets'. The lattice rationals modulo the zeroids yields the infinities plus the 'wheel numbers'. Due to the presence of the 'alternator' @ = 0/-1, double-distribution does not apply, but triple-distribution still does.

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1. Adding infinities and Compensated Addition

This paper began with my desire to add infinities. In particular I wanted the following to be valid:

$$2/0 \;+\; 3/0 \;=\; 5/0$$

This is consistent with the one-denominator rule for adding fractions:

$$a/b \;+\; c/b \;=\; (a+c)/b$$

but not with the two-denominators rule:

$$a/b \;+\; c/d \;=\; (ad+bc)/(bd)$$

for then $2/0 + 3/0 = 0/0$, the indefinite ratio.

How to harmonize the two rules? Well, how about finding a rule that covers both cases?

Consider the following addition:

$$5/12 \;+\; 7/18 \;=\; (5*3 + 7*2)/36 \;=\; 29/36$$

Where did that 36 come from? It is the lowest common multiple of 12 and 18: lcm(12,18)=36. But where did the 3 and the 2 come from? I call these ‘compensating factors’ or ‘compensators’. They compensate for the new denominator:

$$5/12 \;=\; (5*3)/36 \qquad : \qquad 7/18 \;=\; (7*2)/36$$

Therefore let us define the “compensator of b to d”, a.k.a. “b;d”, thus:

$$b;d \quad = \quad \mathrm{lcm}(b,d)\;/\;d \quad = \quad b\;/\;\mathrm{gcf}(b,d)$$

where gcf is greatest common factor. Then we have the rule:

$$a/b + c/d \quad = \quad (\; a*(d;b) + c*(b;d)\;)\;/\;\mathrm{lcm}(b,d)$$

This is the *Compensated Addition Rule*.

2. LCM and compensator for positive numbers

We can define compensator, lcm and gcf from prime factorizations:

$\mathrm{lcm}(\ \mathrm{product}(p_i\char`^a_i),\ \mathrm{product}(p_i\char`^b_i)\ ) = \mathrm{product}(p_i\char`^\ \max(a_i, b_i)\ )$

$\mathrm{gcf}(\ \mathrm{product}(p_i\char`^a_i),\ \mathrm{product}(p_i\char`^b_i)\ ) = \mathrm{product}(p_i\char`^\ \min(a_i, b_i)\ )$

$(\ \mathrm{product}(p_i\char`^a_i)\ ;\ \mathrm{product}(p_i\char`^b_i)\ ) = \mathrm{product}(p_i\char`^\ \max(a_i - b_i\ , 0)\ )$

Compensator, lcm and gcf follow these rules for positive numbers:

lcm(b,d) = (d;b)*gcf(b,d)*(b;d) = (d;b)*b = (b;d)*d

d = (d;b)*gcf(b,d) : b = (b;d)*gcf(b,d)

gcf(b,b) = b ; lcm(b,b) = b ; (b;b) = 1

gcf(b,bc) = b ; lcm(b,bc) = bc ; (b;bc) = 1 ; (bc;b) = c

gcf(1,c) = 1 ; lcm(1,c) = c ; (1;c) = 1 ; (c;1) = c

Distribution laws:

gcf(a, lcm(b,c)) = lcm(gcf(a,b), gcf(ac))

lcm(a, gcf(b,c)) = gcf(lcm(a,b), lcm(ac))

a*lcm(b,c) = lcm(a*b, a*c)

a*gcf(b,c) = gcf(a*b, a*c)

These last two laws are because addition distributes over minimum and maximum, and lcm and gcf are defined by minima and maxima of exponents. For instance:

(2^3 * 3^5) * lcm(2^2 * 3^1, 2^1 * 3^4)

= (2^3 * 3^5) * 2^max(2,1) * 3^max(1,4))

= 2 ^ (3 + max(2,1)) * 3 ^ (5 + max(1,4))

= 2 ^ max(3+2, 3+1)) * 3 ^ max(5+1, 5+4)

= lcm (2^(3+2) * 3^(5+1) , 2^(3+1) * 3^(5+4))

= lcm ((2^3 * 3^5)*(2^2 * 3^1) , (2^3 * 3^5)*(2^1 * 3^4))

Also, the distribution of multiplication over lcm implies *Cancellation*:

(a*b ; a*d) = (b ; d)

Commutativity and *associativity*:

lcm(a,b) = lcm(b,a) ; lcm(a,lcm(b,c)) = lcm(lcm(a,b),c))

gcf(a,b) = gcf(b,a) ; gcf(a,gcf(b,c)) = gcf(gcf(a,b),c))

These laws, plus the compensator definition of addition, yield "*Semicancellation*":

(a*c)/(b*c) = a/b + 0/(bc)

I also call this law "*casting out zeroids*", where a 'zeroid' is a ratio with numerator zero.

This in turn implies "*Compensated Reduction*":

a / b = (a;b) / (b;a) + 0/b

Since lcm(b,b) = b and (b;b)=1, we recover the one-denominator rule:

(a/b) + (c/b) = (a*(b;b) + c*(b;b)) / lcm(b,b) = (a+c)/b

But if b and d are relatively prime, then lcm(b,d)=b*d; (b;d)=b; and (d;b)=d; so we recover the two-denominators rule:

(a/b) + (c/d) = (a*d + c*b) / (b*d)

3. GCF, LCM, compensators and the Euclidean Algorithm

Here are some rules uniting addition with gcf, lcm, and the compensator:

gcf(a,b) = gcf(a-b, b) = gcf(a+b, b) = gcf(a+nb, b) for any integer n.

Define (a mod b) to be the remainder of a when divided by b. (A mod B) equals A+nB for some n; therefore:

gcf(a, b) = gcf(a mod b, b) = gcf(a, b mod a)

This, along with the rule:

gcf(a,0) = gcf(0,a) = a

implies the Euclidean algorithm. For instance:

gcf(52,20) = gcf(12,20) = gcf(12,8) = gcf(4,8) = gcf(4,0) = 4

Modulation ping-pongs across gcf until resolution. From these rules:

Lcm(a,b) = a*b/gcf(a,b)

(a;b) = a / gcf(a,b)

we can derive these Euclidean-algorithm-like rules:

lcm(a,b) = (a/(a mod b)) * lcm(a mod b, b) = (b/(b mod a)) * lcm(a, b mod a)

lcm(ab,a) = lcm(a,ab) = ab

(a;b) = (a ; b mod a) = (a/(a mod b)) * (a mod b ; b)

(a;ab) = 1

(ab;b) = a

Therefore, for instance:

Lcm(52,20) = (52/12)*lcm(12,20) = (52/12)*(20/8)*lcm(12,8) = (52/12)*(20/8)*(12/4)*lcm(4,8) = (52/12)*(20/8)*(12/4)*8 = 260

(52;20) = (52/12)*(12;20) = (52/12)*(12;8) = (52/12)*(12/4)*(4;8) = (52/12)*(12/4)*1 = 13

(20;52) = (20;12) = (20/8)*(8;12) = (20/8)*(8;4) = (20/8)*2 = 5

4. LCM and compensator for zero

To define sums for ratios with denominator zero, we need to define gcf, lcm and compensator for zero. Since every number divides into zero, and zero divides into none, it is at the top of the divisibility lattice; therefore zero is an attractor for lcm and an identity for gcf:

$$\mathrm{lcm}(a, 0) = 0 \qquad ; \qquad \mathrm{gcf}(a,0) = a$$

Since $(a;b) = \mathrm{lcm}(a,b)/b = a/\mathrm{gcf}(a,b)$, it follows that

$$(a;0) = 0/0 = a/a$$

The first equation is useless; 0/0 is indefinite; but a/a equals one; so let us take as a rule:

$$(a;0) = 1$$

Now $(0;a) = 0/a$; this is 0 if a is not zero, indefinite if a=0. So what is (0;0)? If we take the rule that (0;0)=0, then we get the equation:

$$(a/0) + (c/0) = (0/0)$$

This is the two-denominators result. But if we assume that (0;0) = 1, then:

$$(a/0) + (c/0) = (a+c)/0$$

This is the one-denominator result, as requested. Therefore in this paper I shall take the rule:

$(0;a) \quad = \quad 0 \qquad$ if a does not equal zero; and

$(0;0) \quad = \quad 1$

Then in general:

$(a/0) + (b/c) \quad = \quad (a/0) \quad$ if c does not equal zero;

$(a/0) + (b/0) \quad = \quad (a+b)/0$

Infinities absorb finite quantities, but add like integers, by adding numerators.

5. LCM and compensator for negative numbers and ratios

For simplicity's sake, and to resemble the 2-denominator rule as much as possible, I propose that lcm is an odd function, and gcf is an even function:

lcm(-a,b) = lcm(a,-b) = - lcm(a,b)

gcf(-a,b) = gcf(a,-b) = gcf(a,b)

so

lcm(-a,-b) = lcm(a,b) ; gcf(-a,-b) = gcf(a,b)

and as before:

lcm(a,b)*gcf(a,b) = a*b

We no longer have double-distribution of multiplication over lcm:

-1 * lcm(2,3) = -6 but lcm(-2,-3) = 6

But we still have *triple*-distribution:

a * lcm(b,c,d) = lcm(ab,ac,ad)

and lcm and gcf still double-distribute over each other.

We still have:

lcm(ab,ac) = |a|*lcm(b,c)

gcf(ab,ac) = |a|*gcf(b,c)

a * lcm(b,c) = sign(a)*lcm(ab,ac)

We also have "*Alternation*":

lcm(a,a) = gcf(a,a) = |a|

lcm(a,|a|) = gcf(a,|a|) = a

The compensator is odd in the first variable and even in the second:

$(-a;b) = -(a;b)$ for a not equal to zero.

$(a;-b) = (a;b)$

All this, plus the Compensated Addition Rule, implies:

$(a/-b) + (c/-d) = (-a/b) + (-c/d)$

In particular, consider 0/-1; call it the "*alternator*" @. Then:

(a/-b) + @ = (-a/b)

(a/b) + @ = (-a/-b)

Note that 0 and @ form a modulo-2 group under both addition and multiplication:

0+0 = @+@ = 0*0 = @*@ = 0 ;

0+@ = @+0 = 0*@ = @*0 = @

Multiplication by @ does not double-distribute over addition;

@ *(0+0) = @*0 = @ ; but @*0 + @*0 = @+@ = 0

But it does triple-distribute:

@*(a+b+c) = @*a + @*b + @*c

Define lcm, gcf and compensator of ratios as ratios:

Lcm(a/A, b/B) = lcm(a,b) / lcm(A,B)

gcf(a/A, b/B) = gcf(a,b) / gcf(A,B)

(a/A ; b/B) = (a;b) / (A;B)

We get many of the same rules as above; for instance lcm*gcf = product.

6. LCM and compensator: review of definitions and laws

lcm(product(p_i^a_i), product(p_i^b_i)) = product(p_i^ max(a_i, b_i))

gcf(product(p_i^a_i), product(p_i^b_i)) = product(p_i^ min(a_i, b_i))

(product(p_i^a_i) ; product(p_i^b_i)) = product(p_i^ max(a_i - b_i , 0))

(a;b) = lcm(a,b) / a = a / gcf(a,b)

lcm(0,a) = 0

gcf(0,a) = a

(0;a) = 0 if a does not equal zero.

(a;0) = (0;0) = 1

lcm(-a,b) = lcm(a,-b) = - lcm(a,b)

gcf(-a,b) = gcf(a,-b) = gcf(a,b)

(-a;b) = - (a;b)

(a;-b) = (a;b)

lcm(a/A, b/B) = lcm(a,b) / lcm(A,B)

gcf(a/A, b/B) = gcf(a,b) / gcf(A,B)

(a/A ; b/B) = (a;b) / (A;B)

From these definitions, you can derive these laws:

Lcm and gcf are commutative and associative.

Lcm and gcf double-distribute over each other.

Absolute value distribution:

lcm(ab,ac) = |a|*lcm(b,c)

gcf(ab,ac) = |a|*gcf(b,c)

Triple distribution:

a * lcm(b,c,d) = lcm(ab,ac,ad)

Alternation:

lcm(a,a) = gcf(a,a) = |a|

lcm(a,|a|) = gcf(a,|a|) = a

Duality:

lcm(a,b) * gcf(a,b) = a*b

Venn Laws:

a = (a;b)*gcf(a,b)

b = (b;a)*gcf(a,b)

lcm(a,b) = (a;b)*gcf(a,b)*(b;a) = (a;b)*b = (b;a)*a

Consider a Venn diagram; two overlapping disks, representing a and b. The union of the disks corresponds to lcm, the intersection of the disks corresponds to gcf, and the two moon-shaped regions correspond to the compensators.

(b;b) = 1 if b is at least zero; (b;b) = -1 if b<0

(ab;ac) = (b;c) if a is at least zero; (ab;ac) = - (b;c) if a<0

(1;c) = 1

(c;1) = c

(0;c) = 0 if c is not zero; (0;0) = 1

(c;0) = 1 if c>0 ; (c;0) = 1 if c=0 ; (c;0) = -1 if c<0

7. Lattice rational arithmetic; definitions and laws

Define addition and subtraction by the compensator rule:

a/b + c/d = (a*(d;b) + c*(b;d)) / lcm(b,d)

a/b - c/d = (a*(d;b) – c*(b;d)) / lcm(b,d)

Define multiplication, reciprocal and division the usual ways:

(a/b)*(c/d) = (a*c) / (b*d)

1/(c/d) = (d/c)

(a/b)/(c/d) = (a*d)/(b*d)

From these we can define "*reduction*", a.k.a. "*reciprocal addition*":

(a/b) ⊕ (c/d) = 1 / ((1/(a/b)) + (1/(c/d)))

= lcm(a,c) / (b*(a;d) + c*(d;a))

Reduction is like addition, with the roles of numerator and denominator reversed. Therefore they follow similar rules. The following are provable:

(a/A) + (b/B) + (c/C) = (a (lcm(B,C);A) + b (lcm(C,A);B) + c (lcm(A,B);C)) / lcm(A,B,C)

(a/A) ⊕ (b/B)⊕ (c/C) = lcm(a,b,c) / (A (lcm(b,c);a) + B (lcm(c,a);b) + C (lcm(a,b);c))

Proof requires these lemmas:

(A;B)*(C; lcm(A,B)) = (lcm(A,C) ; B)

(lcm(db, dc) ; da) = (lcm(b,c) ; a)

Addition and reduction are commutative.

Proof by symmetry of definitions.

Addition and reduction are associative.

Another proof of symmetry of definitions.

Multiplication triple-distributes over addition and reduction.

Proof involves triple-distribution of multiplication over lcm.

Division distributes from the left, and anti-distributes from the right: (trivial proof)

$$(A+B+C) / D = (A/D) + (B/D) + (C/D)$$

$$(A \oplus B \oplus C) / D = (A/D) \oplus (B/D) \oplus (C/D)$$

$$A / (B+C+D) = (A/B) \oplus (A/C) \oplus (A/D)$$

$$A / (B \oplus C \oplus D) = (A/B) + (A/C) + (A/D)$$

Identities (trivial proof):

$$(a/b) = (a/b) + (0/1) = (a/b) \oplus (1/0) = (a/b) * (1/1)$$

Near-Inverses:

$$(a/b) + (-a/b) = 0/b$$

$$(a/b) \oplus (a/-b) = a/0$$

$$(a/b) * (b/a) = (ab)/(ab) = 1 + 0/(ab)$$

$$x/x = 1 + 0/x$$

0/-1 is known as the "*alternator*" @; its reciprocal is -1/0, negative infinity.

$$a/b + @ \quad = \quad -a/-b \quad = \quad a/b \oplus -\infty$$

Because of them we must weaken distribution to *triple distribution*:

$w(x+y+z)$	$=$	$wx + wy + wz$	if w is finite
$w(x \oplus y \oplus z)$	$=$	$wx \oplus wy \oplus wz$	if w is nonzeroid
$w(x+y)$	$=$	$wx + wy + 0w$	if w is finite
$w(x \oplus y)$	$=$	$wx \oplus wy \oplus w/0$	if w is nonzeroid

$$0x * 0y = 0xy$$

$$0x + 0y = 0(x+y) = 0(\mathrm{lcm}(x,y))$$

$$0/(nm) = 0/n + 0/(nm)$$

$$(nm)/0 = n/0 \oplus (nm)/0$$

0/0 is "*indefinity*", the indefinite ratio. It is an attractor for addition, reduction, multiplication and division for all finite nonzeroids:

$$a/b + 0/0 = a/b \oplus 0/0 = a/b * 0/0 = (a/b)/(0/0) = (0/0)/(a/b) = 0/0$$

if a and b are both not zero.

Indefinity acts like a *generic* finite nonzeroid, even though it is *both* an infinity *and* a zeroid!

0/0 is an identity for adding infinities, and an attractor for reducing infinities:

$$a/0 + 0/0 = a/b$$

$$a/0 + -a/0 = 0/0$$

$$a/0 \oplus 0/0 = 0/0$$

0/0 is an identity for reducing zeroids, and an attractor for adding zeroids:

$$0/a \oplus 0/0 = a/b$$

$$0/a \oplus 0/-a = 0/0$$

$$0/a + 0/0 = 0/0$$

Double Ringlets of Zeroids and Infinities

Recall that in general:

$(a/0) + (b/c) \quad = \quad (a/0)$ if c does not equal zero;

$(a/0) + (b/0) \quad = \quad (a+b)/0$

Infinities absorb finite quantities, but add like integers.

They also multiply like integers, if we use the usual definitions:

$(a/0)*(b/0) \quad = \quad (a*b) / 0$

Now consider these laws for infinities:

$(a/0) + (b/0) \quad = \quad (a+b) / 0$

$(a/0) \oplus (b/0) \quad = \quad \mathrm{lcm}(a,b) / 0$

$(a/0) * (b/0) \quad = \quad (a*b)/0$

Infinities add by adding indices, reduce by lcm on indices, and multiply by multiplying indices. I call this the 'double ringlet of infinities'; for in it multiplication double-distributes over addition in a ring and triple-distributes over reduction in a semi-ring.

Zeroids have these laws:

$(0/a) + (0/b) \quad = \quad 0 / \mathrm{lcm}(a,b)$

$(0/a) \oplus (0/b) \quad = \quad 0 / (a+b)$

$(0/a) * (0/b) \quad = \quad 0 / (a*b)$

Zeroids reduce by adding indices, add by lcm on indices, and multiply by multiplying indices. I call this the 'double ringlet of zeroids'; for in it multiplication double-distributes over reduction in a ring and triple-distributes over addition in a semi-ring.

8. The Wheel Numbers

TheWheel Numbers arise from the lattice rationals if you add the axiom:

$$0/n \quad = \quad 0/1 \qquad \text{for} \quad n>0$$

This is equivalent to “Positive Cancellation”:

$$(ac)/(bc) \quad = \quad a/b \qquad \text{if} \quad c>0$$

Note that the alternator is still unequal to zero, and negatives do not cancel.

The wheel numbers can be divided in mainstream, alternates, and null quotients.

The mainstream numbers are of the form a/b , in lowest terms, with b>0.

The alternate numbers are of the form a/b , in lowest terms, with b<0.

The null quotients are 1/0, infinity; -1/0, negative infinity; and 0/0, indefinity.

The wheel numbers correspond to a circle surrounding a point. Each wheel number corresponds to the slope from the center to the point. The point at the center corresponds to 0/0; the points directly above and below the center correspond to $+\infty$ and $-\infty$; the points directly right and left of the center correspond to 0 and @; the right half of the circle corresponds to the mainstream numbers, the left half of the circle corresponds to the alternate numbers.

		$+\infty$		
	-1+@		1	
@		0/0		0
	1+@		-1	
		$-\infty$		

Note that $\quad -\infty \quad = \quad +\infty + @$

and $\quad \infty \quad = \quad -\infty + @$

Reciprocal flips at the infinities between both sign and alternativity.

The wheel numbers have these laws:

Addition, reduction and multiplication are commutative and associative.

Identities:

$$x + 0/1 = x \oplus 1/0 = x*1/1 = x$$

Attractors:

$x + 1/0 = 1/0$ if x is finite

$x \oplus 0/1 = 0/1$ if x is nonzeroid.

Indefinities:

$$0 \oplus @ = \infty + (-\infty) = 0 * \infty = 0/0$$

Alternator:

$$X + @ = x \oplus -\infty = -x/-1$$

$$X + @ + @ = x \oplus -\infty \oplus -\infty = x$$

Triple Distribution:

$X*(A+B+C) = X*A + X*B + X*C$ if X is finite

$X*(A \oplus B \oplus C) = X*A \oplus X*B \oplus X*C$ if X is nonzeroid

Since @+@ = 0, it's consistent with exponential arithmetic to posit, for A not equal to 0:

$$A\hat{}@ = -1$$

and $A\hat{}(x+@) = -(A\hat{}x)$

So in general we can posit:

$$\log_A(-1) = @$$

and $\log_A(-x) = \log_A(x) + @$

This is a theory of logarithmic negation without reference to the complex numbers.

10. Lattice Rationals modulo zeroids and infinities

Say that x and y are "equal modulo zeroids", a.k.a. "$x =_0 y$", if and only if:

$$x + 0/n \quad = \quad y + 0/n \quad \text{for some positive integer } n$$

Therefore we have positive cancellation modulo zeroids:

$$(ac)/(bc) \quad =_0 \quad a/b \quad \text{if } c>0$$

Equality modulo zeroids is an equivalence relation; reflexive, symmetric, and transitive. Therefore it defines equivalence classes, and operations on those classes for well-defined on those classes. For instance:

If $x =_0 X$ and $y =_0 Y$ then;

$$x + y \quad =_0 \quad X + Y$$

$$x - y \quad =_0 \quad X - Y$$

$$x * y \quad =_0 \quad X * Y$$

But reciprocal is not well-defined on the zeroids;

$0/1 =_0 0/2$; but $1/0$ and $2/0$ are not equal modulo zeroids.

Similarly, reduction is not well-defined for the zeroids.

The equivalence classes, with operations thus defined, equals the wheel numbers plus the double ringlet of infinities. This is an arithmetic of the rationals, plus alternator, plus all the infinities.

Reciprocally, we could define equality modulo infinities:

"$x =_\infty y$" if and only if :

$$x \oplus n/0 \quad = \quad y \oplus n/0 \quad \text{for some positive integer } n$$

This too has positive cancellation:

$$(ac)/(bc) \quad =_\infty \quad a/b \quad \text{if } c>0$$

Reduction and multiplication are well defined modulo infinities; but reciprocal and addition are not well defined on the infinities. The equivalence classes modulo infinities equals the wheel numbers plus the double ringlet of zeroids.