Bhaskaracharya Quadratics and Spade Sequences

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Number-crunching done on T.I. Voyage 200 hand calculator

Abstract

In this article I generalize on a word problem by Bhaskaracharya. The resulting quadratics are trivial to solve; but composing them, so that they have whole number solutions, is not trivial. In this article I discover a class of sequences, which I call "Spade Sequences" as homage to a hero of detective fiction, which generate both Bhaskaracharya quadratics and their solutions. The article ends with a list of such word problems, presented as a problem set with answer key.

Consider this centuries-old word problem, by the Indian algebraist Bhaskaracharya:

Inside a forest, a number of apes equal to the square of one-eighth of the total number of apes in the pack are playing noisy games. The remaining twelve apes, who are of a more serious disposition, are on a nearby hill and irritated by the shrieks coming from the forest. What is the total number of apes in the pack?

The problem leads to a quadratic equation with two solutions:

$$A = (A/8)^2 + 12 \quad ; \quad A = 16 \text{ or } 48.$$

This generalizes to the equation: $A = (A/F)^2 + R$

Call this a "Bhaskaracharya quadratic" if A, F and R are all integers. The solutions are:

$$A = (F^2 \pm F*\mathrm{sqrt}(F^2-4R)) / 2$$

Finding solutions is routine; the tricky part is ensuring that those solutions are whole numbers. So when is sqrt(F^2-4R) an integer? Let it be called "G"; then $F^2 - 4R = G^2$; so

$$(1) \qquad R = (F^2 - G^2) / 4 \quad ; \quad A = F*(F \pm G) / 2$$

If F and G have the same parity, and G<F, then (1) yields positive integer values for A and R.

For instance:

G =	F =	R =	A_-=	A_+=	
2	4	3	4	12	
3	5	4	5	20	
1	5	6	10	15	
5	7	6	7	42	*
3	7	10	14	35	
1	7	12	21	28	
4	12	32	48	96	**

From lines * and ** I compose these word problems:

In the crazy lady's house, a number of cats equal to the square of one-seventh of the total number of cats are orange; the remaining six cats are black. The crazy lady has how many cats?

Answer: 7 or 42.

On the duck pond, the number of males equals the square of one-twelfth of the total number of ducks; and there are 32 females. How many ducks?

Answer: 48 or 96.

From F = 2001 and G = 69 I compose this word problem:

Of the city's voters, the square of one part in 2001 voted for the crook; the remaining 999,810 voted for the fool. How many voters?

Answer: 1,932,966 or 2,071,035.

From F = 42 and G = 0 I compose:

At the science-fiction convention's banquet, the square of one part in 42 of the fans had crottled greeps, and the remaining 441 had spoo. How many fans were at the banquet?

Answer: 882.

From F = 1984 and G = 1930 I compose:

From the funds collected at the political rally, the organizer kept the square of one part in 1984; the remaining $52,839 went to a worthy charity. How much money was collected at the rally?

Answer: \$53,568 or \$3,882,688.

And so on! For every F and G of same parity with G<F, we get a Bhaskaracharya word problem.

Now consider these word problems:

In the kindergarten, a number equal to the square of one-sixth of the total number of children were playing with blocks; the square of one-fifteenth of the total were finger-painting; and the remaining child was taking a nap. How many children in the kindergarten?

Of the monkey troop, a number equal to the square of one-fifteenth of the total number of monkeys were foraging for berries; the square of one-thirty-ninth of the total were grooming each other; and the remaining monkey kept watch from a high tree-branch. How many monkeys in the troop?

In the meadow, a number equal to the square of one-102nd of the total number of mice were tan-and-white; the square of 267th of the total were all black; and the remaining mouse was black-and-tan. How many mice in the meadow?

From the beehive, a number equal to the square of one part in 699 of the total number of bees were out seeking nectar; the square of one part in 1830 were maintaining the hive, and the remaining bee was the Queen. How many bees in the hive?

A countertop was washed with a soap containing antibiotics A and B. The square of one part in 12543 of the total number of bacteria succumbed to antibiotic A; the square of one part in 32838 of the total succumbed to antibiotic B; and the remaining bacterium was resistant to both antibiotics, as were its many descendants. How many bacteria were on the countertop?

They correspond to these equations:

$$x = (x/6)^2 + (x/15)^2 + 1$$

$$x = (x/15)^2 + (x/39)^2 + 1$$

$$x = (x/102)^2 + (x/267)^2 + 1$$

$$x = (x/699)^2 + (x/1830)^2 + 1$$

$$x = (x/12543)^2 + (x/32838)^2 + 1$$

These are Bhaskaracharya quadratics with *two* quadratic terms, and remainder one.

The first two equations have these solutions:

$2^2 + 5^2 + 1 = 30 = 3*2*5$; so $x=30$ solves $x = (x/6)^2+(x/15)^2+1$

$5^2 + 13^2 + 1 = 195 = 3*5*13$; so $x=195$ solves $x = (x/15)^2+(x/39)^2+1$

Let's generalize this. Try:

$a^2 + b^2 + 1 = 3*a*b$; so $x=3ab$ solves $x = (x/(3b))^2+(x/(3a))^2+1$

Then we derive:

$$a^2 - 3ba + (b^2+1) = 0$$

$$a = (3b \pm \text{sqrt}(5b^2-4)) / 2$$

When is this an integer? Here is a table:

$b =$	$a_- =$	$a_+ =$
1	1	2
2	1	5
5	2	13
13	5	34
34	13	89

No other b values in this range yield integers. Note how each a_- equals the previous b, and each a_+ equals the next b. This suggests this recursion:

$a_{n+1}^2 + a_n^2 + 1 = 3 a_{n+1}a_n$;

so $a_{n+1} = (3 a_n \pm \text{sqrt}(5a_n^2-4)) / 2$ by the quadratic formula;

and $x= 3a_{n+1}a_n$ solves the two-term Bhaskaracharya quadratic:

$$x = (x/(3a_n))^2+(x/(3a_{n+1}))^2+1$$

Also: since $a_{n+1}^2 + a_n^2 + 1 = 3 a_{n+1}a_n$

and also $a_{n+2}^2 + a_{n+1}^2 + 1 = 3 a_{n+2}a_{n+1}$

it follows that $a_{n+2}^2 - a_n^2 = 3 a_{n+2}a_{n+1} - 3 a_{n+1}a_n$

and therefore $(a_{n+2} - a_n) (a_{n+2} + a_n) = 3 a_{n+1}(a_{n+2} - a_n)$

and therefore $a_{n+2} + a_n = 3\, a_{n+1}$

and therefore $a_{n+2} = 3\, a_{n+1} - a_n$

Start with 1, 1; then you get the integer sequence

1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418, 514229, 1346269…

You can prove by recursion that these are the odd Fibonacci numbers.

And as for the above word problems:

From a_n, a_{n+1} = 2, 5	we get	30 children;
From a_n, a_{n+1} = 5, 13	we get	195 monkeys;
From a_n, a_{n+1} = 34, 89	we get	9,028 mice;
From a_n, a_{n+1} = 233, 610	we get	426,390 bees;
From a_n, a_{n+1} = 4181, 10946	we get	137,295,678 bacteria.

Now consider this word problem:

"The town I live in is small, but it's big enough for me. The square of one part in 132 of the population is the number of this little burg's honest working folk, God bless 'em. The square of one part in 1572 are not so honest. They're crooks, thieves, street predators. Bad news. But there's worse news, and that's the legal crooks, the puppet-masters, the owners and operators; and they number the square of one part in 5764. And then there's me, Sam Spade, private eye. Now tell me, if you can; how many people in my small town?"

It corresponds to this *three*-term Bhaskaracharya quadratic, with remainder one:

$$x = (x/132)^2 + (x/1572)^2 + (x/5764)^2 + 1$$

We can generate other equations of this sort. For instance:

$1^2 + 1^2 + 1^2 + 1 = 4 = 4*1*1*1;$ so x=4 solves $x = (x/4)^2+(x/4)^2+(x/4)^2+1$

$1^2 + 1^2 + 3^2 + 1 = 12 = 4*1*1*3;$ so x=12 solves $x = (x/12)^2+(x/12)^2+(x/4)^2+1$

$1^2 + 3^2 + 11^2 + 1 = 132 = 4*1*3*11;$ so x=132 solves $x = (x/132)^2+(x/44)^2+(x/12)^2+1$

This suggests a recursion, the "Spade Sequence for three terms":

$a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 1 = 4\, a_{n+2}a_{n+1}a_n$;

this implies $a_{n+2} = 2a_{n+1}a_n + \mathrm{sqrt}(\ (2a_{n+1}a_n)^2 - a_{n+1}^2 - a_n^2 - 1\ )$

and $x = 4\, a_{n+2}a_{n+1}a_n$ solves $x = (x/(4\, a_{n+1}a_n))^2 + (x/(4\, a_{n+2}a_n\ ))^2 + (x/(4\, a_{n+2}a_{n+1}))^2 + 1$

Also: since $a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 1 = 4\, a_{n+2}a_{n+1}a_n$

and also $a_{n+3}^2 + a_{n+2}^2 + a_{n+1}^2 + 1 = 4\, a_{n+3}a_{n+2}a_{n+1}$

it follows that $a_{n+3}^2 - a_n^2 = 4\, a_{n+3}a_{n+2}a_{n+1} - 4\, a_{n+2}a_{n+1}a_n$

and therefore $(a_{n+3} - a_n)\,(a_{n+3} + a_n) = 4\, a_{n+2}\, a_{n+1}\,(a_{n+3} - a_n)$

and therefore $a_{n+3} + a_n = 4\, a_{n+2}\, a_{n+1}$

and therefore $a_{n+3} = 4\, a_{n+2}\, a_{n+1} - a_n$

From initial values 1, 1, 1 we get the integer sequence:

1, 1, 1, 3, 11, 131, 5761, 3018753, 69564144001, 839987873581797251,

233732149587751710483796746251, …

From 3, 11, 131 on the sequence we get

$3^2 + 11^2 + 131^2 + 1 = 17292$

So $x = 17292$ solves $x = (x/132)^2+(x/1572)^2+(x/5764)^2+1$

Sam Spade's home town has 17161 honest folk, 121 crooks, 9 bosses, and Sam.

From 11, 131, 5761 on the sequence I compose this word problem:

"In my golden home state, the square of one part in 5764 of the population are honest working folk; the square of one part in 253484 are street criminals; and the square of one part in 3018764 are white-collar criminal bosses; and then there's me, Sam Spade, private eye. How many of each?"

Answer: 33,189,121 honest folk, 17161 crooks, 121 bosses, and Sam Spade, for a total of 33,206,404.

Now consider this recursion, the “Spade Sequence for four terms”:

$a_{n+3}^2 + a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 1 = 5\, a_{n+3}a_{n+2}a_{n+1}a_n;$ this implies

$a_{n+3} = (5\, a_{n+2}a_{n+1}a_n + \text{sqrt}((5a_{n+2}a_{n+1}a_n)^2 - 4a_{n+2}^2 - 4a_{n+1}^2 - 4a_n^2 - 4)) / 2$

so $x = 5a_{n+3}a_{n+2}a_{n+1}a_n$ solves this four-term Bhaskaracharya quadratic:

$x = (x/(5a_{n+3}a_{n+2}a_{n+1}))^2 + (x/(5a_{n+3}a_{n+2}a_n))^2 + (x/(5a_{n+3}a_{n+1}a_n))^2 + (x/(5a_{n+2}a_{n+1}a_n))^2 + 1$

Also, by reasoning like in the two and three term cases, we derive the recursion:

$$a_{n+4} = 5\, a_{n+3}\, a_{n+2}\, a_{n+1} - a_n$$

From initial values 1, 1, 1, 1 we get the integer sequence:

1, 1, 1, 1, 4, 19, 379, 144019, 5185404091, 1415179768826376436,

5284257989697826589787882104688841, …

From 4, 19, 379 and 144019 I compose this word problem:

“Planet Earth is crowded, here in the 23rd century, but I like it that way. Of Earth’s population, the square of one part in 144020 are honest working folk. The square of one part in 54727220 are not so honest; they’re wheelers, dealers, banksters, billionaire parasites. Bad news. But there’s worse news, and that’s the Insiders. These master manipulators number the square of one part in 1091664020. Worst of all are the Illuminati, secret masters of the secret masters, and they number the square of one part in 5185404095. And then there’s me, Sam Spade, private eye. How many of each?”

Answer: 20,741,472,361 honest Earthlings, 143,641 billionaires, 361 Insiders, 16 Illuminati, and Sam Spade, making a total of 20,741,616,380.

Now consider the “Spade Sequence for five terms”:

$a_{n+4}^2 + a_{n+3}^2 + a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 1 = 6\, a_{n+4}a_{n+3}a_{n+2}a_{n+1}a_n;$ this implies

$a_{n+4} = (3\, a_{n+3}a_{n+2}a_{n+1}a_n + \text{sqrt}((3\, a_{n+3}a_{n+2}a_{n+1}a_n)^2 - a_{n+3}^2 - a_{n+2}^2 - a_{n+1}^2 - a_n^2 - 1))$

so $x = 6\, a_{n+4}a_{n+3}a_{n+2}a_{n+1}a_n$ solves this 5-term Bhaskaracharya quadratic:

$$x = (x/(6\, a_{n+4}a_{n+3}a_{n+2}a_{n+1}))^2 + (x/(6\, a_{n+4}a_{n+3}a_{n+2}a_n))^2 + (x/(6\, a_{n+4}a_{n+3}a_{n+1}a_n))^2$$
$$+ (x/(6\, a_{n+4}a_{n+2}a_{n+1}a_n))^2 + (x/(6\, a_{n+3}a_{n+2}a_{n+1}a_n))^2 + 1$$

Also, by reasoning like in the two and three term cases, we derive the recursion:

$$a_{n+5} = 6\, a_{n+4}\, a_{n+3}\, a_{n+2}\, a_{n+1} - a_n$$

From initial values 1, 1, 1, 1, 1 we get the integer sequence:

1, 1, 1, 1, 1, 5, 29, 869, 756029, 571580604869,

653408775725966974224O1, …

Here is Sam Spade’s latest report:

“That was one Hell of a coin he offered to me. Of the atoms in that disk:

the square of one part in 571580604870 were gold;

the square of one part in 432132084699110070 were silver;

the square of one part in 375954416413099524870 were platinum;

the square of one part in 112656685469994305900 70 were palladium;

the square of one part in 653408775725966974224 06 were rhodium;

and the remaining atom was lead.

How many atoms were in that coin?”

Answer: 326704387862411906507161 atoms of gold; 571579848841 atoms of silver; 755161 atoms of platinum; 841 atoms of palladium; 25 atoms of rhodium; and one atom of lead.

This makes a total of 326704387862983487112030 atoms; or about 0.5424 moles of almost pure gold; therefore weighing about 106.8 grams, or 3.768 ounces. It’s hard to refuse an offer like that!

Continuing in this way, we can define Spade sequences, and corresponding Bhaskaracharya quadratics and word problems, for any number of terms. For k terms the equations are:

$$1 = a_1 = a_2 = \dots = a_k$$

$$a_{n+k-1}^2 + a_{n+k-2}^2 + \dots + a_n^2 + 1 = (k+1)a_{n+k-1}a_{n+k-2}\dots a_n;$$

$a_{n+k-1} = ((k+1)a_{n+k-2}a_{n+k-3}\ldots a_n + \mathrm{sqrt}(4((k+1)a_{n+k-2}a_{n+k-3}\ldots a_n)^2 - 4a_{n+k-2}^{\ 2} - 4a_{n+k-3}^{\ 2} - \ldots -4a_n^{\ 2} - 4)) / 2$

so $x= (k+1)a_{n+k-1}a_{n+k-2}\ldots a_n$ solves this k-term Bhaskaracharya quadratic:

$$x = (x/((k+1)a_{n+k-1}a_{n+k-2}\ldots a_{n+2}a_{n+1}))^2+(x/((k+1)a_{n+k-1}a_{n+k-2}\ldots a_{n+2}a_n))^2+ \ldots$$

$$+ (x/((k+1)a_{n+k-2}a_{n+k-3}\ldots a_{n+1}a_n))^2+ 1$$

And we can derive the recursion

$$a_{n+k} = (k+1)\, a_{n+k-1}\, a_{n+k-2} \ldots a_{n+1} - a_n$$

Starting from 1,1,1,…1, we get an integer sequence.

We can also generalize on the remainder. Consider this recursion:

$$a_{n+1}^{\ 2} + a_n^{\ 2} + 2 = 4\, a_{n+1}a_n;$$

The 4 in the 4 $a_{n+1}a_n$ term allows the solution $a_1 = a_2 = 1$.

$x= 4a_{n+1}a_n$ solves the two-term Bhaskaracharya quadratic with remainder 2:

$$x = (x/(4a_n))^2+(x/(4a_{n+1}))^2+2$$

Since $a_{n+1}^{\ 2} + a_n^{\ 2} + 2 = 4\, a_{n+1}a_n$

and also $a_{n+2}^{\ 2} + a_{n+1}^{\ 2} + 2 = 4\, a_{n+2}a_{n+1}$

it follows that $a_{n+2}^{\ 2} - a_n^{\ 2} = 4\, a_{n+2}a_{n+1} - 4\, a_{n+1}a_n$

and therefore $(a_{n+2} - a_n)\,(a_{n+2} + a_n) = 4\, a_{n+1}(a_{n+2} - a_n)$

and therefore $a_{n+2} + a_n = 4\, a_{n+1}$

and therefore $a_{n+2} = 4\, a_{n+1} - a_n$

Starting from 1, 1, we get the sequence:

1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681, …

From 153 and 571 I compose this word problem:

From my savings account, I spent the square of one part in 612 to pay off my mortgage; I spent the square of one part in 2284 for a new car; and the last two dollars paid for a soda. How much did I spend on what, and how much had been in my now-empty savings account?

Answer: \$326,041 for mortgage; \$ 23,409 for new car; \$ 2 for soda; for a total of \$ 349,452.

Now let's try three terms, remainder 17:

$$a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 17 = 20\, a_{n+2}a_{n+1}a_n;$$

The 20 allows the solution $a_1 = a_2 = a_3 = 1$; also, $x= 20\, a_{n+2}a_{n+1}a_n$ solves the three-term Bhaskaracharya quadratic with remainder 17:

$$x = (x/(20\, a_{n+2}\, a_{n+1}))^2+(x/(20a_{n+2}a_n))^2+(x/(20a_{n+1}a_n)^2+17$$

Since $a_{n+2}^2 + a_{n+1}^2 + a_n^2 + 17 = 20\, a_{n+2}a_{n+1}a_n$

and $a_{n+3}^2 + a_{n+2}^2 + a_{n+1}^2 + 17 = 20\, a_{n+3}a_{n+2}a_{n+1}$

it follows that $a_{n+3}^2 - a_n^2 = 20\, a_{n+3}a_{n+2}a_{n+1} - 20\, a_{n+2}a_{n+1}a_n$

and therefore $(a_{n+3} - a_n)\,(a_{n+3} + a_n) = 20\, a_{n+2}a_{n+1}(a_{n+3} - a_n)$

and therefore $a_{n+3} + a_n = 20\, a_{n+2}a_{n+1}$

and therefore $a_{n+3} = 20\, a_{n+2}a_{n+1} - a_n$

Starting from 1, 1, 1, we get the sequence:

1, 1, 1, 19, 379, 144019, 1091664001, 3144407155200001, …

From 19, 379 and 144019 I compose this word problem:

From their secret budget, the mercenaries spent the square of one part in 144020 in bribes to the locals; the square of one part in 54727220 went for ammunition; the square of one part in 1091664020 was for vodka; and the remaining 17 dollars paid for condoms. How much was spent on what?

Answer: the mercenaries spent \$ 17 for condoms; \$ 361 for vodka; \$ 143,641 for ammunition; and \$ 20,741,472,361 for bribes; making \$ 20,741,616,380 in all.

And so on. For any number of terms k, and any specified remainder R, there is a Spade sequence a_n whose terms define the corresponding Bhaskaracharya quadratic;

$$x = (x/((k+R)a_{n+k-1}a_{n+k-2}\ldots a_{n+2}a_{n+1}))^2 + (x/((k+R)a_{n+k-1}a_{n+k-2}\ldots a_{n+2}a_n))^2 +$$

$$\ldots + (x/((k+R)a_{n+k-2}a_{n+k-3}\ldots a_{n+1}a_n))^2 + R$$

The solution is $x = (k+R)a_{n+k-1}a_{n+k-2}\ldots a_{n+1}a_n$; and the a's can be found from the recursion

$$1 = a_1 = a_2 = \ldots = a_k \quad ; \quad a_{n+k} = (k+R)\, a_{n+k-1}\, a_{n+k-2} \ldots a_{n+1} - a_n$$

I end this paper with these word problems collected together as a problem set, plus answer key.

Quadratic Word Problems

Consider this centuries-old word problem, by the Indian algebraist Bhaskaracharya:

Inside a forest, a number of apes equal to the square of one-eighth of the total number of apes in the pack are playing noisy games. The remaining twelve apes, who are of a more serious disposition, are on a nearby hill and irritated by the shrieks coming from the forest. What is the total number of apes in the pack?

The problem leads to a quadratic equation with two solutions:

$$A = (A/8)^2 + 12 \quad ; \quad A = 16 \text{ or } 48.$$

Now solve the following word problems. All solutions should be whole numbers.

1. In the crazy lady's house, a number of cats equal to the square of one-seventh of the total number of cats are orange; the remaining six cats are black. The crazy lady has how many cats?

2. On the duck pond, the number of males equals the square of one-twelfth of the total number of ducks; and there are 32 females. How many ducks?

3. (2 points.) Of the city's voters, the square of one part in 2001 voted for the crook; the remaining 999,810 voted for the fool. How many voters?

4. At the science-fiction convention's banquet, the square of one part in 42 of the fans had crottled greeps, and the remaining 441 had spoo. How many fans were at the banquet?

5. (2 points.) From the funds collected at the political rally, the organizer kept the square of one part in 1984; the remaining $52,839 went to a worthy charity. How much money was collected at the rally?

6. In the kindergarten, a number equal to the square of one-sixth of the total number of children were playing with blocks; the square of one-fifteenth of the total were finger-painting; and the remaining child was taking a nap. How many children in the kindergarten?

7. Of the monkey troop, a number equal to the square of one-fifteenth of the total number of monkeys were foraging for berries; the square of one-thirty-ninth of the total were grooming each other; and the remaining monkey kept watch from a high tree-branch. How many monkeys in the troop?

8. In the meadow, a number equal to the square of one-102nd of the total number of mice were tan-and-white; the square of 267th of the total were all black; and the remaining mouse was black-and-tan. How many mice in the meadow?

9. (2 points) From the beehive, a number equal to the square of one part in 699 of the total number of bees were out seeking nectar; the square of one part in 1830 were maintaining the hive, and the remaining bee was the Queen. How many bees in the hive?

10. (3 points.) A countertop was washed with a soap containing antibiotics A and B. The square of one part in 12543 of the total number of bacteria succumbed to antibiotic A; the square of one part in 32838 of the total succumbed to antibiotic B; and the remaining bacterium was resistant to both antibiotics, as were its many descendants. How many bacteria were on the countertop?

11. (2 points.) From my savings account, I spent the square of one part in 612 to pay off my mortgage; I spent the square of one part in 2284 for a new car; and the last two dollars paid for a soda. How much did I spend on what, and how much had been in my now-empty savings account?

12. (3 points.) From their secret budget, the mercenaries spent the square of one part in 144020 in bribes to the locals; the square of one part in 54727220 went for ammunition; the square of one part in 1091664020 paid for vodka; and the remaining 17 dollars got condoms. How much was spent on what?

13. “The town I live in is small, but it’s big enough for me. The square of one part in 132 of the population is the number of this little burg’s honest working folk, God bless ‘em. The square of one part in 1572 are not so honest. They’re crooks, thieves, street predators. Bad news. But there’s worse news, and that’s the legal crooks, the puppet-masters, the owners and operators; and they number the square of one part in 5764. And then there’s me, Sam Spade, private eye. Now tell me, if you can; how many people in my small town?”

14. (2 points.) “In my golden home state, the square of one part in 5764 of the population are honest working folk; the square of one part in 253484 are street criminals; and the square of one part in 3018764 are white-collar criminal bosses; and then there’s me, Sam Spade, private eye. How many of each?”

15. (3 points.) “Planet Earth is crowded, here in the 23rd century, but I like it that way. Of Earth’s population, the square of one part in 144020 are honest working folk. The square of one part in 54727220 are not so honest; they’re wheelers, dealers, banksters, billionaire parasites. Bad news. But there’s worse news, and that’s the Insiders. These master manipulators number the square of one part in 1091664020. Worst of all are the Illuminati, secret masters of the secret masters, and they number the square of one part in 5185404095. And then there’s me, Sam Spade, private eye. How many of each?”

16. (7 points.) Here is Sam Spade’s latest report:

    “That was one Hell of a coin he offered to me. Of the atoms in that disk:

    the square of one part in 571580604870 were gold;

    the square of one part in 432132084699110070 were silver;

    the square of one part in 375954416413099524870 were platinum;

    the square of one part in 1126566854699943059070 were palladium;

    the square of one part in 65340877572596697422406 were rhodium;

    and the remaining atom was lead.

    How many atoms were in that coin?”

1. 7 or 42 cats.
2. 48 or 96 ducks.
3. 1,932,966 or 2,071,035 voters.
4. 882 fans.
5. $53,568 or $3,882,688.
6. 30 children.
7. 195 monkeys.
8. 9,028 mice.
9. 426,390 bees.
10. 137,295,678 bacteria.
11. $326,041 for mortgage; $ 23,409 for new car; $ 2 for soda; for a total of $ 349,452.
12. The mercenaries spent $ 17 for condoms; $ 361 for vodka; $ 143,641 for ammunition; and $20,741,472,361 for bribes; making $ 20,741,616,380 in all.
13. Sam Spade's home town has 17161 honest folk, 121 crooks, 9 bosses, and Sam, for a total of 17292.
14. 33,189,121 honest folk, 17161 crooks, 121 bosses, and Sam Spade, for a total of 33,206,404.
15. 20,741,472,361 honest Earthlings, 143,641 billionaires, 361 Insiders, 16 Illuminati, and Sam Spade, making a total of 20,741,616,380.
16. 326704387862411906507161 atoms of gold; 571579848841 atoms of silver; 755161 atoms of platinum; 841 atoms of palladium; 25 atoms of rhodium; and one atom of lead. This makes a total of 326704387862983487112030 atoms; or about 0.5424 moles of almost pure gold; therefore weighing about 106.8 grams, or 3.768 ounces. It's hard to refuse an offer like that!