Do experiment and the correspondence principle oblige revision of relativistic quantum theory?

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Abstract

Recent preliminary data gathered by the Fermilab MINOS Collaboration suggest with 95% confidence that the mass of the muon neutrino differs from that of its antineutrino partner, which contradicts the entrenched relativistic quantum theory notion that a free antiparticle is a negative-energy free particle compelled to travel backwards in time. Also a discrepancy of about five standard deviations in the value of the proton charge radius recently obtained from muonic hydrogen versus that previously obtained from electronic hydrogen casts doubt on the calculation of the dominant relativistic QED contributions to the effects that are actually measured (e.g., the Lamb shift): these QED contributions dominate proton charge radius contributions less in muonic hydrogen than in electronic hydrogen. The negative-energy "free particles" of entrenched relativistic quantum theory are well-known features of the Klein-Gordon and Dirac equations, which are shown to have many other unphysical features as well. The correspondence principle for relativistic particles is incompatible with these two equations, produces no unphysical features and implies only positive energies for free particles, which eliminates the very basis of the entrenched notion of antiparticles, as well as of the CPT theorem. This principle thus requires antiparticles to arise from charge conjugation (or more generally CP) invariance, whose known breaking is naturally expected to produce mass splitting between particle and antiparticle, in consonance with the preliminary MINOS data. It also requires revamping of relativistic QED, which is in accord with the doubt cast on it by the proton charge radius results, and implies that QED is nonlocal, i.e. has no Hamiltonian density.

Introduction

Recent data gathered by two very different experiments have cast a shadow of doubt over the validity of relativistic quantum precepts that have become well-entrenched over almost nine decades. Preliminary data from the Fermilab MINOS Collaboration presented on June 14, 2010 at the Neutrino 2010 conference in Athens, Greece suggest with 95% confidence that the muon neutrino does not have the same mass as the muon antineutrino [1, 2]. If the symmetry which relates particle to antiparticle were deemed to be a multi-particle one of the overlying field theory, as is the symmetry which relates the two members of an isospin doublet, such a mass splitting between neutrino and antineutrino would be *no more remarkable* than is the mass splitting between proton and neutron: after all, just as electromagnetism *breaks* isospin symmetry, there is a physical agency which *breaks* particle-antiparticle symmetry—that is clear from particle domination of the composition of the visible universe.

The issue, however, is that the entrenched approach to relativistic quantum theory has it that the relation between particle and antiparticle is *not* a mere multi-particle symmetry of the overlying field theory, but that particle and antiparticle are in fact *two members of the very same species*: a free antiparticle is deemed by entrenched theory to be a free *negative*-energy particle which, due to its negative energy, is somehow obliged to travel backwards in time—although no deduction from established physics which justifies this astounding contention of time-flow reversal for negative-energy free particles is proffered. This particular (and certainly peculiar) "species identity" of particle with antiparticle in entrenched theory precludes their masses from differing *at all*, and it as well lies at the *very heart* of the "celebrated" CPT theorem.

The above-noted *negative*-energy free particles of course *arise* from the ostensible "quantum relativistic" Klein-Gordon and Dirac equations of entrenched theory. These negative energies *have no lower bound*, and therefore at first glance comprise a source of severe theoretical physics embarrassment for the Klein-Gordon and Dirac free-particle equations—not to mention that free particles of negative energy are not observed. Putting the Klein-Gordon and Dirac negative-energy free particles "at the service" of a phenomenon that actually *is* observed, namely antiparticles, by arbitrarily imposing on them the mind-boggling requirement that they *also* travel backwards in time turned out to be astonishingly well-received by the physics community. This almost certainly was due to great reluctance on the part of this community to *discard* the Klein-Gordon and Dirac equations, notwithstanding that the negative free-particle energies are *just one* of a *list* of egregiously unphysical properties which these equations possess [3]: the Klein-Gordon and Dirac equations have a decided attraction for working theorists because they tend to be *very tractable in calculations*, in part because they are purely *local* in configuration representation.

Indeed fondness for the calculational tractability of the Klein-Gordon and Dirac equations acted as a strong distraction from even *awareness* of the *basic requirement* which the *correspondence principle* imposes on the quantum theory of relativistic free particles, namely that the *classical* relativistic Hamiltonian for a free particle of positive mass m,

$$H_{\text{free}} = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}}, \tag{1a}$$

is to be *quantized*, upon which it straightforwardly becomes the *positive-definite* free-particle relativistic Hamiltonian *operator*,

$$\widehat{H}_{\text{free}} = (m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}, \tag{1b}$$

which is, of course, the essential input to the time-dependent Schrödinger equation for the relativistic free particle of positive mass m,

$$i\hbar\partial(|\psi(t)\rangle)/\partial t = (m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}|\psi(t)\rangle. \tag{1c}$$

Heeding the correspondence principle for relativistic free particles therefore *requires* that the time-dependent relativistic free-particle Schrödinger equation of Eq. (1c) *must supplant* the "more tractable" free-particle Klein-Gordon equation. It *even* must supplant the free-particle Dirac equation: the *nonrelativistic* Pauli equation for the spin $\frac{1}{2}$ particle has *no spin dependence whatsoever* when that particle is *free*, and furthermore there *always* exists an inertial frame in which a free particle moves *nonrelativistically* (or is even at rest). It is of course clear that the relativistic free-particle Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (1b) *has no negative eigenenergies whatsoever*. Thus *enforcement* of the correspondence principle *automatically* makes it *impossible* to even *speak about* the mind-boggling notion of "negative-energy free particles that travel backwards in time", which *forecloses* any possibility of characterizing antiparticles as such, eliminating the basis of the CPT theorem and its corollaries.

In the context of *respecting* the correspondence principle, antiparticles obviously must be introduced at the multi-particle level via the imposition of CP invariance on the field-theory Hamiltonian—in the longer run the nature of the physical mechanisms which in fact *break* CP invariance will need to be discovered. Of course in this context of *respecting* the correspondence principle there is no reason at all why these CP-breaking mechanisms should not produce particle-antiparticle mass splitting in consonance with what the preliminary data from MINOS appear to indicate.

The relativistic free-particle Klein-Gordon and Dirac equations, notwithstanding their associated lists of unphysical features [3], have, of course, been clung to by those who do calculations partly because they are *local* in configuration representation, which inter alia results in *local* field theories. The relativistic free-particle time-dependent Schrödinger equation of Eq. (1c), which follows from the correspondence principle, has *no* unphysical features that correspond to either those of the Klein-Gordon or the Dirac equation [3], but its representation in configuration space is *nonlocal*, so consequent quantum field theories will *as well* be formally nonlocal, i.e., their field-theory Hamiltonians will *not* have underlying *Hamiltonian densities* in the configuration regime.

Indeed there are *other* details of specifically the *quantum electrodynamics* which the correspondence principle implies that *must differ* from those of the present theory, in which the Dirac equation figures so prominently. In particular, the Dirac equation in the presence of an external electromagnetic field needs to be *replaced* by a time-dependent Schrödinger equation which smoothly *reduces* to the time-dependent relativistic *free-particle* Schrödinger equation that is given by Eq. (1c) when that external electromagnetic field is switched off. That time-dependent Schrödinger equation must *also* smoothly reduce to the nonrelativistic Pauli equation in the nonrelativistic limit. Such an equation has indeed been developed from the nonrelativistic Pauli equation by systematically applying to it fully relativistic upgrading techniques which are guided by the basic observation that there always exists an inertial frame in which a positive-mass solitary particle is instantaneously moving nonrelativistically [3]. As pointed out above, antiparticles must be brought into correspondence-principle compatible quantum electrodynamics by imposing charge-conjugation invariance on the field-theory Hamiltonian (parity, of course, is *conserved* in electrodynamics). After this is done, particle-antiparticle pair production and annihilation is made possible by the imposition on the field-theory Hamiltonian of a *further* symmetry, namely its invariance under the interchange of particle annihilation with antiparticle creation as well as under the interchange of antiparticle annihilation with particle creation.

Of course the correspondence-principle compatible quantum electrodynamics of a relativistic spin 0 charged particle of positive mass is to be handled in closely similar fashion; there the analogous systematic relativistic upgrade of the nonrelativistic Hamiltonian of a spinless, positive-mass charged particle in completely nonrelativistic interaction with an electric potential neatly results in precisely the fully relativistic Hamiltonian from which Hamilton's classical equations of motion produce the fully relativistic version of the Lorentz force law [3]. This brings to light subtle and important physics that the Klein-Gordon equation, which inherently reflects only the *square* of a Hamiltonian [3], is obviously incapable of fully emulating.

Elucidation of the *full* structure of the modified quantum electrodynamics that is rooted in the requirements of the correspondence principle, right up to and including its "Feynman rules", requires a quite massive investment of time, patience, and ingenuity on the part of a host of contributors. It is furthermore naturally to be expected that the predictions of the modified theory will *deviate* somewhat from the predictions of the currently existing quantum electrodynamics in which the physically problematic Dirac or Klein-Gordon equations figure so prominently.

It is very interesting is this regard that a recent effort to obtain the value of the charge radius of the proton to high precision from measurement of the Lamb shift in muonic hydrogen has produced a result which is incompatible with the value of this charge radius that is obtained from combining precision spectroscopy of electronic hydrogen with bound-state quantum electrodynamics [4]. The Lamb shift itself is, of course, overwhelmingly due to a bound-state quantum electrodynamics effect (it vanishes in the nonrelativistic Schrödinger and in the relativistic Dirac equation models of the hydrogen atom), with only a very small percentage contribution to it arising from the charge radius of the proton, albeit that very small proton charge radius percentage contribution is clearly very much greater (up to 2% [4]) for muonic hydrogen than it would be for electronic hydrogen, whose Bohr radius is about two hundred times larger. Generally speaking, this very big Bohr radius difference implies that the importance of quantum electrodynamics calculations for the extraction of the charge radius of the proton from hydrogen atomic spectroscopy looms very much larger for electronic than for muonic hydrogen, notwithstanding that it is already very important for the latter. The above-mentioned two incompatible results (about five standard deviations discrepancy [4]) for the proton charge radius naturally casts suspicion on the present theoretical form of quantum electrodynamics in light of the far larger contribution made by quantum electrodynamics than by the proton charge radius itself to the effects that are actually measured—*especially* in view of the fact that the quantum electrodynamics contributions are systematically even *more* dominant over the proton charge radius contribution in electronic hydrogen than they are in muonic hydrogen.

In the following sections key theoretical physics issues alluded to in the preceding paragraphs are treated at length along the lines expounded in Ref. [3]. We begin by pointing out the *natural compatibility* of solitary-particle quantum mechanics with special relativity, which consequently *reaffirms the validity* of the *correspon-*

dence principle in the domain of solitary-particle relativistic quantum mechanics, and we *also* point out the reason why *only* the Hamiltonian of Eq. (1a) is suitable for a relativistic classical free particle of positive mass m.

Solitary-particle quantum mechanics' inherent compatibility with relativity

The compatibility of solitary-particle quantum mechanics with special relativity is a straightforward consequence Schrödinger's two basic postulates for the wave function [5, 6], namely $\langle\mathbf{r}|\psi(t)\rangle$. The *first* Schrödinger postulate is the wave-function rule for the *operator quantization* of the particle's *canonical three-momentum*,

$$\langle\mathbf{r}|\widehat{\mathbf{p}}|\psi(t)\rangle = -i\hbar\nabla_{\mathbf{r}}(\langle\mathbf{r}|\psi(t)\rangle), \tag{2a}$$

which is as well, of course, a result of Dirac's postulated canonical commutation rule [7].

The *second* Schrödinger wave-function postulate is the famed *time-dependent Schrödinger wave equation* [5, 7, 6],

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial t = \langle\mathbf{r}|\widehat{H}|\psi(t)\rangle, \tag{2b}$$

which treats the *operator quantization* $\widehat{H}$ of the particle's classical Hamiltonian H in a manner that is *formally parallel* to the way in which Eq. (2a) treats the *operator quantization* of the particle's canonical three-momentum. The straightforward theoretical physics *implication* of Eqs. (2a) and (2b) is simply that the operators $\widehat{\mathbf{p}}$ and $\widehat{H}$ are the *generators* of the wave function's *infinitesimal space and time translations*, respectively. Therefore, in anticipation of the *restriction on such generators which special relativity imposes*, these two equations are usefully *combined* into the *single* formally *four-vector* Schrödinger equation for the wave function,

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial x_\mu = \langle\mathbf{r}|\widehat{p^\mu}|\psi(t)\rangle, \tag{2c}$$

where the *contravariant four-vector space-time partial derivative operator* $\partial/\partial x_\mu$ is defined as $\partial/\partial x_\mu \stackrel{\text{def}}{=} (c^{-1}\partial/\partial t, -\nabla_{\mathbf{r}})$, and the *formal* "contravariant four-vector" energy-momentum operator $\widehat{p^\mu}$ is defined as $\widehat{p^\mu} \stackrel{\text{def}}{=} (\widehat{H}/c, \widehat{\mathbf{p}})$. Since special relativity *requires* the contravariant space-time partial derivative four-vector operator $\partial/\partial x_\mu$ to transform between inertial frames in *Lorentz-covariant* fashion, it is apparent from Eq. (2c) that the *Hamiltonian operator* $\widehat{H}$ will be *compatible with special relativity* if it is related to the canonical three-momentum operator $\widehat{\mathbf{p}}$ in such a way that *also* makes the energy-momentum operator $\widehat{p^\mu}$ a contravariant four-vector which transforms between inertial frames in *Lorentz-covariant* fashion. This property of the Hamiltonian operator will, of course, be *automatically satisfied* if it is *the quantization of the Hamiltonian of a properly relativistic classical theory*. Therefore the *correspondence principle* definitely *remains valid* in the solitary-particle special-relativistic domain!

Now for a relativistic classical free particle of positive mass m, the logic of the Lorentz transformation from its *rest frame*, where it has four-momentum $(mc, \mathbf{0})$, to a frame where it has velocity $\mathbf{v}$(where $|\mathbf{v}| < c$) leaves *no freedom at all in the choice of its classical Hamiltonian*. That Lorentz boost takes this particle's four-momentum to,

$$(mc(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}},\ m\mathbf{v}(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}) = (E(\mathbf{v})/c,\ \mathbf{p}(\mathbf{v})), \tag{3a}$$

which, together with the *identity*,

$$mc^2(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}} = \sqrt{m^2c^4 + |cm\mathbf{v}|^2(1 - |\mathbf{v}|^2/c^2)^{-1}}, \tag{3b}$$

implies that,

$$E(\mathbf{v}) = \sqrt{m^2c^4 + |c\mathbf{p}(\mathbf{v})|^2} = H_{\text{free}}(\mathbf{p}(\mathbf{v})). \tag{3c}$$

Therefore the *only* physically suitable Hamiltonian for the relativistic classical free particle of positive mass m is the H_{free} of Eq. (1a). Thus *adherence to the correspondence principle*, together with the *categorical implication* of Eqs. (3), *determines* the Hamiltonian operator for the relativistic free particle of positive mass

m to be the square-root operator given by Eq. (1b), namely,

$$\widehat{H}_{\rm free} = \sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2},$$

which implies that the time-dependent Schrödinger equation for the relativistic free particle of positive mass m is that of Eq. (1c).

Since Eq. (1c) is therefore the *only quantum physically correct* time-dependent description of the relativistic free particle of positive mass m, the free-particle Klein-Gordon and Dirac equations ipso facto *must be quantum physically defective.* We now proceed to *analyze the sources* of those physical defects and also to *list* some of the *unphysical consequences* of the free-particle Klein-Gordon and Dirac equations.

The physically unsuitable Hamiltonian-squared basis of the free-particle Klein-Gordon equation

Because the square-root Hamiltonian operator $\widehat{H}_{\rm free}$ of Eq. (1b) for the positive-mass relativistic free particle is nonlocal in configuration representation, which might conceivably present an awkward calculational hurdle at a later stage when interactions with an external field are added, Klein, Gordon and Schrödinger *rejected* the *physically correct* positive-mass relativistic free-particle time-dependent Schrödinger equation of Eq. (1c) in favor of its *iteration*, which *squares* its square-root Hamiltonian operator $\widehat{H}_{\rm free}$, and, in conjunction with Schrödinger's canonical three-momentum quantization rule of Eq. (2a), yields,

$$-\hbar^2\partial^2(\langle\mathbf{r}|\psi(t)\rangle)/\partial t^2 = (m^2c^4 - \hbar^2c^2\nabla^2_{\mathbf{r}})\langle\mathbf{r}|\psi(t)\rangle, \tag{4a}$$

which is readily rewritten in the customary form for the free-particle Klein-Gordon equation,

$$(\partial^2/(\partial x^\mu\partial x_\mu) + (mc/\hbar)^2)\langle\mathbf{r}|\psi(t)\rangle = 0. \tag{4b}$$

To each stationary eigensolution $e^{-i\sqrt{m^2c^4+|c\mathbf{p}|^2}\,t/\hbar}\langle\mathbf{r}|\mathbf{p}\rangle$ of eigenmomentum $\mathbf{p}$ of the physically correct time-dependent relativistic free-particle Schrödinger equation, given by Eq. (1c), Eq. (4a) *adds* an *extraneous negative-energy partner solution* $e^{+i\sqrt{m^2c^4+|c\mathbf{p}|^2}\,t/\hbar}\langle\mathbf{r}|\mathbf{p}\rangle$ of the *same* momentum, whose *sole reason for existing* is the *entirely gratuitous iteration* of Eq. (1c)! These *completely* extraneous *negative* "free solitary-particle" energies, $-\sqrt{m^2c^4 + |c\mathbf{p}|^2}$, do *not* correspond to *anything* that exists in the *classical* dynamics of a free relativistic solitary particle, and by their negatively *unbounded* character threaten to spawn unstable runaway phenomena should the *free* Klein-Gordon equation be sufficiently perturbed (the Klein paradox) [6].

Due to the fact that the free-particle Klein-Gordon equation *lacks* a corresponding Hamiltonian operator—it depends on *only* the *square* of the Hamiltonian operator $\widehat{H}_{\rm free}$, as is seen from Eq. (4a) in conjunction with Eq. (2a)—it turns out, as is easily verified, that the *two* solutions of the *same momentum* $\mathbf{p}$ which have *opposite-sign* energies, i.e., $\pm\sqrt{m^2c^4 + |c\mathbf{p}|^2}$, *fail to be orthogonal to each other*, which *outright violates a key property* of orthodox quantum mechanics! *Without this property* the probablity interpretation of quantum mechanics *cannot be sustained*, and the Klein-Gordon equation is unsurprisingly *diseased* in that regard, yielding, inter alia, *negative probabilities* [6].

Furthermore, free-particle Klein-Gordon theory, depending as it does on *only* the *square* of the Hamiltonian operator $\widehat{H}_{\rm free}$ of Eq. (1b), rather than on that Hamiltonian operator *itself*, is thereby *cut adrift* from the normal quantum mechanical relationship to the Heisenberg picture, Heisenberg's equations of motion and the Ehrenfest theorem.

The fact of the matter is that the *square* of a Hamiltonian operator, *unlike* that Hamiltonian operator *itelf*, has *no cogent physical meaning*! That is the *source* of the above list of *unphysical consequences* of the free-particle Klein-Gordon equation.

Space-time mishandling of Schrödinger's equation that engenders Dirac's free-particle equation

Dirac pondered the foregoing list of the free-particle Klein-Gordon equation's unphysical properties, especially its failure to have a probability interpretation, and concluded that its dependence on *only* the *square* of the Hamiltonian operator $\widehat{H}_{\rm free}$ of Eq. (1b) was not tenable, but that the time-dependent description of a quantum mechanical system instead *must* be couched in terms of a time-dependent Schrödinder equation of the form of Eq. (2b) with a Hermitian Hamiltonian operator $\widehat{H}$. Very unfortunately indeed, notwithstanding that the correspondence principle *mandates* that this $\widehat{H}$ *must* equal the $\widehat{H}_{\rm free}$ of Eq. (1b) for the case of a positive-mass relativistic free particle, Dirac, emulating Klein, Gordon and Schrödinger, continued to *reject* the *physically* *correct* square-root Hamiltonian operator $\widehat{H}_{\mathrm{free}}$ of Eq. (1b) for the positive-mass relativistic free particle out of concern that its nonlocality in configuration representation might present an awkward calculational hurdle at a later stage when interactions of that particle with an external field are included.

Casting about for a more compelling theoretical "justification" than mere concerns over conceivable calculational hurdles for his quantum-physically *untenable* rejection of the square-root Hamiltonian operator $\widehat{H}_{\mathrm{free}}$, Dirac hit upon a spurious "relativistic need" for the time-dependent Schrödinger equation of Eq. (2b) to *by itself* exhibit "space-time coordinate symmetry" [8, 9, 6].

It is, of course, abundantly clear that it is the *four-vector* Schrödinger equation *system* of Eq. (2c) which *in fact manifests just this space-time coordinate symmetry* when its Hamiltonian operator $\widehat{H}$ is related to the canonical three-momentum operator $\widehat{\mathbf{p}}$ in such a way that the energy-momentum operator $\widehat{p^{\mu}} = (\widehat{H}/c, \widehat{\mathbf{p}})$ is a contravariant four-vector which transforms between inertial frames in *Lorentz-covariant* fashion, a property of $\widehat{H}$ which is *automatically satisfied* when it is *the quantization of a Hamiltonian H of a properly relativistic classical theory*! Thus the time-dependent Schrödinger equation of Eq. (2b) upon which Dirac myopically fastened his "space-time coordinate symmetry" gaze is the mere *time component* of a Lorentz-covariant *four-vector equation system*, and, as such, is *not space-time coordinate symmetric at all* since it is *completely skewed toward time*!

The fact of this *utter skewing toward time* of the time-dependent Schrödinger equation of Eq. (2b) is *driven home* by the *theoretical physics content* which its mathematical presentation *unmistakably conveys*, namely that the Hamiltonian operator is the generator of the *time translations* of the wave function. To attempt to *force* "space-time coordinate symmetry" on an equation which is *so completely skewed toward time* as is the time-dependent Schrödinger equation is a *classic* instance of attempting to "jam a square peg into a round hole", and can *only* result in a plenitude of unphysical consequences.

Blithely insensitive to the *necessarily* completely time-skewed nature of the time-dependent Schrödinger equation of Eq. (2b), Dirac noted that its *left-hand side* is *proportional* the *time-derivative* operator $\partial/\partial t$, and therefore sought to impose space-time coordinate symmetry on it by requiring its *right-hand side* to be (inhomogeneously) *linear* in the *spatial gradient* operator $\nabla_{\mathbf{r}}$. Of course the right-hand side of the time-dependent Schrödinger equation of Eq. (2b) only involves the *Hamiltonian operator $\widehat{H}$ in configuration representation*, whose (inhomogeneous) *linearity* in $\nabla_{\mathbf{r}}$ guarantees its *local nature*, which of course was Dirac's *overriding consideration from the very beginning*!

More abstractly, Dirac's imposition of space-time coordinate symmetry on the configuration-representation time-dependent Schrödinger equation of Eq. (2b) implies that its Hamiltonian operator $\widehat{H}$ is (inhomogeneously) *linear* in the *momentum operator* $\widehat{\mathbf{p}}$. If we now calculate the particle *velocity operator* $d\widehat{\mathbf{r}}/dt$ that is *implied* by such a Hamiltonian operator, i.e., one which is *linear* in the momentum operator $\widehat{\mathbf{p}}$, by using Heisenberg's equation of motion, we immediately obtain that this *velocity operator* $d\widehat{\mathbf{r}}/dt$ is *completely independent* of the *momentum operator* $\widehat{\mathbf{p}}$. However, we know very well that for the postive-mass relativistic *free particle in the nonrelativistic regime* the velocity operator $d\widehat{\mathbf{r}}/dt$ is *proportional* to $\widehat{\mathbf{p}}$ (i.e., equals $\widehat{\mathbf{p}}/m$), and, more generally, the *relativistic free particle* is *always* expected to have its velocity operator $d\widehat{\mathbf{r}}/dt$ *parallel* to the momentum operator $\widehat{\mathbf{p}}$, but this is obviously *impossible* if $d\widehat{\mathbf{r}}/dt$ is *independent* of $\widehat{\mathbf{p}}$, which is the *clear consequence* of Dirac's *physically misconceived* effort to *force* space-time coordinate symmetry on the time-dependent Schrödinger equation of Eq. (2b). In *stark contrast*, if we use for the Hamiltonian operator $\widehat{H}$ in the time-dependent Schrödinger equation of Eq. (2b) the positive-mass relativistic free-particle *square-root* Hamiltonian operator $\widehat{H}_{\mathrm{free}}$ of Eq. (1b) that is *mandated by the correspondence principle*, Heisenberg's equation of motion yields,

$$d\widehat{\mathbf{r}}/dt = \widehat{\mathbf{p}}/(m^2 + |\widehat{\mathbf{p}}/c|^2)^{\frac{1}{2}},$$

which is *obviously the correct result*! In other words, the *upshot* of the *squirming* by Klein, Gordon, Schrödinger, and Dirac to *evade the mandate of the correspondence principle* only results in *gratuitous theoretical grief* in the *completely unnecessary form of obviously unphysical results.*

It is clear that *continuing* with Dirac's physically misconceived approach is *counterproductive* from the standpoint of attaining physically correct understanding of positive-mass relativistic solitary-particle quantum mechanics. However, it is the case that textbooks [6, 9, 10] have simply *not presented* the most *strikingly* unphysical consequences of Dirac's approach to the positive-mass relativistic free particle, which makes it worthwhile to continue with Dirac's development in order to *expose those results to the light of day.*

Dirac's physically misconceived imposition of space-time coordinate symmetry on the time-dependent solitary-particle Schrödinger equation of Eq. (2b) does *not* fully determine its Hamiltonian operator $\widehat{H}$; it *only* determines that $\widehat{H}$ is (inhomogeneously) *linear* in the components of the momentum operator $\widehat{\mathbf{p}}$. For the *free* particle of positive mass m, we can write such a $\widehat{H}$ as,

$$\widehat{H}_D = \vec{\alpha} \cdot \widehat{\mathbf{p}}c + \beta mc^2, \tag{5a}$$

where what is known about β and the components of $\vec{\alpha}$ is that they are obviously *dimensionless*, and, because the solitary particle is *free*, they *won't* depend on the particle's coordinate operator $\widehat{\mathbf{r}}$, and so are *constants* in the particle's quantized phase-space vector operator $(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})$. Since that is *all* that can be said about β and $\vec{\alpha}$ *without* any further assumption, Dirac decided to *make* an assumption which essentially *determines* β and $\vec{\alpha}$. Having up to this point *deliberately snubbed* the positive-mass relativistic free- particle *square-root* Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (1b)—which is *mandated* by the correspondence principle to in fact be the *physically correct one*—Dirac now decided to *pull* $\widehat{H}_{\text{free}}$ *into the proceedings* by making it a *requirement* that,

$$(\widehat{H}_D)^2 = (\widehat{H}_{\text{free}})^2 = m^2c^4 + |c\widehat{\mathbf{p}}|^2. \tag{5b}$$

Notwithstanding that this requirement *superficially* appears to be a plausible one, Dirac *failed to note* that the *square* of a Hamiltonian operator *has no cogent physical meaning*, just as Klein, Gordon and Schrödinger had *earlier failed to note this very same pertinent fact*! Setting equal two mathematical entities which each lack definite physical meaning would seem *at least* as likely to generate *unphysical* consequences as physically legitimate ones. Indeed the requirement of Eq. (5b) turns out to be *directly responsible* for the fact that the eigenenergy spectrum of $\widehat{H}_D$ *exactly matches* the energies of the solutions of the free-particle Klein-Gordon equation, *including* that equation's extraneous negative energies which are unbounded below! So the *full* theory of the free-particle Dirac Hamiltonian $\widehat{H}_D$ is underlain by not merely *one*, but by *two* physically misconceived requirements. It is perhaps little wonder, then, as we shall shortly see, that $\widehat{H}_D$ gives rise to some stunningly unphysical predictions.

The upshot of the requirement of Eq. (5b) turns out to be that β and the three components of $\vec{\alpha}$ are Hermitian matrices (because $\widehat{H}_D$ is required to be a Hermitian operator) which each square to the identity matrix and which all mutually anticommute. These properties of themselves imply that these four matrices are all as well *traceless* [6], which implies that $\widehat{H}_D$ is traceless as well. Therefore $\widehat{H}_D$ *must* have *negative* eigenvalues if it has *positive* ones (and conversely). This fact, taken together with Eq. (5b) itself, implies the aforementioned identity of the eigenenergy spectrum of $\widehat{H}_D$ with the energies of the solutions of the free-particle Klein-Gordon equation, *including* that equation's *extraneous negative energies*, which are *unbounded below.*

Returning now to the issue that was broached above concerning the free Dirac particle's *velocity operator*, we obtain from Eq. (5a) and the Heisenberg equation of motion that,

$$d\widehat{\mathbf{r}}/dt = \vec{\alpha}c, \tag{6a}$$

which has the *highly unphysical* property of being *completely independent* of the particle's momentum operator $\widehat{\mathbf{p}}$, as was already pointed out above. Even *worse*, the free particle's *speed operator* comes out to be,

$$|d\widehat{\mathbf{r}}/dt| = \sqrt{3}\,c\mathrm{I}, \tag{6b}$$

which stunningly has a *but a single eigenvalue that exceeds the speed of light by* 73%! It is most interesting that while it is not uncommon for textbooks to at least *mention* the velocity operator result of Eq. (6a) [6]—and then to rapidly turn away from it—there is apparently not a single textbook which *uses* Eq. (6a) to obtain the very simple consequent *speed* operator result of Eq. (6b), which is, of course, utterly unphysical to an extent that is breathtaking. But underlain as the free-particle Dirac Hamiltonian operator $\widehat{H}_D$ is by not merely *one* but actually *two* requirements that are *not physically sensible*, namely the imposition of space-time symmetry on its time-dependent Schrödinger equation and the imposition on it of Eq. (5b), it is perhaps not surprising that it can give rise to such a blatantly relativistically-forbidden consequence.

Newton's first law of motion implies that the acceleration of a free particle vanishes identically. If we calculate $d^2\widehat{\mathbf{r}}/dt^2$ from the positive-mass relativistic free-particle square-root Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (1b), which is mandated by the correspondence principle, by applying Heisenberg's equation of motion twice in succession, we indeed obtain that this acceleration operator vanishes identically. It is a *very different story*, however, when we switch this calculation to the free-particle Dirac Hamiltonian $\widehat{H}_D$ of Eq. (5a). In that case, Heisenberg's equation of motion yields,

$$d^2\widehat{\mathbf{r}}/dt^2 = (imc^3/\hbar)(2\beta\vec{\alpha} + ((\vec{\alpha} \times \vec{\alpha}) \times \widehat{\mathbf{p}})/(mc)), \tag{7a}$$

which *fails* to vanish. Note that the matrix cross product $(\vec{\alpha} \times \vec{\alpha})$ does *not* vanish because the three components of $\vec{\alpha}$ mutually *anticommute*. From Eq. (7a) we can calculate the magnitude of the free Dirac particle's spontaneous acceleration,

$$|d^2\widehat{\mathbf{r}}/dt^2| = (2\sqrt{3}\, mc^3/\hbar)(1 + (2/3)(|\widehat{\mathbf{p}}|/(mc))^2)^{\frac{1}{2}}, \tag{7b}$$

whose *minimum* value, $(2\sqrt{3}\, mc^3/\hbar)$, is, for the case of the electron, well in excess of $10^{28}g$, where g is the acceleration of gravity at the earth's surface. This dumbfounding spontaneous acceleration of the "free Dirac electron", which stupendously violates Newton's first law of motion, again drives home the lesson of *just how unphysical* the Dirac free-particle Hamiltonian $\widehat{H}_D$ is—but this result as well seems to have escaped the notice of textbooks.

It is readily shown that the orbital angular momentum operator $\widehat{\mathbf{L}} \stackrel{\text{def}}{=} \widehat{\mathbf{r}}\times\widehat{\mathbf{p}}$ commutes with the positive-mass relativistic free-particle square-root Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (1b) that is mandated by the correspondence principle. It commutes as well with the nonrelativistic free-particle Pauli Hamiltonian operator—which is simply $|\widehat{\mathbf{p}}|/(2m)$ for that free-particle case. However it does *not* commute with the free-particle Dirac Hamiltonian $\widehat{H}_D$, which yields the nonvanishing spontaneous spin-orbit torque operator,

$$d\widehat{\mathbf{L}}/dt = \vec{\alpha} \times \widehat{\mathbf{p}}c, \tag{8a}$$

whose magnitude is,

$$|d\widehat{\mathbf{L}}/dt| = \sqrt{2}\, |\widehat{\mathbf{p}}|c, \tag{8b}$$

Now the relativistic free particle's *kinetic energy* is,

$$\widehat{T} = (m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}} - mc^2 = ((\widehat{H}_D)^2)^{\frac{1}{2}} - mc^2. \tag{8c}$$

If we take the *dimensionless ratio* of the Dirac particle's spontaneous spin-orbit torque magnitude to its kinetic energy, we obtain,

$$|d\widehat{\mathbf{L}}/dt|/\widehat{T} = \sqrt{2}\, ((1 + (mc/|\widehat{\mathbf{p}}|)^2)^{\frac{1}{2}} + (mc/|\widehat{\mathbf{p}}|)), \tag{8d}$$

which *increases monotonically without bound* from its ultrarelativistic asymptotic value of $\sqrt{2}$ as $|\widehat{\mathbf{p}}|$ *decreases*. This free-particle Dirac-theory result is, of course, *completely inconsistent* with the free-particle Pauli theory, where this ratio always *vanishes identically* for nonvanishing $|\widehat{\mathbf{p}}|$.

So the Dirac theory certainly *does not* reduce to the Pauli theory merely by going to sufficiently small nonzero values of momentum. That was *already clear*, of course, from the fact that the Dirac particle's *speed* always has the value $\sqrt{3}\, c$ *irrespective* of its momentum, which *doesn't accord* with the free-particle Pauli theory speed operator $|\widehat{\mathbf{p}}|/m$ *at all* when $|\widehat{\mathbf{p}}| \ll mc$. The *highly anomalous* spontaneous spin-orbit coupling of the *free* Dirac particle that we discussed above seems to have eluded the notice of textbooks as well.

The examples of astoundingly unphysical results which emerge from the Dirac free-particle Hamiltonian $\widehat{H}_D$ can apparently be multiplied almost at will: e.g., the noncommutativity of orthogonal components of the Dirac velocity operator of Eq. (6a) has *surpassingly unphysical* systematic characterics,

$$[(d\widehat{\mathbf{r}}/dt)_x, (d\widehat{\mathbf{r}}/dt)_y] = 2c^2\alpha_x\alpha_y. \tag{9}$$

This orthogonal velocity-component commutator *refuses to vanish* even in the *classical limit* that $\hbar \to 0$, in defiance of everything known about classical velocity. If one then struggles for a glimmer of physical comprehension of this orthogonal velocity-component commutator by going to the *nonrelativistic limit* $c \to \infty$, where it obviously *also* should vanish, it instead *diverges*! The *highly unphysical behavior* of the *commutators* of a list of *observables* in the free-particle Dirac theory has apparently not been noticed by textbooks either.

Relativistic solitary-particle quantum mechanics in an electromagnetic potential

The preceding subsections have made it abundantly clear that the Klein-Gordon and Dirac theories cannot sensibly describe the positive-mass relativistic free particle, but the straightforward square-root Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (1b), *which is mandated by the correspondence principle for this task*, describes the positive-mass relativistic free particle flawlessly. We shall now present in detail the *extensions* of $\widehat{H}_{\text{free}}$ which were developed in Refs. [11, 3] for the cases of a solitary relativistic spin 0 and spin $\frac{1}{2}$ particle of charge e and positive mass m in the presence of an external electromagnetic potential $A^\mu(\mathbf{r}, t)$.

The underlying idea is that if one has a trustworthy description of the physics experienced by a solitary particle that moves *nonrelativistically*, the physics that it experiences when it moves relativistically boils down to Lorentz transformations from an appropriate succession of inertial frames in each of which it instantaneously moves nonrelativistically.

However, instead of trying to model a self-consistently nonrelativistic succession of inertial frames, and then carrying out the corresponding Lorentz transformations, the technical approach adopted here is rather to try to associate each individual term of the solitary particle's *nonrelativistic Hamiltonian* with a fully Lorentz-covariant four-momentum whose time component *reduces to that particular nonrelativistic Hamiltonian term* in any inertial frame where the particle is moving sufficiently slowly. All those individual Lorentz-covariant four-momenta are then *summed* to produce the solitary particle's Lorentz-covariant *total* four-momentum. The total three-momentum *part* of the solitary particle's total four-momentum is obviously identified as the generator of the solitary particle's spatial translations, and therefore as the solitary particle's relativistic *canonical* three-momentum. Of course the solitary particle's *relativistic total energy*, when expressed as function of its relativistic *canonical* three-momentum, the time, and that particle's three space coordinates comprises that particle's *relativistic Hamiltonian. Initially*, of course, the *individual* four-momenta that contribute to the solitary particle's *total* four-momentum will be couched in the language of the particle's three space coordinates, the time, and the particle's relativistic *kinetic* three-momentum. After *identification* of the particle's *canonical* (i.e., *total*) three-momentum, it is *necessary* to solve for its *kinetic* three-momentum as a *function* of its *canonical* three-momentum in order to be able to *reexpress* its total energy as its Hamiltonian. Unfortunately, there is no guarantee that the particle's relativistic *kinetic* three-momentum can be worked out as a function of its relativistic *canonical* three-momentum *in closed form*. Thus the solitary particle's relativistic Hamiltonian *itself* could *conceivably* only be available as a sequence of approximations.

We begin by applying this program to a spin 0 solitary particle of positive mass m and charge e in the presence of an external electromagnetic potential $A^\mu(\mathbf{r}, t)$. Note that all *magnetic* effects of such a potential on the spin 0 charged particle's motion *vanish entirely* in the particle's rest frame, and are, more generally, of order $O(1/c)$, but in nonrelativistic physics the speed of light c is regarded as an asymptotically large parameter. Thus the *strictly nonrelativistic* Hamiltonian operator for this particle involves *only* the electromagnetic potential's time component $A^0(\mathbf{r}, t)$,

$$\widehat{H}^{(\text{NR})}_{\text{EM};0} = |\widehat{\mathbf{p}}|^2/(2m) + eA^0(\widehat{\mathbf{r}}, t). \tag{10a}$$

Because of the technical issue regarding the choice of ordering of noncommuting operators (whose resolution we allude to below), it is convenient to develop the relativistic four-momentum as a function of *classical* $(\mathbf{r}, \mathbf{p})$ phase space *rather* than as a function of the *already quantized* $(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})$ phase space of Eq. (10a). The solitary particle's nonrelativistic kinetic energy $|\mathbf{p}|^2/(2m)$, plus its rest mass energy mc^2, is well-known to correspond to c times its Lorentz-covariant free-particle *kinetic* four-momentum p^μ,

$$p^\mu \stackrel{\text{def}}{=} ((m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}, \mathbf{p}),$$

where, of course, $\mathbf{p}$ is the particle's relativistic *kinetic* three-momentum, which was carefully distinguished in the discussion above from its relativistic *total* (i.e., *canonical*) three-momentum. It is apparent that in the nonrelativistic limit $|\mathbf{p}| \ll mc$, the time component times c of p^μ does indeed, as just mentioned, behave as,

$$cp^0 \approx mc^2 + |\mathbf{p}|^2/(2m).$$

The potential energy term $eA^0(\mathbf{r}, t)$ of $H^{(\text{NR})}_{\text{EM};0}$, divided by c, is obviously the time component of the Lorentz-covariant four-momentum $eA^\mu(\mathbf{r}, t)/c$. Therefore adding eA^μ/c to p^μ produces a fully Lorentz-covariant *total* four-momentum whose time component times c reduces, in any inertial frame in which the nonzero-mass charged spin 0 solitary particle instantaneously has a sufficiently slow speed (i.e., $|\mathbf{p}| \ll mc$), to this particle's nonrelativistic classical Hamiltonian $H^{(\text{NR})}_{\text{EM};0}$ (which corresponds to the quantized Hamiltonian operator $\widehat{H}^{(\text{NR})}_{\text{EM};0}$ of Eq. (10a)) plus this particle's rest mass energy mc^2. We therefore regard,

$$P^\mu \stackrel{\text{def}}{=} p^\mu + eA^\mu(\mathbf{r}, t)/c, \tag{10b}$$

as this solitary particle's *total* four-momentum. Eq. (10b) implies that this particle's relativistic *total* three-momentum is,

$$\mathbf{P} = \mathbf{p} + e\mathbf{A}(\mathbf{r}, t)/c, \tag{10c}$$

and that its relativistic total energy is,

$$E(\mathbf{r}, \mathbf{p}, t) = cP^0 = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}} + eA^0(\mathbf{r}, t). \tag{10d}$$

Here we are in the fortunate position of being able to *solve* Eq. (10c) for the particle's relativistic *kinetic* three-momentum $\mathbf{p}$ as a *function* of its relativistic *total* (i.e., *canonical*) three-momentum $\mathbf{P}$ in *closed form*, i.e.,

$$\mathbf{p}(\mathbf{P}) = \mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c, \tag{10e}$$

which we must now *substitute* into Eq. (10d) for the relativistic total energy in order to *reexpress* that total energy as the *relativistic Hamiltonian* $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$, i.e.,

$$H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t) \stackrel{\mathrm{def}}{=} E(\mathbf{r}, \mathbf{p}(\mathbf{P}), t).$$

With this we obtain from Eqs. (10d) and (10e) the fully relativistic classical Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$, that uniquely corresponds to our original nonrelativistic Hamiltonian operator $\widehat{H}^{(\mathrm{NR})}_{\mathrm{EM};0}$ of Eq. (10a),

$$H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t) = (m^2c^4 + |c\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)|^2)^{\frac{1}{2}} + eA^0(\mathbf{r}, t). \tag{10f}$$

Because of the *presence of the square root* in Eq. (10f) for $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$, there could conceivably be an issue regarding the *ordering* of the mutually noncommuting operators $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{P}}$ when one attempts *quantize* this classical Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$ to become the *Hamiltonian operator* $\widehat{H}^{(\mathrm{REL})}_{\mathrm{EM};0}$. Use of the *Hamiltonian phase-space path integral* [12] with $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$ in its *classical form as given by* Eq. (10f) provides one *definitive solution* to any such operator-ordering issue. Another *completely equivalent* solution to this issue lies with a *natural slight strengthening* of Dirac's canonical commutation rule such that *it remains self-consistent* [13]. From *either* of these approaches the resulting *unambiguous operator-ordering rule* turns out to be the one of Born and Jordan [14].

It is well worth noting that the relativistic classical Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$ of Eq. (10f) for the solitary spin 0 charged particle, when inserted into Hamilton's classical equations of motion, yields, after taking Eq. (10c) into account, *the fully relativistic version of the Lorentz-force law.* In other words, the relativistic solitary charged-particle Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$ of Eq. (10f) embodies *precisely* the well-known classical relativistic physics of the charged particle developed by H. A. Lorentz [15]. We also note that in the limit that the solitary-particle charge e goes to zero, $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$ reduces to the relativistic free-particle Hamiltonian H_{free} of Eq. (1a), as it indeed must. These results buttress confidence that the above-described systematic approach to upgrading physically trustworthy nonrelativistic solitary-particle Hamiltonians to fully relativistic ones is physically sound.

We now turn to the positive-mass spin $\frac{1}{2}$ solitary charged particle in the presence of an external electromagnetic potential $A^\mu(\mathbf{r}, t)$. Its nonrelativistic Hamiltonian $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ is the *same* as the nonrelativistic Hamiltonian $H^{(\mathrm{NR})}_{\mathrm{EM};0}$ of the spin 0 solitary charged particle *except for an additional interaction energy between the external magnetic field and the spin* $\frac{1}{2}$ *particle's magnetic dipole moment due to its intrinsic spin*, i.e., its Pauli spin magnetic dipole energy. Notwithstanding that this Pauli energy is customarily *formally written* as being proportional to $(1/c)$, *it must nonetheless be kept in the nonrelativistic limit* because it *fails to vanish* in the spin $\frac{1}{2}$ particle's *rest frame*,

$$H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}} = |\mathbf{p}|^2/(2m) + (ge/(mc))(\hbar/2)\vec{\sigma} \cdot (\nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r}, t)) + eA^0(\mathbf{r}, t). \tag{11a}$$

Just as in the case of $H^{(\mathrm{NR})}_{\mathrm{EM};0}$, we *deliberately refrain for the time being* from quantizing $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ in its conventional phase-space degrees of freedom $(\mathbf{r}, \mathbf{p})$ in order to facilitate the derivation of its natural fully relativistic upgrade. We cannot, however, switch off the inherently *quantum* nature of the spin $\frac{1}{2}$ particle's intrinsic spin without *causing its physical effects to disappear altogether*, so we have *no choice* but to accept the Hamiltonian $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ of Eq. (11a) as a *two-by-two Hermitian matrix* whose four entries are (complex-valued) classical dynamical variables. The Pauli spin magnetic dipole energy contribution to $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ is, however, the *only part* of this nonrelativistic Hamiltonian which is *not* a multiple of the two-by-two *identity* matrix. Now the Lorentz-covariant four-momenta that we shall be developing in the course of deriving the natural relativistic upgrade of $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ will of course *themselves* naturally come out to be four-vectors of two-by-two matrices, but this should not present an issue *insofar as their four components always mutually commute*. To *ensure* that this is the case, we shall "quarantine" the *non-identity* Pauli spin magnetic dipole energy matrix into a *Lorentz scalar*. We can then render this entity *dimensionless* by dividing it by mc^2. If we now *multiply* this *dimensionless Lorentz scalar* by the particle's *kinetic* four-momentum $p^\mu = ((m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}, \mathbf{p})$, we will indeed have a Lorentz-covariant four-momentum contribution whose time component times c reduces to the Pauli energy matrix in the particle rest frame, which is precisely what we require.

There remains, of course, the challenging problem of reexpressing the complicated Pauli energy matrix term of Eq. (11a) as a Lorentz scalar. In relativistic tensor language, the magnetic field axial vector $(\nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r}, t))$ that appears in the Pauli energy matrix term of Eq. (11a) comprises a certain *three-dimensional part* of the four-dimensional relativistic second-rank antisymmetric electromagnetic *field tensor* $F^{\mu\nu}(\mathbf{r}, t) = \partial^\mu A^\nu(\mathbf{r}, t) - \partial^\nu A^\mu(\mathbf{r}, t)$. Now *if* we can manage to reexpress the spin $\frac{1}{2}$ angular-momentum axial vector $(\hbar/2)\vec{\sigma}$ that appears in the Pauli energy matrix term of Eq. (11a) as a "matching" three-dimensional part of a four-dimensional relativistic second-rank antisymmetric tensor $s^{\mu\nu}$, hopefully the Pauli energy matrix term of Eq. (11a) will end up being proportional to to their *Lorentz-scalar contraction* $s^{\mu\nu}F_{\mu\nu}(\mathbf{r}, t)$. As the first step toward this goal, we define the natural *three-dimensional* second-rank antisymmetric spin $\frac{1}{2}$ tensor S^{ij} in terms of the spin $\frac{1}{2}$ angular momentum axial vector $(\hbar/2)\vec{\sigma}$,

$$S^{ij} \stackrel{\mathrm{def}}{=} (\hbar/2)\epsilon^{ijk}\sigma^k,$$

and then note that the most complicated factor in the Pauli energy matrix term of Eq. (11a) neatly reduces to a contraction of S^{ij} with the well-known *magnetic-field* three-dimensional part $F^{ij}(\mathbf{r}, t)$ of $F^{\mu\nu}(\mathbf{r}, t)$, i.e.,

$$(\hbar/2)\vec{\sigma} \cdot (\nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r}, t)) = (1/2)S^{ij}F^{ij}(\mathbf{r}, t).$$

This allows us to reexpress the *nonrelativistic* Hamiltonian matrix $H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}}$ of Eq. (11a) in the *relativistically more suggestive form*,

$$mc^2 + H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}} = mc^2[1 + |\mathbf{p}|^2/(2m^2c^2) + (g/2)(e/(m^2c^3))S^{ij}F^{ij}(\mathbf{r}, t)] + eA^0(\mathbf{r}, t). \qquad (11b)$$

Of course we need to go *beyond* S^{ij} to the spin $\frac{1}{2}$ particle's fully covariant *four-dimensional* antisymmetric spin tensor $s^{\mu\nu}$. In the particle rest frame, namely in the special inertial frame where the particle kinetic three-momentum $\mathbf{p}$ vanishes, the nine space-space components of $s^{\mu\nu}$ must clearly be the nine components of S^{ij}, and its remaining seven components must be *filled out with zeros*, i.e.,

$$s^{\mu\nu}(\mathbf{p} = \mathbf{0}) \stackrel{\mathrm{def}}{=} \delta^\mu_i \delta^\nu_j S^{ij},$$

because this *ensures* that, in the particle rest frame,

$$s^{\mu\nu}(\mathbf{p} = \mathbf{0})F_{\mu\nu}(\mathbf{r}, t) = S^{ij}F^{ij}(\mathbf{r}, t).$$

Once a tensor is *fully determined* in *one* inertial frame, it is fully determined in *all* inertial frames by application of the appropriate Lorentz transformation to its indices. To get from the particle rest frame to the inertial frame where the particle has kinetic three-momentum $\mathbf{p}$ simply requires the appropriate Lorentz-boost four-dimensional matrix $\Lambda^\mu_\alpha(\mathbf{v}(\mathbf{p})/c)$ that is characterised by the corresponding dimensionless scaled relativistic particle velocity,

$$\mathbf{v}(\mathbf{p})/c = (\mathbf{p}/(mc))/(1 + |\mathbf{p}/(mc)|^2)^{\frac{1}{2}},$$

and its accompanying dimensionless time-dilation factor,

$$\gamma(\mathbf{p}) = (1 + |\mathbf{p}/(mc)|^2)^{\frac{1}{2}},$$

so that, in general,

$$s^{\mu\nu}(\mathbf{p}) = \Lambda^{\mu}_{i}(\mathbf{v}(\mathbf{p})/c)\Lambda^{\nu}_{j}(\mathbf{v}(\mathbf{p})/c)S^{ij},$$

which, of course, *ensures* that $s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)$ is a Lorentz scalar that Lorentz-invariantly *conveys* the spin $\frac{1}{2}$ particle's *rest-frame value of* $S^{ij}F^{ij}(\mathbf{r},t)$.

With that, we are in the position to be able to write down the Lorentz-covariant *total* four-momentum matrix P^{μ} for the spin $\frac{1}{2}$ particle in the presence of the external electromagnetic potential $A^{\mu}(\mathbf{r},t)$ that corresponds to its nonrelativistic energy matrix of Eq. (11b) in the same way that the Lorentz-covariant total four-momentum P^{μ} of Eq. (10b) for the spin 0 particle in the presence of $A^{\mu}(\mathbf{r},t)$ corresponds to *its* nonrelativistic energy $(mc^2 + H^{(\mathrm{NR})}_{\mathrm{EM};0})$,

$$P^{\mu} \stackrel{\mathrm{def}}{=} p^{\mu}[1 + (g/2)(e/(m^2c^3))s^{\alpha\beta}(\mathbf{p})F_{\alpha\beta}(\mathbf{r},t)] + eA^{\mu}(\mathbf{r},t)/c. \tag{11c}$$

From P^{μ} we obtain the spin $\frac{1}{2}$ particle's relativistic total energy matrix,

$$E(\mathbf{r},\mathbf{p},t) = cP^0 = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)] + eA^0(\mathbf{r},t), \tag{11d}$$

and also its relativistic total (i.e., *canonical*) three-momentum matrix,

$$\mathbf{P} = \mathbf{p}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)] + e\mathbf{A}(\mathbf{r},t)/c. \tag{11e}$$

It is apparent from Eq. (11e) that we *cannot solve* for the spin $\frac{1}{2}$ particle's kinetic momentum matrix $\mathbf{p}(\mathbf{P})$ as a *function* of its *canonical* momentum matrix $\mathbf{P}$ in *closed form*, but we *can* express $\mathbf{p}(\mathbf{P})$ in the "iteration-ready" form,

$$\mathbf{p}(\mathbf{P}) = (\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)]^{-1}. \tag{11f}$$

Furthermore, the spin $\frac{1}{2}$ particle's relativistic total energy matrix $E(\mathbf{r},\mathbf{p},t)$ of Eq. (11d) yields the *schematic form* of its *relativistic Hamiltonian matrix* $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ as simply $E(\mathbf{r},\mathbf{p}(\mathbf{P}),t)$,

$$H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) = (m^2c^4 + |c\mathbf{p}(\mathbf{P})|^2)^{\frac{1}{2}}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)] + eA^0(\mathbf{r},t). \tag{11g}$$

If we take the limit $g \to 0$ in Eqs. (11f) and (11g), then $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) \to H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r},\mathbf{P},t)$, as is easily checked from Eq. (10f). Of course it is nothing more than basic common sense that fully relativistic spin $\frac{1}{2}$ theory must reduce to fully relativistic spin 0 theory when the spin coupling of the single particle to the external field is switched off, but analogous cross-checking between the Dirac and Klein-Gordon theories is never so much as discussed! It is certainly possible to add a term to the Dirac Hamiltonian that *cancels out* it's *supposed* $g = 2$ spin coupling to the magnetic field, but the result of doing this bears *very little resemblance* to the Klein-Gordon equation in the presence of the external electromagnetic potential! Elementary consistency checks are obviously *not* the strong suit of those two "theories"! If we similarly take the limit $e \to 0$ in Eqs. (11f) and (11g), then $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) \to (m^2c^4 + |c\mathbf{P}|^2)^{\frac{1}{2}}$, the free-particle Hamiltonian of Eq. (1a), as is physically required.

It is unfortunate that Eq. (11f) for $\mathbf{p}(\mathbf{P})$ is not amenable to closed-form solution, but if we assume that the spin coupling term, $(g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)$, which is a dimensionless Hermitian two-by-two matrix, effectively has the magnitudes of both of its eigenvalues much smaller than unity (which should be a very safe assumption for atomic physics), then we can approximate $\mathbf{p}(\mathbf{P})$ via successive iterations of Eq. (11f), which produces the approximation $(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)$ for $\mathbf{p}(\mathbf{P})$ through zeroth order in the spin coupling and,

$$\mathbf{p}(\mathbf{P}) \approx (\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)F_{\mu\nu}(\mathbf{r},t)]^{-1},$$

through first order in the spin coupling. We wish to interject at this point that since $s^{\mu\nu}(\mathbf{p}(\mathbf{P}))$ is an anti-symmetric tensor, the tensor contraction $s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)$ is equal to $2s^{\mu\nu}(\mathbf{p}(\mathbf{P}))\partial_{\mu}A_{\nu}(\mathbf{r},t)$, which is often a more transparent form. Now if we simply use the approximation $(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)$ through zeroth order in the spin coupling for the kinetic three-momentum matrix $\mathbf{p}(\mathbf{P})$ of Eq. (11f), we obtain the following approximation to the spin $\frac{1}{2}$ relativistic Hamiltonian matrix $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ of Eq. (11g),

$$H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) \approx (m^2c^4 + |c\mathbf{P} - e\mathbf{A}(\mathbf{r},t)|^2)^{\frac{1}{2}}[1 + (ge/(m^2c^3))s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)\partial_{\mu}A_{\nu}(\mathbf{r},t)] + eA^0(\mathbf{r},t). \tag{11h}$$

The *approximation* on the right-hand side of Eq. (11h) to the Hamiltonian matrix $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ (whose schematic form is given by Eq. (11g)) for the positive-mass spin $\frac{1}{2}$ charged relativistic solitary particle in the presence of the external electromagnetic potential $A^{\mu}(\mathbf{r},t)$, is a two-by-two matrix whose four entries are (complex-valued) *classical dynamical variables.* These four entries must each be *quantized* in accordance with *the Born-Jordan operator-ordering rule*, analogously to the case of the spin 0 relativistic solitary-particle Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r},\mathbf{P},t)$ of Eq. (10f). Of course higher-order approximations in the spin coupling to the spin $\frac{1}{2}$ solitary-particle Hamiltonian matrix $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ must *likewise* be quantized.

Antiparticles from field-theory symmetry instead of from negative energy

Let us denote the just-mentioned Born-Jordan *quantization* of the ij entry ($i,j=1,2$) of the Hamiltonian matrix $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ (given schematically by Eq. (11g)) for the relativistic spin $\frac{1}{2}$ solitary particle of charge e and positive mass m in the presence of the external electromagnetic potential $A^{\mu}(\mathbf{r},t)$ as $\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(e,m,[A^{\mu}])\big)_{ij}$. Then a basic quantum field-theory model for *electrons alone*, which all have charge $-e$ and mass m_-, in the presence of the external electromagnetic potential $A^{\mu}(\mathbf{r},t)$ is given by the Hamiltonian operator,

$$\widehat{H}^{(-)}_F = \int d^3\mathbf{r}\sum_{i=1}^{2}\int d^3\mathbf{r}'\sum_{j=1}^{2}(\psi^{\dagger}_-(\mathbf{r}))_i\langle\mathbf{r}|\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(-e,m_-,[A^{\mu}])\big)_{ij}|\mathbf{r}'\rangle(\psi_-(\mathbf{r}'))_j. \tag{12a}$$

Since the relativistic solitary-particle Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t)$ has square roots whose arguments involve the canonical momentum $\mathbf{P}$, the above-utilized *configuration representation* of its *quantization* $\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(e,m,[A^{\mu}])$ will be *nonlocal*, and therefore the quantum field-theory Hamiltonian operator $\widehat{H}^{(-)}_F$ clearly *cannot* be expressed in terms of a local Hamiltonian *density* in the *configuration regime* utilized in Eq. (12a).

Now a quantum field-theory model which involves electrons *alone* is obviously extremely *charge asymmetric.* To *extend* our basic quantum field-theory model to one which manifests the symmetry of charge conjugation invariance, we are *compelled* to postulate the existence of *another particle* that has the *opposite charge* to that of the electron, but is *otherwise identical in all respects to the electron.* Denoting the creation fields of this postulated *positron* as $(\psi^{\dagger}_+(\mathbf{r}))_i$, we readily write down a minimally extended basic quantum field-theory Hamiltonian operator that indeed manifests charge conjugation invariance,

$$\begin{gathered}\widehat{H}^{(-+)}_F = \int d^3\mathbf{r}\sum_{i=1}^{2}\int d^3\mathbf{r}'\sum_{j=1}^{2}\big[\\ (\psi^{\dagger}_-(\mathbf{r}))_i\langle\mathbf{r}|\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(-e,m_-,[A^{\mu}])\big)_{ij}|\mathbf{r}'\rangle(\psi_-(\mathbf{r}'))_j+\\ (\psi^{\dagger}_+(\mathbf{r}))_i\langle\mathbf{r}|\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(+e,m_-,[A^{\mu}])\big)_{ij}|\mathbf{r}'\rangle(\psi_+(\mathbf{r}'))_j\big].\end{gathered} \tag{12b}$$

The minimally extended basic quantum field-theory Hamiltonian operator $\widehat{H}^{(-+)}_F$ of Eq. (12b) describes the *scattering* (or *binding*) of both relativistic electrons and relativistic positrons by the external electromagnetic potential $A^{\mu}(\mathbf{r},t)$. We know, however, that in principle such a potential could, if it were sufficiently rapidly-varying and strong, produce (or annihilate) electron-positron *pairs.* We can open a theoretical door to the occurrence of these electron-positron *pair processes* by imposing a *further charge-related symmetry* on the quantum field-theory Hamiltonian of Eq. (12b), namely its invariance under interchange of electron annihilation with positron creation and also under interchange of positron annihilation with electron creation. The simplest extension of $\widehat{H}^{(-+)}_F$ which manifests this "charge equivalence" symmetry under the interchanges $(\psi_-(\mathbf{r}))_i \leftrightarrow (\psi^{\dagger}_+(\mathbf{r}))_i$ and $(\psi_+(\mathbf{r}))_i \leftrightarrow (\psi^{\dagger}_-(\mathbf{r}))_i$, and which *as well* maintains the charge conjugation invariance symmetry, is given by the Hamiltonian operator,

$$\begin{gathered}\widehat{H}^{(-\leftrightarrow+^{\dagger})}_F = \tfrac{1}{2}\int d^3\mathbf{r}\sum_{i=1}^{2}\int d^3\mathbf{r}'\sum_{j=1}^{2}\big[\\ ((\psi_-(\mathbf{r}))_i+(\psi^{\dagger}_+(\mathbf{r}))_i)^{\dagger}\langle\mathbf{r}|\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(-e,m_-,[A^{\mu}])\big)_{ij}|\mathbf{r}'\rangle((\psi_-(\mathbf{r}'))_j+(\psi^{\dagger}_+(\mathbf{r}'))_j)+\\ ((\psi_+(\mathbf{r}))_i+(\psi^{\dagger}_-(\mathbf{r}))_i)^{\dagger}\langle\mathbf{r}|\big(\widehat{H}^{(\mathrm{REL})}_{\frac{1}{2}}(+e,m_-,[A^{\mu}])\big)_{ij}|\mathbf{r}'\rangle((\psi_+(\mathbf{r}'))_j+(\psi^{\dagger}_-(\mathbf{r}'))_j)\big].\end{gathered} \tag{12c}$$

It is apparent from Eq. (12c) that the imposition of the twin symmetries of charge conjugation invariance and "charge equivalence" does indeed produce a quantum field-theory model for electron-positron pair creation and annihilation by the external electromagnetic potential $A^\mu(\mathbf{r}, t)$, as well as electron and positron scattering (or binding) by that potential.

Now the visible universe is obviously skewed toward the preponderance of electrons over positrons, so we certainly expect that there is a physical agency which *breaks* charge conjugation invariance. While there is experimental evidence of such an agency, contemporary theoretical physics has not yet understood it in more than phenomenological fashion, but one would suppose that there must exist fields whose effective interaction strength magnitudes with electron and positron are *unequal.* There is, of course, *no* apparent theoretical reason why such a charge conjugation invariance *breaking* mechanism shouldn't generate *disparate corrections* to the electron and positron *masses.* In fact, from the "laziest" phenomenological standpoint, the *simplest* way to introduce charge conjugation invariance *breaking* into our model field-theory Hamiltonian operator $\widehat{H}_F^{(-\leftrightarrow+^\dagger)}$ of Eq. (12c) is to insert into it *exactly* such a *mass difference* δm between positron and electron,

$$\widehat{H}_F^{(-\leftrightarrow+^\dagger)\text{broken}}(\delta m) = \tfrac{1}{2}\int d^3\mathbf{r}\sum_{i=1}^{2}\int d^3\mathbf{r}'\sum_{j=1}^{2}\big[$$

$$((\psi_-(\mathbf{r}))_i + (\psi_+^\dagger(\mathbf{r}))_i)^\dagger\langle\mathbf{r}|\big(\widehat{H}_{\frac{1}{2}}^{(\text{REL})}(-e, m_-, [A^\mu])\big)_{ij}|\mathbf{r}'\rangle((\psi_-(\mathbf{r}'))_j + (\psi_+^\dagger(\mathbf{r}'))_j) + \tag{12d}$$

$$((\psi_+(\mathbf{r}))_i + (\psi_-^\dagger(\mathbf{r}))_i)^\dagger\langle\mathbf{r}|\big(\widehat{H}_{\frac{1}{2}}^{(\text{REL})}(+e, m_- + \delta m, [A^\mu])\big)_{ij}|\mathbf{r}'\rangle((\psi_+(\mathbf{r}'))_j + (\psi_-^\dagger(\mathbf{r}'))_j)\big].$$

This simple-minded model of charge conjugation invariance breaking drives home the point that in the *absence* of the theoretically ill-founded Klein-Gordon and Dirac equations—with their unphysical free particles of unboundedly negative energies (amongst a *plethora* of other unphysical features), upon which is further erected, via the mind-boggling assumption of compulsory negative-energy free-particle travel backwards in time, the "species identity" of antiparticle with particle which implies their perfect mass equality and the CPT theorem—there is *simply no compelling reason whatsoever* to expect the *exact equality* of particle and antiparticle masses. On the contrary, it would be *entirely unexpected* for the breaking of particle-antiparticle symmetry to fail to *naturally split* particle and antiparticle masses. Thus the embrace of the correspondence principle in relativistic quantum theory *removes* that discipline's categorical incompatibility with the preliminary MINOS finding of a mass difference between muon antineutrino and neutrino [1, 2]—as well as removing the CPT theorem and the configuration-regime *locality* of relativistic quantum field theory.

The relativistic quantum electrodynamics implied by embrace of the correspondence principle clearly differs in detail from the "orthodox" discipline bearing that name, with the pervasive influence of the "minimally coupled" Dirac Hamiltonian operator obviously supplanted by the Hamiltonian operators $\widehat{H}_{\frac{1}{2}}^{(\text{REL})}(\mp e, m_-, [A^\mu])$ that feature in Eq. (12c) above. The disagreement between the electronic hydrogen and muonic hydrogen approaches to measuring the charge radius of the proton [4], with their differing degrees of dependence on calculated quantum electrodynamics contributions, might turn out to be a harbinger of the need to reposition relativistic quantum electrodynamics firmly on the foundation of the correspondence principle.

Conclusion

An immense amount of work will need to be carried out in order to give birth to a comprehensive relativistic quantum electrodynamics (or other relativistic quantum field theory) that is properly founded on the correspondence principle. The proximate task is to upgrade the model field-theory Hamiltonian operator $\widehat{H}_F^{(-\leftrightarrow+^\dagger)}$ of Eq. (12c) to accommodate the *quantized* electromagnetic potential. This is a tricky undertaking: because electromagnetism is a gauge theory, only a *part* of it is dynamical and quantizable, but its nondynamical, nonquantizable potential part still has *physical consequences*, while *relativistically compatible gauge fixing* is needed to block spurious *unphysical* consequences [16]. After electromagnetism has been (hopefully) successfully dealt with, the "Feynman rules" threaten to be a tangled web indeed: the $H_{\text{EM};\frac{1}{2}}^{(\text{REL})}(\mathbf{r}, \mathbf{P}, t)$ schematically given by Eq. (11g) can itself only be obtained *iteratively*, and, even that aside, its square root structure, taken in conjunction with Eq. (11f), *already* guarantees that it depends on the electron charge e_- to arbitrarily high order. To this must be added the requirement of its Born-Jordan quantization to obtain $\widehat{H}_{\frac{1}{2}}^{(\text{REL})}(e, m, [A^\mu])$, on *top* of which comes perturbative development of the consequent quantum field theory in order to calculate transition amplitudes! There can be no question that sustained, patient and ingenious efforts by many contributors over a very extended period of time will be essential to obtaining results from relativistic quantum electrodynamics founded on the correspondence principle.

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