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The Tetron Model as a Lattice Structure: Applications to Astrophysics

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Abstract

The tetron model is reinterpreted as an inner symmetry lattice model where quarks, leptons and gauge fields arise as lattice excitations. On this basis a modification of the standard Big Bang scenario is proposed, where the advent of a spacetime manifold is connected to the appearance of a permutation lattice. The metric tensor is constructed from lattice excitations and a possible reason for cosmic inflation is elucidated. Furthermore, there are natural dark matter candidates in the tetron model. The ratio of ordinary to dark matter in the universe is estimated to be 1:5.

1 Introduction

Particle physics phenomena can be described, for example, by the left-right symmetric Standard Model with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ [1] and 24 left-handed and 24 right-handed fermion fields which including antiparticles amounts to 96 degrees of freedom, i.e. this model has right handed neutrinos as well as righthanded weak interactions.

In recent papers [2, 3, 4] a new ordering scheme for the observed spectrum of quarks, leptons and gauge bosons was presented, which relies on the structure of the permutation group S_4, and a mechanism was proposed, how 'germs' of the Standard Model interactions might be buried in the representations A_1, A_2, E, T_1 and T_2 of this group.

In the present paper I will argue that this model is not just a strange observation in the realm of particle physics, but has a more fundamental meaning, so that also gravitational and astrophysical effects can also be understood on the tetron basis.

In modern cosmology there are 3 outstanding phenomena not completely understood: the underlying reason for inflation, the ratio of dark to ordinary matter and the appearance of dark energy:

i) Cosmic inflation [5] is the widely accepted hypothesis that the nascent universe passed through a phase of exponential expansion that was driven by a vacuum energy density of negative pressure. It resolves several problems in the Big Bang cosmology that were pointed out in the 1970s, like the horizon problem, the flatness problem and the magnetic monopole problem.

ii) Gravitational effects in the rotation of galaxies as well as other observations (see e.g. [6]) suggest the existence of dark matter with an amount 4 or 5 times larger than ordinary matter which appears in stars, dust and gas.

iii) The present universe is appearantly undergoing a phase of accelerated expansion (see e.g. [7]). This can be explained either by a modification of the Einstein Lagrangian, the so called F(R) gravities, see [8] and references therein, or by the presence of dark energy, see e.g. [9], either in the form of a positive cosmological constant or of a scalar field, sometimes called 'quintessence' [10], that drives the accelaration and acts not unlike the 'inflaton' which is often introduced to drive inflation.

In the present paper I want to analyze these phenomena in the light of the tetron model. Tetron interactions will be assumed to describe the deepest level of matter, just above the Planck scale. I will argue that

i) the appearance of a tetron 'permutation lattice' may affect the inflationary scenario.

ii) some tetron bound states naturally contribute to the dark matter of the universe.

iii) tetron interactions may be related to the formation of spacetime and the appearance of gravitational forces and of dark energy (in the form of a quintessence field).

2 The Tetron Idea as a Lattice Model

The starting point of refs. [2, 3, 4] was the observation that there is a natural one-to-one correspondence between the quarks and leptons and the elements of the permutation group S_4, as made explicit in table 1 and natural in the sense that the color, isospin and family structure correspond to the K, Z_2 and Z_3 subgroups of S_4, where Z_n is the cyclic group of n elements and K is the so-called Kleinsche Vierergruppe which consists of the 3 even permutations $\overline{2143}$, $\overline{3412}$, $\overline{4321}$, where 2 pairs of numbers are interchanged, plus the identity. Note that permutations $\sigma \in S_4$ will be denoted $\overline{abcd}, a, b, c, d \in \{1, 2, 3, 4\}$.

S_4 is a semi-direct product $S_4 = K \diamond Z_3 \diamond Z_2$ where the Z_3 factor is the family symmetry and Z_2 and K can be considered to be the 'germs' of weak isospin and color symmetry (cf. [3]). At low energies this product cannot be distinguished from the direct product $K \times Z_3 \times Z_2$ but has the advantage of being a simple group and having a rich geometric and group theoretical interpretation as the rotational symmetry group of a regular tetrahedron and, up to a parity factor, the symmetry group of a 3-dimensional cubic lattice. Furthermore it does not only describe quarks and leptons (table 1) but also leads to a new ordering scheme for the Standard Model and some GUT-like vector bosons, cf. ref. [2]. In fact, 12 GUT-like heavy vector bosons can be constructed in the tetron model, which behave similar, though not identical, as the ones appearing in the standard SU(5) model.

Actually, the assignments in table 1 are only heuristic. One has to take linear combinations instead, corresponding to the 5 representations A_1, A_2, E, T_1 and T_2 of S_4 [3].

In refs. [2, 3, 4] a constituent picture was suggested where quarks and leptons are assumed to be built from 4 tetron 'flavors' a, b, c and d, whose interchanges generate the permutation group S_4. In the present paper I follow a somewhat different approach which relies on the fact that S_4 is also the symmetry group of a tetrahedral lattice or of a fluctuating S_4-permutation (quantum) lattice. In this approach the inner symmetry space is not continuous (with a continuous symmetry group) but has instead the discrete structure of a tetrahedral or S_4-permutation lattice.

The observed quarks and leptons can then be interpreted as excitations on this lattice and characterized by representations of the lattice symmetry group S_4, i.e. by $A_1+A_2+2E+3T_1+3T_2$ or $2G_1+2G_2+4H$, just as in the 'classical' tetron model [2, 3, 4], and the original dynamics is governed by some unknown lattice interaction instead of by four real tetron constituents.

The lattice ansatz also naturally explains the selection rule mentioned in ref. [2] that all physical states must be permutation states: just because the lattice excitations must transform under representations of S_4.

In the following I will make the additional assumption that not only the inner symmetry is discrete but that physical space is a lattice, too. The reason for this assumption is that although theories with a discrete inner symmetry over a continuous base manifold have been examined [22] they seem to me rather artificial because they usually lead to domain walls and other discontinuities. In addition, this line of thought takes up an old dream that field theoretical UV-infinities and renormalization problems can eventually be avoided by considering a fundamental theory living on a discretized instead of a continuous spacetime, with the average lattice spacing typically of the order of the Planck scale.

	...1234... family 1	...1423... family 2	...1243... family 3
	τ, $b_{1,2,3}$	μ, $s_{1,2,3}$	e, $d_{1,2,3}$
ν	$\overline{1234}(id)$	$\overline{2314}$	$\overline{3124}$
u_1	$\overline{2143}(k_1)$	$\overline{3241}$	$\overline{1342}$
u_2	$\overline{3412}(k_2)$	$\overline{1423}$	$\overline{2431}$
u_3	$\overline{4321}(k_3)$	$\overline{4132}$	$\overline{4213}$
	ν_τ, $t_{1,2,3}$	ν_μ, $c_{1,2,3}$	ν_e, $u_{1,2,3}$
l	$\overline{3214}(1 \leftrightarrow 3)$	$\overline{1324}(2 \leftrightarrow 3)$	$\overline{2134}(1 \leftrightarrow 2)$
d_1	$\overline{2341}$	$\overline{3142}$	$\overline{1243}(3 \leftrightarrow 4)$
d_2	$\overline{1432}(2 \leftrightarrow 4)$	$\overline{2413}$	$\overline{3421}$
d_3	$\overline{4123}$	$\overline{4231}(1 \leftrightarrow 4)$	$\overline{4312}$

Table 1: List of elements of S_4 ordered in 3 fermion families. k_i denote the elements of K and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. Permutations with a 4 at the last position form a S_3 subgroup of S_4 and may be thought of giving the set of lepton states. It should be noted that this is only a heuristic assignment. Actually one has to consider linear combinations of permutation states as discussed in section 2.

To distinguish the inner S_4-symmetry from the symmetries of the spatial lattice I will denote it by S_4^{in} in the following.

Quantum theory dictates that there is an uncertainty in the position of the lattice points. Therefore instead of a fixed spatial lattice one should allow the lattice points to fluctuate, with the fluctuations following some (quantum) stochastic process [23]. Working in a semiclassical approximation one may neglect these fluctuations to first order and consider a fixed lattice with tetrahedral symmetry.

To understand fermions (quarks and leptons) as excitations on this lattice, one has to note that besides the 5 above mentioned ordinary representations, S_4 has 3 irreducible projective representations (representations of the covering group $\tilde{S}_4$), namely G_1, G_2 and H of dimensions 2, 2 and 4, respectively [11]. The sum 4+4+16 of the dimensions squared accounts for the 24 additional elements due to the Z_2 covering of S_4. Among them, G_1 uniquely corresponds to spin-$\frac{1}{2}$, i.e. is obtained as the restriction of the fundamental SU(2) representation to $\tilde{S}_4$. Similarly, H can be obtained from the spin-$\frac{3}{2}$ representation of SU(2), whereas G_2 is obtained from G_1 by reversing the sign for odd permutations. The combination $G_2 + H$ corresponds to a restriction of the spin-$\frac{5}{2}$ representation of SU(2) to $\tilde{S}_4$.

For the understanding of the following arguments a a short digression on quaternions and its usefulness for describing nonrelativistic spin-$\frac{1}{2}$ fermions will be helpful:

Quaternions [12, 13, 14] are a non-commutative extension of the complex numbers and play a special role in mathematics, because they form one of only three finite-dimensional division algebra containing the real numbers as a subalgebra. (The other two are the complex numbers and the octonions.) As a vector space they are generated by 4 basis elementes 1, I, J and K which fulfill $I^2 = J^2 = K^2 = IJK = -1$, where K can be obtained as a product $K = IJ$ from I and J. Quaternions are non-commutative in the sense IJ=-JI. Any quaternion q has an expansion of the form

$$\begin{aligned} q &= c_1 + Jc_2 \\ &= r_1 + Ir_2 + Jr_3 + Kr_4 \end{aligned} \tag{1}$$

with real r_i and complex $c_1 = r_1 + Ir_2$ and $c_2 = r_3 - Ir_4$.

In order to describe spin-$\frac{1}{2}$ bound states one should use the symmetry function of the representation G_1. This function will also be called G_1 in the following and can be given as linear combination of the G_1 representation matrices (=unit quaternions):

$$\begin{aligned} G_1 &= g(1,2,3,4) + Ug(2,3,1,4) + U^2 g(3,1,2,4) \\ &+ Ig(2,1,4,3) + Sg(3,2,4,1) + R^2 g(1,3,4,2) \\ &+ Jg(3,4,1,2) + Rg(1,4,2,3) + T^2 g(2,4,3,1) \\ &+ Kg(4,3,2,1) + Tg(4,1,3,2) + S^2 g(4,2,1,3) \\ &+ \frac{I+K}{\sqrt{2}} g(3,2,1,4) + \frac{I-J}{\sqrt{2}} g(1,3,2,4) + \frac{J+K}{\sqrt{2}} g(2,1,3,4) \\ &+ \frac{1-J}{\sqrt{2}} g(2,3,4,1) + \frac{1-K}{\sqrt{2}} g(3,1,4,2) + \frac{J-K}{\sqrt{2}} g(1,2,4,3) \\ &+ \frac{I-K}{\sqrt{2}} g(1,4,3,2) + \frac{1+K}{\sqrt{2}} g(2,4,1,3) + \frac{1+I}{\sqrt{2}} g(3,4,2,1) \\ &+ \frac{1+J}{\sqrt{2}} g(4,1,2,3) + \frac{I+J}{\sqrt{2}} g(4,2,3,1) + \frac{1-I}{\sqrt{2}} g(4,3,1,2) \end{aligned} \tag{2}$$

where $R = \frac{1}{2}(1 - I - J - K), S = \frac{1}{2}(1 - I + J + K), T = \frac{1}{2}(1 + I - J + K)$ and $U = \frac{1}{2}(1 + I + J - K)$.

Eq. (2) should be considered as the spin factor of the quark and lepton states, whereas the A_1, A_2, E, T_1 and T_2-functions of the ordinary S_4 representations account for the inner symmetry 'flavor' factor. (Those functions can be found, for example, in ref. [2].) The full quark and lepton spectrum of table 1 including spatial and inner symmetries can then be written as

$$(A_1 + A_2 + 2E + 3T_1 + 3T_2)_{in} \otimes G_{1sp} = 24G_1 \tag{3}$$

where *in* stands for the inner and *sp* for the spatial part of the wave function, and the factor of 24 on the r.h.s. accounts for the 24 degrees fo freedom of 3 fermion families.

One could ask, why the (inner) lattice structure is seen in the flavour spectrum part of eq. (3) whereas the spatial part G_{1sp} to a human observer appears as spin-$\frac{1}{2}$ representations of the *continuous* rotation group. The point is that with respect to the spatial lattice present physical experiments always work at distances much larger than the lattice spacing ($\cong M_P$) whereas for the inner symmetry lattice we do *not* encounter the continuum limit, so that the representations $A_{1,2}$, E and $T_{1,2}$ remain relevant for the particle spectrum.

A drawback of the lattice picture as compared to the tetron constituent model, is that because nothing is known about the underlying it is still less specific and there is a larger amount of arbitrariness concerning the origin of the observed spectrum $(A_1 + A_2 + 2E + 3T_1 + 3T_2)_{in}$ for quarks and leptons. One may, for example, assume the existence of 'elementary' excitations g_{1in}, t_{1in} and h_{in} on the inner symmetry lattice (transforming with respect to the representations G_1, T_1 and H, respectively) from which the quark and lepton spectrum is built according to

$$g_{1in} \otimes t_{1in} \otimes h_{in} = (A_1 + A_2 + 2E + 3T_1 + 3T_2)_{in} \tag{4}$$

However, the physical meaning of the 'elementary' excitations is rather unclear.

One may speculate whether a unification of the spatial and inner symmetry sector could remedy the arbitrariness. What I have in mind is a compactification scenario where one starts with a n-dimensional lattice (or n+1 in a relativistic scenario to include a time variable), n-3 dimensions of which being compactified. The most natural choice seems to be n=7 because it allows spinorial structures which is inherited to the n=3 base manifold in the process of compactification. Due to lack of time I have not yet analyzed this promising possibility in detail.

3 Dark Matter from Tetrons

Dark matter is a hypothetical type of matter that is undetectable by its emitted radiation, but which can be inferred only from gravitational effects. Its presence is postulated to explain the flat rotation curves of spiral galaxies and other evidence of missing mass in the universe. According to present observations, there exists between 4 and 6 times more dark matter than ordinary matter in the universe. Further it is known, that it must be composed of mostly cold, i.e. nonrelativistic, particles.

We have seen in the last section, how the spin-$\frac{1}{2}$ nature of quarks and leptons can be constructed using the G_1 representation of the permutation group. In the following I want to make use of these considerations to show that there are natural dark matter candidates in the tetron model responsible for the bulk of the observed dark matter in the universe. Namely, if this approach has some meaning it is tempting that besides G_1 also the two other half-integer spin representations of $\tilde{S}_4$ (H and G_2) play a role in nature, or in other words, that together with ordinary (G_1-)matter sets of particle families with spin $\frac{3}{2}$ (H) and spin $\frac{5}{2}$ ($G_2 + H$) should have been produced during cosmogenesis. In fact, eq. (3) naturally extends to

$$(A_1 + A_2 + 2E + 3T_1 + 3T_2)_{in} \otimes (G_1 + G_2 + 2H)_{sp} = 24(G_1 + G_2 + 2H) \qquad (5)$$

where the rules for S_4 tensor products have been used, e.g. $T_2 \otimes G_1 = G_2 + H$ etc. As before, *in* stands for the inner and *sp* for the spatial S_4 index set and the factor of 24 on the r.h.s. accounts for the 24 'flavor' degrees of freedom of 3 times 3 fermion families for G_1, H and $G_2 + H$ each with particle masses of roughly comparable size.

Next, it will be assumed that - apart from gravity forces - the new (G_2 and H) fermions decouple from ordinary (G_1) fermions, i.e. that spin-$\frac{3}{2}$ and spin-$\frac{5}{2}$ matter have interactions completely separate from those of ordinary matter. This assumption is implied by the way, interaction bosons are constructed in the tetron model[2]:

when an electron-positron pair annihilates, a photon of type G_1 appears, and this can only annihilate into a fermion-antifermion scattering state of G_1-fermions.[1]

Assuming further, that initially all matter fields are produced at uniform rates, one expects a ratio of 1:5 for the relative distribution of matter (including neutrinos) and dark matter in the universe. This ratio is obtained by counting the spin degrees of freedom 2:(4+6) of spin-$\frac{1}{2}$, -$\frac{3}{2}$ and -$\frac{5}{2}$ objects or equivalently from the ratio of dimensions $dim(G_1) : dim(G_2 + 2H)$ and should be considered as one of the main results of the present paper. The fact that only 3 representations are involved has to do with the fact that S_4 is a finite group with a finite number of representations.

The idea behind this consideration is, that at Big Bang energies where masses play no role, all 3 matter types (G_1, G_2 and H) are produced in equal amount corresponding to a mass energy ratio of ordinary to dark matter $dim(G_1) : dim(G_2 + 2H) = 1 : 5$ and that this ratio has not changed since that time because apart from gravity there are no interactions between the 3 matter types. In other words, all decays and transitions take place only within one of the matter types and do not disturb the ratio 1:5.

One may object, that these assumptions are not truly valid and the 'prediction' of 1:5 can be considered only as a very crude approximation and is not more than an order of magnitude estimate. In fact, in a precise calculation there will be corrections due to the thermal evolution of the universe, to the primordial baryon asymmetry, the proton-antiproton annihilation cross section and the energy density in other species around the time that the cosmic temperature fell below the mass of the proton.

However, the aim of this section was not to present a complete calculation but to point out to what kind of conclusions the input from the tetron model will lead in the field of dark matter astrophysics.

[1] It is an interesting question how the interactions among the dark matter (G_2 and H) fermions look like and whether they lead to atomic and molecular binding states similar to what we are used from ordinary matter, or whether the spin-$\frac{3}{2}$ and spin-$\frac{5}{2}$ quarks will not be confined and exist as free particles.

4 Gravitons, Quintessence and the Permutation Lattice

In section 2 we have seen how quarks and leptons may arise as excitations of an (inner and spatial) permutation lattice.

Similar considerations apply to their vectorboson interactions, which also follow S_4 permutation symmetry [2]. I do not want to discuss this here but will now concentrate on the gravitational field and on the idea that it can be described as a permutation lattice excitation, too.

Since the graviton is flavor independent, there is no inner symmetry contribution to its wave function. One therefore has to consider only the spatial part. Then, in the same way as fermion states were written down with the help of the representation G_1 in eq. (2), the gravitational field can be expanded with the help of spin-2 representation matrices $R_{\mu\nu}(ijkl)$ of the spatial lattice permutation symmetry S_4.

These may be explicitly calculated, for example, by formally considering a product of 2 vector representations

$$T_1 \otimes T_1 = A_1 + T_1 + E + T_2 \tag{6}$$

of two S_4 vector representations T_1, where A_1, T_1 and $E + T_2$ represent the spin-0, spin-1 and spin-2 contributions to the product, respectively. Furthermore, the temporal gauge $g_{0\mu} = 0$ will be used which, at least in the weak field approaximation, is known to be compatibel with the harmonic gauge often used in relativistic calculations [21].

The metric tensor then takes the form

$$g_{\mu\nu} = \begin{pmatrix} -t_{XX} - t_{YY} - t_{ZZ} & 0 & 0 & 0 \\ 0 & t_{XX} & t_{XY} & t_{XZ} \\ 0 & t_{YX} & t_{YY} & t_{YZ} \\ 0 & t_{ZX} & t_{ZY} & t_{ZZ} \end{pmatrix} \tag{7}$$

Here I have allowed for a nonvanishing g_{00} contribution due to the singlet A_1 which may represent the quintessence scalar ϕ_q [10] appearing in solutions to the dark energy problem and a possible antisymmetric component U_μ stemming from the spin-1 contribution T_1 on the r.h.s. of eq. (6). The antisymmetric components may play a role in the so-called scalar-vector-tensor model [17] and in gravity with torsion [18]. Making use of the appropriate Clebsch-Gordon coefficients [16] the relation of $g_{\mu\nu}$ eq. (7) to the known S_4 representation matrices [2] is given by

$$A_1 = t_{XX} + t_{YY} + t_{ZZ} \tag{8}$$

$$E_{11} = (t_{XX} - t_{YY})/2 \tag{9}$$

$$E_{12} = (t_{XX} + t_{YY} - 2t_{ZZ})/\sqrt{6} \tag{10}$$

$$T_{2,11} = (t_{XY} + t_{YX})/2 \tag{11}$$

$$T_{1,11} = (t_{XY} - t_{YX})/2 \tag{12}$$

etc.

One may summarize this construction by saying that the graviton and its companions are excitations within the permutation lattice of the form

$$1_{in} \otimes (A_1 + A_2 + 2E + 3T_1 + 3T_2)_{sp} \tag{13}$$

Since - in contrast to eqs. (3) and (5) - there is no inner symmetry index, only one A_1, one A_2, one T_1 and one $E + T_2$ field emerge on the ground state level. This corresponds to a scalar field ϕ_q, an axial scalar ϕ_a, a spin-1 vector U_μ and a spin-2 tensor field. In the massless limit the transversal modes of the spin-1 and spin-2 excitations will vanish and a graviton and a vector field each with 2 helicities appear.

The effective interactions between these compound states can be written down in the standard way, because the requirement of local Lorentz invariance at distances much larger than the lattice spacings more or less fixes the Lagrangian to be

$$L = \frac{1}{2}\sqrt{-g}M_P^2 R + L(\phi_q) + L(\phi_a) + L(U_\mu) + L_{WW} \tag{14}$$

where R is the Ricci scalar associated with the graviton, g is the determinant of the (symmetric) metric tensor and $M_P = 1/\sqrt{8\pi G}$ the reduced Planck mass.

$$L(\phi_q) = \frac{1}{2}\partial_\mu\phi_q\partial^\mu\phi_q - V(\phi_q) \tag{15}$$

denotes the quintessence part of the Lagrangian. Similarly for $L(\phi_a)$ and $L(U_\mu)$, whereas L_{WW} denotes interactions among the various fields[17].

Exploring the phenomenology of eqs. (14) and (15) requires (among other things) a form for the potential $V(\phi_q)$. In order to account for the dark energy component of the total cosmic mass energy, this is usually chosen in such a way that the field stress-energy tensor approximates the effect of a cosmological constant[10, 19, 20].

5 Conclusions

In the present paper astrophysical consequences of the tetron idea have been discussed. It was shown how the hypothesis of a S_4 permutation symmetry naturally leads to (cold) dark matter particles with spin $\frac{3}{2}$ and $\frac{5}{2}$ and that it allows, under certain assumptions, to calculate the ratio of ordinary to dark matter. Furthermore, a suggestion has been made, how the gravitational interactions can be constructed on the basis of the tetron model.

There are several objections which may be raised against the tetron model. One is that at its current state it relies mainly on group theoretical arguments and not much can be said about the dynamical behavior that lead to the lattice excitations. What seems to be certain, however, is that since no continuum limit is taken, the ultraviolet treatment of the theory will be quite different from the renormalization one usually encounters at small distances in quantum field theories.

There is some relation of the presented ideas to other models which involve a fundamental length scale, like quantum foam models, which however assume gravity to play the central role in producing the new length scale, while in the tetron model gravitational interactions and cosmological phenomena appear only as byproducts of the spin lattice interactions.

Finally, it should be noted that talking about 'lattice excitations' more or less amounts to interpreting fermions and gauge fields as emergent phenomena (in contrast to the more reductionist models of refs. [2, 3, 4]). In a future publication I will present a specific dynamical model for this scenario. Use will be made of lattice spin models which have been shown [24] to possess gapless excitations corresponding to Dirac fermions and gauge theory structures.

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