Generalized Gravity in Clifford Spaces, Vacuum Energy and Grand Unification

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Abstract

Polyvector-valued gauge field theories in Clifford spaces are used to construct a novel $Cl(3,2)$ gauge theory of gravity that furnishes $modified$ curvature and torsion tensors leading to important $modifications$ of the standard gravitational action with a cosmological constant. Vacuum solutions exist which allow a *cancellation* of the contributions of a very large cosmological constant term and the extra terms present in the modified field equations. Generalized gravitational actions in Clifford-spaces are provided and some of their physical implications are discussed. It is shown how the 16 fermions and their masses in each family can be accommodated within a $Cl(4)$ gauge field theory. In particular, the Higgs fields admit a natural Clifford-space interpretation that differs from the one in the Chamseddine-Connes spectral action model of Noncommutative geometry. We finalize with a discussion on the relationship with the Pati-Salam color-flavor model group $SU(4)_C \times SU(4)_F$ and its symmetry breaking patterns. An Appendix is included with useful Clifford algebraic relations.

1 Introduction

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [1] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds.

*Dedicated to the memory of Gustavo Ponce

C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion; it permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing [1]. It resolves the ordering ambiguities in QFT [2]; the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality [3], [1]. The relativity of signatures of the underlying spacetime results from taking different slices of C-space [4], [1]. Ideas very close to the extended Relativity in Clifford spaces have been considered by [6] and [7].

The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural scale changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations [1]. A generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces involves extended charges coupled to antisymmetric tensor fields of arbitrary rank and where the analog of photons are tensionless p-branes. The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and the program behind a Clifford Group Geometric Unification was advanced by [8].

Furthermore, there is no EPR paradox in Clifford spaces [9] and Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories and the Standard Model allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [10], [11]. Clifford-spaces can also be extended to Clifford-Superspaces by including both orthogonal and symplectic Clifford algebras and generalizing the Clifford super-differential exterior calculus in ordinary superspace to the full fledged Clifford-Superspace outlined in [12]. Clifford-Superspace is far richer than ordinary superspace and Clifford Supergravity involving polyvector-valued extensions of Poincare and (Anti) de Sitter supergravity (antisymmetric tensorial charges of higher rank) is a very relevant generalization of ordinary supergravity with applications in M-theory.

Grand-Unification models in 4D based on the exceptional E_8 Lie algebra have been known for sometime [14]. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [15]. Supersymmetric non-linear sigma models of Exceptional Kahler coset spaces are known to contain three generations of quarks and leptons as (quasi) Nambu-Goldstone superfields [16]. A Chern-Simons E_8 Gauge theory of Gravity was proposed [17] as a unified field theory (at the Planck scale) of a Lanczos-Lovelock Gravitational theory with a E_8 Generalized Yang-Mills field theory which is defined in the 15D boundary of a 16D bulk space. In particular, it was discussed in [12] how an E_8 Yang-Mills in 8D, after a sequence of symmetry breaking processes $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow SO(8,2)$, leads to a Conformal gravitational theory in 8D based on gauging the conformal group $SO(8,2)$ in 8D. Upon performing a Kaluza-Klein-Batakis [18] compactification on CP^2, involving a nontrivial torsion, leads to a Conformal Gravity-Yang-Mills unified theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in 4D. Furthermore, it was shown [12] how a conformal (super) gravity and (super) Yang-Mills unified theory in any dimension can be embedded into a (super) Clifford-algebra-valued gauge field theory by choosing the appropriate Clifford group.

A candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ was constructed [19]. It was obtained by recasting the E_8 group as the semi-direct product of $GL(8, R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of rank two and three. Other actions were proposed, like the quartic E_8 group-invariant action in 8D associated with the Chern-Simons E_8 gauge theory defined on the 7-dim boundary of a 8D bulk. To finalize, it was shown how the E_8 gauge theory of gravity can be embedded into a more general extended gravitational theory in Clifford spaces associated with the Cl(16) algebra.

Quantum gravity models in $4D$ based on gauging the (covering of the) $GL(4, R)$ group were shown to be renormalizable by [20] however, due to the presence of fourth-derivatives terms in the metric which appeared in the quantum effective action, upon including gauge fixing terms and ghost terms, the prospects of unitarity were spoiled. The key question remains if this novel gravitational model based on gauging the E_8 group may still be renormalizable without spoiling unitarity at the quantum level.

Most recently it was shown in [21] how a Conformal Gravity and $U(4) \times U(4)$ Yang-Mills Grand Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Upon taking a real slice, and after symmetry breaking, it leads to ordinary Gravity and a Yang-Mills theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in four real dimensions. Other approaches to unification based on Clifford algebras can be found in [13].

Having presented some of the relevant issues behind the role of Clifford algebras we outline the contents of this work. In **2** we construct a novel $Cl(3, 2)$ gauge theory of gravity that furnishes $modified$ curvature and torsion tensors leading to important $modifications$ of the standard gravitational action with a cosmological constant. Vacuum solutions exist which allow a cancellation of the contributions of very large cosmological constant term with the extra terms present in the modified field equations. Generalized gravitational actions in C-spaces are provided and some of their physical implications are discussed. In the final section we describe how the 16 fermions and their masses in each family can be accommodated within a $Cl(4)$ gauge field theory. In particular, how the Higgs fields admit a natural C-space interpretation that differs from the one in the Chamseddine-Connes spectral action model of Noncommutative geometry [37]. We finalize with a discussion on the relationship with the Pati-Salam color-flavor model $SU(4)_C \times SU(4)_F$ and its symmetry breaking patterns. An Appendix is included with useful Clifford algebraic relations.

2 Extended-Gravity in Clifford Spaces as a Gauge Field Theory based on the Clifford Group

A model of Emergent Gravity with the observed Cosmological Constant from a BF-Chern-Simons-Higgs Model was recently revisited [24] which allowed to show how a Conformal Gravity, Maxwell and $SU(2) \times SU(2) \times U(1) \times U(1)$ Yang-Mills Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory in a very natural and geometric fashion. In particular [21] a Conformal Gravity-Maxwell model can be constructed from a Clifford gauge field theory based on a $Cl(1,3)$ algebra. Chamseddine [27] has studied the $U(2,2)$ gravity model using the $Cl(2,2)$ algebra generators.

The $Cl(3,1)$ algebra-valued anti-Hermitian one-form was defined as

$$\mathbf{A} \;=\; \left(i\, a_\mu\, \mathbf{1} + \; b_\mu\, \Gamma_5 \;+\; e^a_\mu\, \Gamma_a \;+\; f^{a5}_\mu\, \Gamma_a\, \Gamma_5 \;+\; \frac{1}{4}\omega^{ab}_\mu\, \Gamma_{ab} \right)\, dx^\mu. \quad (2.1)$$

where $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4, (\Gamma_5)^2 = -1$. The fields e^a_μ, f^a_μ are related to the *physical* vielbein field (tetrad) V^a_μ that gauges the translation P_a symmetry, and the *physical* field V^{a5}_μ that gauges the conformal boosts K_a transformations, as follows

$$e^a_\mu\, \Gamma_a + f^{a5}_\mu\, \Gamma_a \Gamma_5 \;=\; V^a_\mu\, P_a + V^{a5}_\mu\, K_a \;\Rightarrow\; e^a_\mu \;=\; \frac{1}{2}(V^a_\mu + V^{a5}_\mu);\;\; f^a_\mu \;=\; \frac{i}{2}(V^a_\mu - V^{a5}_\mu); \quad (2.2)$$

The above relations are found after recurring to a Clifford algebra realization of the translation and conformal boost generators given by

$$P_a \;=\; \frac{1}{2}\, \Gamma_a\, (1 + i\, \Gamma_5);\;\; K_a \;=\; \frac{1}{2}\, \Gamma_a\, (1 - i\, \Gamma_5);\;\; (\Gamma_5)^2 \;=\; -1;\;\; a = 1,2,3,4 \quad (2.3)$$

such that $[P_a, P_b] = [K_a, K_b] = 0$ and $[P_a, K_b] \sim \eta_{ab} D + \mathcal{M}_{ab}$. The Lorentz generators and dilations are realized as $\mathcal{M}^{ab} = \frac{1}{2}\Gamma^{ab}$ and $D = i\Gamma_5$. The field a_μ was identified with the Maxwell field and b_μ with the Weyl gauge field of dilations.

The *problem* (caveat) with the above realization of the momentum operator $P_a = \frac{1}{2}\Gamma_a(1 + i\Gamma_5)$ is that it constrains P_a to be *nilpotent* $P_1P_1 = P_2P_2 = P_3P_3 = P_4P_4 = 0$, and also $P_aP_b = 0$ for any pair of indices $a \neq b$, due to the conditions $\{\Gamma_a, \Gamma_5\} = 0$ and $(1+i\Gamma_5)(1-i\Gamma_5) = 1+(\Gamma_5)^2 = 1-1 = 0$. Therefore, such realization (2.3) leads to a *trivial* commutator $[P_a, P_b] = P_aP_b - P_bP_a = 0 - 0 = 0$. The same results apply to the conformal boosts generators as well $K_1K_1 = ... = K_4K_4 = 0$ and $K_aK_b = 0$. In the case of the $Cl(2,2)$ algebra one has $(\Gamma_5)^2 = 1$, and the operator realizations $P_a = \Gamma_a(1 + \Gamma_5)$, $K_a = \Gamma_a(1 - \Gamma_5)$ lead to the same conclusions. The same occurs for the other Clifford algebras $Cl(1,3), Cl(4,0), Cl(0,4)$, irrespective of the signature.

To solve this problem, while obtaining an extended gravitational theory in C-spaces from a gauge field theory, we shall recur to the $Cl(3,2)$ algebra and write the gauge connection $\mathcal{A}_M = \mathcal{A}^I_M \ \Gamma_I, I = 1,2,3, ..., 32$ in terms of the 32 $Cl(3,2)$ algebra generators

$$\Gamma_I \ : \ \mathbf{1}; \ \Gamma_a = \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4, \ \Gamma_5; \ \Gamma_{a_1 a_2} = \frac{1}{2}\Gamma_{a_1} \wedge \Gamma_{a_2} = \frac{1}{2}[\Gamma_{a_1}, \Gamma_{a_2}];$$

$$\Gamma_{a_1 a_2 a_3} \ = \ \frac{1}{3!} \ \Gamma_{a_1} \wedge \Gamma_{a_2} \wedge \Gamma_{a_3}; \ ...., \ \Gamma_{a_1 a_2 ......a_5} \ = \ \frac{1}{5!} \ \Gamma_{a_1} \wedge \Gamma_{a_2} \wedge .... \wedge \Gamma_{a_5} \tag{2.4}$$

The decomposition of the connection $\mathcal{A}_M = \mathcal{A}^I_M \ \Gamma_I$ contains Hermitian and anti-Hermitian components. By suitably introducing i factors in the appropriate terms one may render all the components Hermitian or anti-Hermitian if desired.

It is common practice to split the de Sitter/Anti de Sitter algebra gauge connection in $4D$ into a (Lorentz) rotational piece $\omega_\mu^{a_1 a_2} \Gamma_{a_1 a_2}$ where $a_1, a_2 = 1,2,3,4$; $\mu, \nu = 1,2,3,4$, and a momentum piece $\omega_\mu^{a5} \Gamma_{a5} = \frac{1}{l} V^a_\mu P_a$, where V^a is the physical vielbein field, l is the de Sitter/Anti de Sitter throat size, and P_a is the momentum generator with $a = 1,2,3,4$. One may proceed in the same fashion in the Clifford algebra $Cl(3,2), Cl(4,1), ...$ case. The poly-momentum generator corresponds to those poly-rotations with a component along the 5-th direction in the *internal* space.

In *odd* dimensions there is *no* chirality operator. The spinorial representations in $D = 2n+1$ and $D = 2n$ are both 2^n dimensional. One cannot represent the $2^5 = 32$ generators of the Clifford algebra $Cl(5)$ in terms of 32 independent $(2^2 \times 2^2 = 4 \times 4)$ matrices because only 16 matrices (out of the 32) would have been *independent*. For example, the 6 generators of the semi-simple algebra $so(4) = su(2) \oplus su(2)$ cannot be represented in terms of 6 independent (2×2) matrices. Nevertheless, they can be represented in terms of 6 independent (4×4) matrices of the form

$$\mathcal{L}^i \ = \ \frac{1}{2} \ (\sigma^i \ \otimes \ \mathbf{1}_{2\times 2}); \quad \mathcal{R}^j \ = \ \frac{1}{2} \ (\mathbf{1}_{2\times 2} \ \otimes \ \sigma^j); \quad i,j \ = \ 1,2,3. \tag{2.5}$$

given in terms of the tensor products of the unit 2×2 matrix $\mathbf{1}$ and the three Pauli spin 2×2 matrices $\sigma^i, i = 1,2,3$ obeying $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$. The commutators are

$$[\mathcal{L}_i, \mathcal{L}_j] \ = \ i \ \epsilon_{ijk} \ \mathcal{L}_k; \ \ [\mathcal{R}_i, \mathcal{R}_j] \ = \ i \ \epsilon_{ijk} \ \mathcal{R}_k; \ \ [\mathcal{L}_i, \mathcal{R}_j] \ = \ 0 \tag{2.6}$$

Similarly, one may represent the 32 generators of the $Cl(3,2) = Cl(3,1)_L \oplus Cl(3,1)_R$ algebra in terms of $16+16 = 32$ *independent* 8×8 matrices $\mathcal{L}_A, \mathcal{R}_A$ (instead of 4×4 matrices) obtained from the tensor products of the 16 (4×4) matrices Γ_A with the unit 2×2 matrix $\mathbf{1}$ as follows

$$\mathcal{L}_A \ = \ \frac{1}{2} \ (\Gamma_A \otimes \mathbf{1}_{2\times 2}); \ \ \mathcal{R}_B \ = \ \frac{1}{2} \ (\mathbf{1}_{2\times 2} \otimes \Gamma_B); \ A, B = 1,2,3, ..., 16. \tag{2.7a}$$

such that

$$[\mathcal{L}_A, \mathcal{L}_B] \ = \ f_{ABC} \ \mathcal{L}_C; \ \ [\mathcal{R}_A, \mathcal{R}_B] \ = \ f_{ABC} \ \mathcal{R}_C; \ \ [\mathcal{L}_A, \mathcal{R}_B] \ = \ 0 \qquad (2.7b)$$

where f_{ABC} are the structure constants of the $Cl(3,1)$ algebra. We must emphasize that in this section we will *not* be concerned with finding matrix representations of the $Cl(3,2)$ gauge algebra but with the $Cl(3,2)$ algebra commutators per se. Therefore, one may assign

$$\Gamma_5 \ = \ P_{\mathbf{0}}; \ \ \Gamma_{a5} \ = l \ P_a, \ a = 1,2,3,4; \ \ \Gamma_{a_1 a_2 5} = l^2 \ P_{a_1 a_2}, \ a_1, a_2 = 1,2,3,4$$

$$\Gamma_{a_1 a_2 a_3 5} \ = \ l^3 \ P_{a_1 a_2 a_3}, \ a_1, a_2, a_3 = 1,2,3,4$$

$$\Gamma_{a_1 a_2 a_3 a_4 5} \ = \ l^4 \ P_{a_1 a_2 a_3 a_4}, \ a_1, a_2, a_3, a_4 = 1,2,3,4; \qquad (2.8)$$

In this way the 16 components of the (*noncommutative*) poly-momentum operator $P_A = P_{\mathbf{0}}, P_a, P_{a_1 a_2}, P_{a_1 a_2 a_3}, P_{a_1 a_2 a_3 a_4}$ are identified with those poly-rotations with a component along the 5-th direction in the *internal* space. A length scale l is needed to match dimensions.

$P_{\mathbf{0}}$ does not transform as a $Cl(3,2)$ algebra scalar, but as a vector. P_a does not transform as a $Cl(3,2)$ vector but as a bivector. $P_{a_1 a_2}$ does not transform as $Cl(3,2)$ bivector but as a trivector, etc.... What about under $Cl(3,1)$ transformations ? One can notice $[\Gamma_{ab}, \Gamma_5] = [\Gamma_{ab}, P_0] = 0$ when $a, b = 1,2,3,4$. Thus under rotations along the four dimensional subspace, $\Gamma_5 = P_0$ is inert, it behaves like a scalar from the four-dimensional point of view. This justifies the labeling of Γ_5 as P_0. The commutator

$$[\Gamma_{ab}, \ \Gamma_{c5}] \ = \ [\Gamma_{ab}, \ l \ P_c] \ = \ -\eta_{ac}\Gamma_{b5} \ + \ \eta_{bc}\Gamma_{a5} \ = \ -\eta_{ac} \ l \ P_b \ + \ \eta_{bc} \ l \ P_a \quad (2.9)$$

so that $\Gamma_{c5} = l P_c$ does behave like a vector under rotations along the four-dim subspace. Thus this justifies the labeling of Γ_{c5} as $l P_c$, etc...

To sum up, one has split the $Cl(3,2)$ gauge algebra generators into two sectors. One sector represented by $\mathcal{M}$ which comprises poly-rotations along the $four$-dim subspace involving the generators

$$\mathbf{1}; \ \ \Gamma_{a_1}; \ \Gamma_{a_1 a_2}; \ \ \Gamma_{a_1 a_2 a_3}; \ \ \Gamma_{a_1 a_2 a_3 a_4}, \ \ a_1, a_2, a_3, a_4 = 1,2,3,4. \qquad (2.10)$$

and another sector represented by $\mathcal{P}$ involving poly-rotations with one coordinate pointing along the internal 5-th direction as displayed in (2.8).

Thus their commutation relations are of the form

$$[\mathcal{P}, \ \mathcal{P}] \ \sim \ \mathcal{M}; \ \ [\mathcal{M}, \ \mathcal{M}] \ \sim \ \mathcal{M}; \ \ [\mathcal{M}, \ \mathcal{P}] \ \sim \ \mathcal{P}. \qquad (2.11)$$

which are compatible with the commutators of the Anti de Sitter, de Sitter algebra $SO(3,2), SO(4,1)$ respectively. To sum up, we have decomposed the $Cl(3,2)$ gauge connection one-form in C-space as

$$\mathcal{A}_M \, dX^M \ = \ \mathcal{A}^I_M \Gamma_I \, dX^M \ = \ \left(\Omega^A_M \ \Gamma_A \ + \ E^A_M \ P_A \right) \, dX^M; \ \ \Gamma_A \subset \mathcal{M}, \ \ P_A \subset \mathcal{P} \qquad (2.12)$$

where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate

$$\mathbf{X} \ = \ s \, \mathbf{1} \, + \, x_\mu \, \gamma^\mu \, + \, x_{\mu_1 \mu_2} \, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \, + \, x_{\mu_1 \mu_2 \mu_3} \, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \, + \, ... \quad (2.13)$$

In order to match dimensions in each term of (2.3) a length scale parameter must be suitably introduced. In [1] we introduced the Planck scale as the expansion parameter in (2.3). The scalar component s of the spacetime poly-vector valued coordinate $\mathbf{X}$ was interpreted by [5] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

Denoting the derivatives with respect to the poly-vector valued coordinates by ∂_M, the analog of the Abelian $U(1)$ field strength sector is $\mathcal{R}^{\mathbf{0}}_{MN} = \partial_{[M} \, \Omega^0_{N]}$. The other relevant components of the $Cl(3,2)$-valued gauge field strengths F^A_{MN} that are written as $\mathcal{R}^A_{MN}$, for reasons that will become clear below, are given by

$$\mathcal{R}^a_{MN} \ = \ \partial_{[M} \, \Omega^a_{N]} + \Omega^{mn}_M \, \Omega^r_N \ < \ [\gamma_{mn}, \gamma_r] \, \gamma^a \ > \, + \Omega^{mnpq}_M \, \Omega^{rst}_N \ < \ [\gamma_{mnpq}, \gamma_{rst}] \, \gamma^a \ > . \quad (2.14)$$

where the brackets $< \ [\gamma_{mn}, \gamma_r] \ \gamma^a \ >, < \ [\gamma_{mnpq}, \gamma_{rst}] \ \gamma^a \ >$ in (2.14) indicate the *scalar* part of the product of the $Cl(3,2)$ algebra elements; i.e it extracts the $Cl(3,2)$ invariant contribution. For example,

$$< \ [\gamma_{mn}, \gamma_r] \, \gamma^a \ > \ = \ < \ -\eta_{mr} \, \gamma_n \, \gamma^a \ > \, + \, < \ \eta_{nr} \gamma_m \gamma^a \ > \ = \ -\eta_{mr} \, \delta^a_n + \eta_{nr} \, \delta^a_m. \quad (2.15)$$

The commutation relations among the gamma generators of any rank and in any dimension are provided in the Appendix. The C-space version of the curvature two-form is

$$\mathcal{R}^{a_1 a_2}_{MN} \ = \ \partial_{[M} \, \Omega^{a_1 a_2}_{N]} + \Omega^m_M \, \Omega^r_N \ < \ [\gamma_m, \gamma_r] \, \gamma^{a_1 a_2} \ > \, + \Omega^{mn}_M \, \Omega^{rs}_N \ < \ [\gamma_{mn}, \gamma_{rs}] \, \gamma^{a_1 a_2} \ > \, +$$

$$\Omega^{mnp}_M \, \Omega^{rst}_N \ < \ [\gamma_{mnp}, \gamma_{rst}] \, \gamma^{a_1 a_2} \ > \, + \Omega^{mnpq}_M \, \Omega^{rstu}_N \ < \ [\gamma_{mnpq}, \gamma_{rstu}] \, \gamma^{a_1 a_2} \ > \, +$$

$$\Omega^{mnpqk}_M \, \Omega^{rstuv}_N \ < \ [\gamma_{mnpqk}, \gamma_{rstuv}] \, \gamma^{a_1 a_2} \ > . \quad (2.16)$$

To evaluate the torsion component $\mathcal{T}^0_{MN} P_0 = \mathcal{R}^5_{MN} \Gamma_5$ requires writing

$$\mathcal{R}^5_{MN} \ = \ \partial_{[M} \, \Omega^5_{N]} \ + \ f^5_{BC} \, \Omega^B_M \ \wedge \ \Omega^C_N \ = \ \partial_{[M} \, \Omega^5_{N]} \ +$$

$$\Omega^{mn}_M \, \Omega^r_N \ < \ [\gamma_{mn}, \gamma_r] \, \gamma^5 \ > \ + \ \Omega^{mnpq}_M \, \Omega^{rst}_N \ < \ [\gamma_{mnpq}, \gamma_{rst}] \, \gamma^5 \ > . \quad (2.17)$$

To evaluate the torsion component $\mathcal{T}^a_{MN} P_a = \frac{1}{l} \mathcal{R}^{a5}_{MN} \Gamma_{a5}$ requires writing

$$\mathcal{T}^a_{MN} \ = \mathcal{R}^{a5}_{MN} \ = \partial_{[M} \, \Omega^{a5}_{N]} \ + \ \Omega^m_M \, \Omega^r_N \ < \ [\gamma_m, \gamma_r] \, \gamma^{a5} \ > \ +$$

$$\Omega^{mn}_M \, \Omega^{rs}_N \ < \ [\gamma_{mn}, \gamma_{rs}] \, \gamma^{a5} \ > \ +$$

$$\Omega^{mnp}_M \, \Omega^{rst}_N \ < \ [\gamma_{mnp}, \gamma_{rst}] \, \gamma^{a5} \ > \ + \ \Omega^{mnpq}_M \, \Omega^{rstu}_N \ < \ [\gamma_{mnpq}, \gamma_{rstu}] \, \gamma^{a5} \ > \ +$$

$$\Omega^{mnpqk}_M \, \Omega^{rstuv}_N \ < \ [\gamma_{mnpqk}, \gamma_{rstuv}] \, \gamma^{a5} \ > . \quad (2.18)$$

For example, when M, N are both vector indices one arrives at the *modified* torsion

$$\mathcal{T}^a_{\mu\ \nu} = \mathcal{R}^{a5}_{\mu\ \nu} = \partial_{[\mu}\ \Omega^{a5}_{\nu]} +$$
$$\Omega^m_\mu\ \Omega^r_\nu\ < [\gamma_m, \gamma_r]\ \gamma^{a5}\ > + \Omega^{mn}_\mu\ \Omega^{rs}_\nu\ < [\gamma_{mn}, \gamma_{rs}]\ \gamma^{a5}\ > +$$
$$\Omega^{mnp}_\mu\ \Omega^{rst}_\nu\ < [\gamma_{mnp}, \gamma_{rst}]\ \gamma^{a5}\ > + \Omega^{mnpq}_\mu\ \Omega^{rstu}_\nu\ < [\gamma_{mnpq}, \gamma_{rstu}]\ \gamma^{a5}\ > +$$
$$\Omega^{mnpqk}_\mu\ \Omega^{rstuv}_\nu\ < [\gamma_{mnpqk}, \gamma_{rstuv}]\ \gamma^{a5}\ > . \tag{2.19}$$

Form (2.19) one can see that the $Cl(3,2)$-algebraic expression for the torsion $\mathcal{T}^a_{\mu\ \nu}$ contains many *more* terms than the standard expression for the torsion in Riemann-Cartan spacetimes

$$T^a_{\mu\nu}\ dx^\mu\ \wedge\ dx^\nu\ =\ R^{a5}_{\mu\nu}\ dx^\mu\ \wedge\ dx^\nu\ =\ l\ (\mathbf{d}\ \Omega^{a5}\ +\ \Omega^a{}_b \wedge \Omega^{b5}\ )\ =$$
$$\mathbf{d}\ V^a\ +\ \Omega^a{}_b \wedge V^b. \tag{2.20}$$

The vielbein one-form is $V^a = V^a_\mu dx^\mu = l\ \Omega^{a5}_\mu dx^\mu$ and the spin connection one-form is $\Omega^{ab} = \Omega^{ab}_\mu dx^\mu$ (it is customary to denote the spin connection by ω^{ab}_μ instead).

For example, when M is a bivector index and N is a scalar index, there is a curvature term of the form

$$\mathcal{R}^{a_1a_2}_{\mu_1\mu_2\ \mathbf{0}}\ =\ \partial_{[\mu_1\mu_2}\ \Omega^{a_1a_2}_{\mathbf{0}]}\ +\ \Omega^m_{\mu_1\mu_2}\ \Omega^r_{\mathbf{0}}\ < [\gamma_m, \gamma_r]\ \gamma^{a_1a_2}\ > +$$
$$\Omega^{mn}_{\mu_1\mu_2}\ \Omega^{rs}_{\mathbf{0}}\ < [\gamma_{mn}, \gamma_{rs}]\ \gamma^{a_1a_2}\ > + \Omega^{mnp}_{\mu_1\mu_2}\ \Omega^{rst}_{\mathbf{0}}\ < [\gamma_{mnp}, \gamma_{rst}]\ \gamma^{a_1a_2}\ > +$$
$$\Omega^{mnpq}_{\mu_1\mu_2}\ \Omega^{rstu}_{\mathbf{0}}\ < [\gamma_{mnpq}, \gamma_{rstu}]\ \gamma^{a_1a_2}\ > + \Omega^{mnpqk}_{\mu_1\mu_2}\ \Omega^{rstuv}_{\mathbf{0}}\ < [\gamma_{mnpqk}, \gamma_{rstuv}]\ \gamma^{a_1a_2}\ > . \tag{2.21}$$

where the *bivector* derivative in C-space is

$$\partial_{\mu_1\mu_2}\ =\ \frac{\partial}{\partial x^{\mu_1\mu_2}} \tag{2.22}$$

The components $\mathcal{R}^{a_1a_2}_{\mu_1\mu_2\ \mathbf{0}}$ must *not* be confused with the components of the *modified* curvature tensor

$$\mathcal{R}^{a_1a_2}_{\mu\ \nu}\ =\ \partial_{[\mu}\ \Omega^{a_1a_2}_{\nu]} + \Omega^m_\mu\ \Omega^r_\nu\ < [\gamma_m, \gamma_r]\ \gamma^{a_1a_2}\ > + \Omega^{mn}_\mu\ \Omega^{rs}_\nu\ < [\gamma_{mn}, \gamma_{rs}]\ \gamma^{a_1a_2}\ > +$$
$$\Omega^{mnp}_\mu\ \Omega^{rst}_\nu\ < [\gamma_{mnp}, \gamma_{rst}]\ \gamma^{a_1a_2}\ > + \Omega^{mnpq}_\mu\ \Omega^{rstu}_\nu\ < [\gamma_{mnpq}, \gamma_{rstu}]\ \gamma^{a_1a_2}\ > +$$
$$\Omega^{mnpqk}_\mu\ \Omega^{rstuv}_\nu\ < [\gamma_{mnpqk}, \gamma_{rstuv}]\ \gamma^{a_1a_2}\ > . \tag{2.23}$$

The standard curvature tensor is given by

$$R^{a_1a_2}_{\mu\ \nu}\ =\ \partial_{[\mu}\ \Omega^{a_1a_2}_{\nu]}\ +\ \Omega^{mn}_\mu\ \Omega^{rs}_\nu\ < [\gamma_{mn}, \gamma_{rs}]\ \gamma^{a_1a_2}\ > . \tag{2.24}$$

which clearly differs from the modified expression in (2.23).

Since the indices m, n, r, s in general run from $1, 2, 3, 4, 5$ the standard curvature two-form becomes

$$R_{\mu\nu}^{a_1 a_2} \ dx^\mu \ \wedge \ dx^\nu \ = \ \mathbf{d}\Omega^{a_1 a_2} \ + \ \Omega^{a_1}_{\ c} \ \wedge \ \Omega^{c a_2} \ - \eta_{55} \ \Omega^{a_1 5} \ \wedge \ \Omega^{a_2 5} \ =$$

$$\mathbf{d}\Omega^{a_1 a_2} \ + \ \Omega^{a_1}_{\ c} \ \wedge \ \Omega^{c a_2} \ - \eta_{55} \ \frac{1}{l^2} V^{a_1} \ \wedge \ V^{a_2}; \quad \Omega^{a5} = \frac{1}{l} V^a \tag{2.25}$$

where the vielbein one-form is $V^a = V^a_\mu dx^\mu$. In the $l \to \infty$ limit the last terms $\frac{1}{l^2} V^{a_1} \wedge V^{a_2}$ in (2.25) decouple and one recovers the standard Riemmanian curvature two-form in terms of the spin connection one form $\omega^{a_1 a_2} = \omega_\mu^{a_1 a_2} dx^\mu$ and the exterior derivative operator $\mathbf{d} = dx^\mu \partial_\mu$. From (2.25) one infers that a vacuum solution $R_{\mu\nu}^{a_1 a_2} = 0$ in de Sitter/ Anti de Sitter gravity leads to the relation

$$R^{a_1 a_2}(\omega) \ \equiv \ \mathbf{d}\omega^{a_1 a_2} \ + \ \omega^{a_1}_{\ c} \ \wedge \ \omega^{c a_2} \ = \ \frac{1}{l^2} \ \eta_{55} \ V^{a_1} \ \wedge \ V^{a_2} \tag{2.26}$$

which is tantamount to having a constant Riemannian scalar curvature in $4D$ $R(\omega) = \pm(12/l^2)$ and a cosmological constant $\Lambda = \pm(3/l^2)$; the positive (negative) sign corresponds to de Sitter (anti de Sitter space) respectively ; i.e. the de Sitter/ Anti de Sitter gravitational *vacuum* solutions are solutions of the Einstein field equations *with* a non-vanishing cosmological constant.

A different approach to the cosmological constant problem can be taken as follows. The *modified* curvature tensor in (2.23) is

$$\mathcal{R}_{\mu \ \nu}^{a_1 a_2} \ = \ R_{\mu\nu}^{a_1 a_2} \ + \ extra \ terms \ =$$

$$\mathbf{d}\omega^{a_1 a_2} \ + \ \omega^{a_1}_{\ c} \ \wedge \ \omega^{c a_2} \ - \eta_{55} \ \frac{1}{l^2} V^{a_1} \ \wedge \ V^{a_2} \ + \ extra \ terms \tag{2.27}$$

vacuum solutions $\mathcal{R}_{\mu\nu}^{a_1 a_2} = 0$ imply

$$\mathbf{d}\omega^{a_1 a_2} \ + \ \omega^{a_1}_{\ c} \ \wedge \ \omega^{c a_2} \ = \ \frac{1}{l^2} \ \eta_{55} \ V^{a_1} \ \wedge \ V^{a_2} \ - \ extra \ terms. \tag{2.28}$$

Consequently, as a result of the *extra* terms in the right hand side of (2.28) obtained from the extra terms in the definition of $\mathcal{R}_{\mu\nu}^{a_1 a_2}$ in (2.23), it could be possible to have a cancellation of a cosmological constant term associated to a very large vacuum energy density $\rho \sim (L_{Planck})^{-4}$; i.e. one would have an $effective$ zero value of the cosmological constant.

For instance, one could have a cancellation (after neglecting the terms of higher order rank in eq-(2.28)) to the contribution of the cosmological constant as follows

$$\Omega^m_\mu \ \Omega^n_\nu \ < \ [\gamma_m, \gamma_r] \ \gamma^{a_1 a_2} \ > \ + \ \Omega^{m5}_\mu \ \Omega^{r5}_\nu \ < \ [\gamma_{m5}, \gamma_{r5}] \ \gamma^{a_1 a_2} \ > \ = \ 0 \ \Rightarrow$$

$$\Omega^{a_1} \ \wedge \ \Omega^{a_2} \ - \eta_{55} \ \Omega^{a_1 5} \ \wedge \ \Omega^{a_2 5} = \ 0. \tag{2.29a}$$

Since the $Cl(3,2)$ algebra corresponds to the Anti de Sitter algebra $SO(3,2)$ case one has

$$\eta_{55} = -1 \Rightarrow \frac{V^a}{l} = \Omega_\mu^{a5} = \pm i\ \Omega_\mu^a \qquad (2.29b)$$

Hence, one can attain a cancellation of a very large cosmological constant term in (2.29) if $\Omega_\mu^{a5} = \pm i\ \Omega_\mu^a$. In the de Sitter case, $\eta_{55} = 1$ and one would have instead the condition $\Omega_\mu^{a5} = \pm \Omega_\mu^a$. Having an imaginary value for Ω_μ^a in the Anti de Sitter case fits into a gravitational theory involving a complex Hermitian metric $G_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ which is associated to a complex tetrad $E_\mu^a = \frac{1}{\sqrt{2}}(\tilde{e}_\mu^a + i \tilde{f}_\mu^a)$ such that $G_{\mu\nu} = (E_\mu^a)^* E_\nu^b \eta_{ab}$ and the fields are constrained to obey $\tilde{e}_\mu^a = V_\mu^a; i\tilde{f}_\mu^a = i V_\mu^a = \mp l\ \Omega_\mu^a$. For further details on complex metrics (gravity) in connection to Born's reciprocity principle of relativity [22], [23] involving a maximal speed and maximum proper force see [24] and references therein.

It is desirable to solve the full-fledge field equations in C-space and afterwards verify whether or not such condition (2.29) is consistent with the solutions to the full set of field equations. Most likely, it would be necessary to include all the higher order rank terms in eq-(2.28). In order to solve the C-space gravitational field equations one must evaluate all of the remaining components of the $Cl(3,2)$ curvature (field strength) $\mathcal{R}_{MN}^A$ and torsion $\mathcal{T}_{MN}^A$ in C-space, where M, N are poly-vector valued coordinate indices $X^M = s, x^\mu, x^{\mu\nu}, ..., x^{\mu\nu\rho\tau}$ associated with the C-space corresponding to the $Cl(3,1)$ four-dim spacetime algebra. These expressions are very complicated. Once the expressions for $\mathcal{R}_{MN}^A, \mathcal{T}_{MN}^A$ are known one can construct many actions in C-space that are invariant under the internal $Cl(3,2)$ gauge transformations as well as invariant under the $Cl(3,1)$ transformations associated with the poly-vector valued coordinates X^M of the underlying C-space base manifold. The integration measure in C-space is

$$\mathbf{DX} = ds\ (\prod dx^\mu)\ (\prod dx^{\mu\nu})\ (\prod dx^{\mu\nu\rho})\ dx^{\mu\nu\rho\tau} \qquad (2.30)$$

a quadratic curvature/torsion invariant action in C-space, up to numerical factors required to match units, is given by

$$S = \int \mathbf{DX} \sqrt{|det\ \mathbf{G}|}\ \delta_{AB}\ (\ \mathcal{R}_{M_1N_1}^A\ \mathcal{R}_{M_2N_2}^B\ +\ \mathcal{T}_{M_1N_1}^A\ \mathcal{T}_{M_2N_2}^B\ )\ G^{M_1M_2}\ G^{N_1N_2}. \qquad (2.31)$$

The C-space metric G^{MN} has for components

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n} = g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ signed\ permutations \qquad (2.32a)$$

The components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n \times n$ matrix whose entries are $g^{\mu_I\nu_J}$ as follows

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n} = \frac{1}{n!}\ \epsilon_{i_1i_2.....i_n}\ \epsilon_{j_1j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \qquad (2.32b)$$

and the range of indices is $i_1, i_2, ..., i_n \subset I = 1, 2, ..., D$ and $j_1, j_2, ..., j_n \subset J = 1, 2, ..., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field). The expression for *det* $\mathbf{G}$ involves the product of the determinants associated with $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ for $n = 0, 1, 2, ..., D$.

To simplify matters, we may just concentrate in the ordinary $4D$ spacetime actions comprised of the vector coordinates x^μ. Even in this case, one finds clear *modifications* to the standard gravitational actions due to the $Cl(3, 2)$ algebraic structure. One can introduce an $SO(3, 2)$-valued scalar multiplet $\phi^1, \phi^2, ..., \phi^5$ and construct an $SO(3, 2)$ invariant action of the form

$$S = \int_M d^4x \ \left(\phi^5 \ \mathcal{R}^{ab}_{\mu\nu} \ \mathcal{R}^{cd}_{\rho\sigma} \ + \ \phi^a \ \mathcal{R}^{bc}_{\mu\nu} \ \mathcal{R}^{d5}_{\rho\sigma} \ + \ .... \right) \ \epsilon_{abcd5} \ \epsilon^{\mu\nu\rho\sigma}. \quad (2.33)$$

as described above the *modified* curvature two-form $\mathcal{R}^{ab}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ is given by the standard expression $R^{ab}_{\mu\nu}(\omega)\ dx^\mu \wedge dx^\nu + \frac{1}{l^2} V^a_\mu dx^\mu \wedge V^b_\nu dx^\nu$ *plus* the addition of many *extra* terms as shown in (2.23). Also the modified torsion $\mathcal{R}^{a5}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ in (2.18) is given by the standard torsion expression *plus* extra terms. Therefore, by a simple inspection, the action (2.33) after setting $\phi^a = 0, \phi^5 = \phi^5_o = constant$ which breaks the $SO(3, 2)$ symmetry down to the Lorentz group symmetry $SO(3, 1)$, contains many *more* terms than the Macdowell-Mansouri-Chamseddine-West gravitational action given by (after supressing spacetime indices for convenience)

$$S \ = \ \phi^5_o \int d^4x \ (\ R^{ab}(\omega) + \frac{1}{l^2} \ V^a \wedge V^b \ ) \wedge (\ R^{cd}(\omega) + \frac{1}{l^2} \ V^c \wedge V^d \ ) \ \epsilon_{abcd}. \quad (2.34)$$

it is comprised of the topological invariant Gauss-Bonnet term $R^{ab}(\omega) \wedge R^{cd}(\omega)\epsilon_{abcd}$; the Einstein-Hilbert term $\frac{1}{l^2} R^{ab}(\omega) \wedge V^c \wedge V^d \epsilon_{abcd}$, and the cosmological constant term $\frac{1}{l^4} V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}$.

A quadratic $Cl(3, 2)$ gauge invariant action in a $4D$ spacetime involving the modified curvature $\mathcal{R}^A_{\mu\nu}$ and torsion terms $\mathcal{T}^A_{\mu\nu}$ in eqs-(2.18, 2.23), is given by

$$\int \ d^4x \ \sqrt{|g|} \ [\ (\mathcal{R}^{\mathbf{0}}_{\mu\nu})^2 + (\mathcal{R}^a_{\mu\nu})^2 + (\mathcal{R}^{a_1a_2}_{\mu\nu})^2 + \ .... \ (\mathcal{R}^{a_1a_2a_3a_4}_{\mu\nu})^2 \ +$$

$$(\mathcal{R}^{\mathbf{5}}_{\mu\nu})^2 + (\mathcal{R}^{a5}_{\mu\nu})^2 + (\mathcal{R}^{a_1a_5}_{\mu\nu})^2 + \ .... \ (\mathcal{R}^{a_1a_2a_35}_{\mu\nu})^2 + (\mathcal{R}^{a_1a_2a_3a_45}_{\mu\nu})^2 \ ] \quad (2.35)$$

The modifications to the ordinary scalar Riemmanian curvature $R(\omega)$ is given in terms of the inverse vielbein V^μ_a by the expression $\mathcal{R}^{a_1a_2}_{\mu\nu} V^{[\mu}_{[a_1} V^{\nu]}_{a_2]}$ which is comprised of $R(\omega)$, plus the cosmological constant term , plus the extra terms stemming from the additional connection pieces in (2.23)

$$\Omega^{a_1} \wedge \Omega^{a_2}, \ \ \Omega^{a_1}{}_{b_1b_2} \wedge \Omega^{b_1b_2a_2}, \ ...., \ \Omega^{a_1}{}_{b_1b_2b_3b_4} \wedge \Omega^{b_1b_2b_3b_4a_2} \quad (2.36)$$

one of many $SO(3,2)$ invariant actions in ordinary spacetime, linear in the curvature is

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} \mathcal{R}^{a_1 a_2}_{\mu\nu} V^{[\mu}_{[a_1} V^{\nu]}_{a_2]}; \quad g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab}, \quad |g| = |det\ g_{\mu\nu}|. \tag{2.37}$$

where $\kappa^2 = 8\pi G_N$, G_N is the Newtonian gravitational constant and the components of the curvature two-form are antisymmetric under the exchange of indices by construction $\mathcal{R}^{a_1a_2}_{\mu\nu} = -\mathcal{R}^{a_1a_2}_{\nu\mu}$, $\mathcal{R}^{a_1a_2}_{\mu\nu} = -\mathcal{R}^{a_2a_1}_{\mu\nu}$. The action (2.37) contains clear $modifications$ to the Einstein-Hilbert action with a cosmological constant due to the extra terms (2.36) stemming from the higher rank connection elements.

The components of the gauge connection $\Omega^{a5}_{\mu_1\mu_2}$, $\Omega^{ab}_{\mu_1\mu_2}$ in C-space must not be identified in general with the ordinary torsion and curvature two-form in Riemann-Cartan spaces, despite the correspondence

$$\Omega^{a5}_{\mu_1\mu_2}\ dx^{\mu_1\mu_2} \longleftrightarrow T^a_{\mu_1\mu_2}\ dx^{\mu_1} \wedge dx^{\mu_2} \tag{2.38}$$

$$\Omega^{a_1a_2}_{\mu_1\mu_2}\ dx^{\mu_1\mu_2} \longleftrightarrow R^{a_1a_2}_{\mu_1\mu_2}\ dx^{\mu_1} \wedge dx^{\mu_2} \tag{2.39}$$

where $dx^{\mu_1\mu_2}$ is a *bivector* differential form involving *areas* in C-space. If an identification is made in eqs-(2.38, 2.39) the generalized gravitational action in C-space (given by the Clifford space scalar-curvature version of the Einstein-Hilbert Lagrangian) can be decomposed into sums of terms involving higher powers of the ordinary curvature and torsion [1]. Such actions are comprised of higher derivatives. The Lanczos-Lovelock gravitational actions are based on higher powers of the curvature tensor but with the key feature that the field equations do not contain terms of higher derivatives than order two for the metric tensor. Gravitational actions in Noncommutative spaces can also be constructed based on star products and the results obtained for poly-vector valued gauge field theories in Noncommutative C-spaces [25]. Noncommutative Clifford-space gravity as a poly-vector-valued gauge theory of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras. Generalized poly-vector-valued supersymmetry algebras in C spaces based on antisymmetric tensor-spinorial coordinates have been recently studied in [26]. These novel algebraic structures and the study of generalized super-gravitational theories in supersymmetric Clifford spaces deserve further investigation.

One of the most salient features of the $Cl(3,2)$ algebra modifications to gravity is the very plausible cancellation mechanism of a very large vacuum energy density as described in eqs-(2.27-2.29). This procedure is very different than the other approaches to the resolution to the cosmological constant problem based on scaling/conformal symmetry [28], for example.

3 Yang-Mills, Fermion Masses and Unification

It was recently shown [21] how an unification of Conformal Gravity and a $U(4) \times U(4)$ Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have been studied in [1], [10] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate. A Clifford geometric basis of the standard model has been advanced by [29]. In this last section we describe how the 16 fermions in each family and their masses can be accommodated within a $Cl(4)$ gauge field theory and how the Higgs fields admit a natural C-space interpretation that differs from the one in the Chamseddine-Connes spectral action model of Noncommutative geometry [37].

The 16 fermions (quarks and leptons) of the first generation can be arranged into the 16 entries of the 4×4 matrix associated with the $A = 1, 2, 3, ..., 16$ degrees of freedom corresponding to the $Cl(4)$ gauge algebra as follows

$$\Psi^A_\alpha\,(\Gamma_A)^{mn} \equiv \begin{pmatrix} \nu_e & u_r & u_b & u_g \\ e & d_r & d_b & d_g \\ \nu^c_e & u^c_r & u^c_b & u^c_g \\ e^c & d^c_r & d^c_b & d^c_g \end{pmatrix}, \quad \bar\Psi^A_\alpha\,(\Gamma_A)^{mn} \equiv \begin{pmatrix} \bar\nu_e & \bar u_r & \bar u_b & \bar u_g \\ \bar e & \bar d_r & \bar d_b & \bar d_g \\ \bar\nu^c_e & \bar u^c_r & \bar u^c_b & \bar u^c_g \\ \bar e^c & \bar d^c_r & \bar d^c_b & \bar d^c_g \end{pmatrix} \tag{3.1}$$

where we have omitted the spacetime spinorial indices $\alpha = 1, 2, 3, 4$ in each one of the entries of the above 4×4 matrices whose row and column indices are $m, n = 1, 2, 3, 4$. In particular, e, ν_e denote the electron and its neutrino. The subscripts r, b, g denote the red, blue, green color of the up and down quarks, u, d. The superscript c denotes their anti-particles. The Dirac adjoint of each spacetime spinor entry inside the second 4×4 matrix is denoted by $\bar e, \bar\nu_e, \bar u^r, ...$ and is defined as usual $\bar\Psi_\alpha = \Psi^\dagger_\beta (\Gamma_o)^\beta_\alpha$. One must not confuse the gamma matrices $(\Gamma_\mu)^{\alpha\beta}$ associated with the spacetime Dirac $Cl(3, 1)$ algebra and the internal $Cl(4)$ gauge algebra matrices $(\Gamma_a)^{mn}, (\Gamma_{a_1 a_2})^{mn}, ...;\ a = 1, 2, 3, 4$.

By writing $\Psi^{mn}_\alpha = \Psi^C_\alpha (\Gamma_C)^{mn}$, $\bar\Psi^{mn}_\alpha = \bar\Psi^A_\alpha (\Gamma_A)^{mn}$, and attaching an extra index $i = 1, 2, 3, ..., n_f$ indicating the fermion family, the fermionic matter kinetic terms is given by the expression involving a *trace* over the 4×4 matrix indices as

$$\mathcal{L}_m = \sum_{i=1}^{n_f} \bar\Psi^{mn}_{\alpha i}\, \Gamma^\mu_{\alpha\beta}\, (\ \delta^{np}\, i\, \partial_\mu + g \mathcal{A}^{np}_\mu\ )\, \Psi^{pm}_{\beta i};\quad \mathcal{A}^{np}_\mu = \mathcal{A}^B_\mu (\Gamma_B)^{np}. \tag{3.2}$$

and can be rewritten as

$$\mathcal{L}_m = \sum_{i=1}^{n_f} \bar\Psi^A_{\alpha i}\, \Gamma^\mu_{\alpha\beta}\, \delta_{AC}\, (i\, \partial_\mu \Psi^C_{\beta i}) + \sum_{i=1}^{n_f} g\, \bar\Psi^A_{\alpha i}\, \Gamma^\mu_{\alpha\beta}\, \mathcal{A}^B_\mu\, \Psi^C_{\beta i} < \Gamma_A \Gamma_B \Gamma_C > =$$

$$\sum_{i=1}^{n_f} \bar{\Psi}^A_{\alpha i} \, \Gamma^{\mu}_{\alpha\beta} \, (\, \delta_{AC} \, i \, \partial_\mu \, + \, g \, h_{ABC} \, \mathcal{A}^B_\mu \,) \, \Psi^C_{\beta i}. \tag{3.3}$$

where the indices $i = 1, 2, 3, ...n_f$ extend over the number of generations (families) and $A, B, C = 1, 2, 3, ..., 16$. g is the coupling constant. h_{ABC} is the scalar part of the Clifford product $< \Gamma_A \Gamma_B \Gamma_C >$ which can be written in terms of the (anti) commutators structure constants of the $Cl(4)$ gauge algebra as follows

$$h_{ABC} \; = \; \frac{1}{2} \, (f_{ABC} \, + \, d_{ABC}); \;\; [\Gamma_A, \Gamma_B] = f_{ABC} \, \Gamma^C; \;\; \{\Gamma_A, \Gamma_B\} \; = \; d_{ABC} \, \Gamma^C \tag{3.4}$$

Given the definition

$$\Psi^{mn}_\alpha \; = \; \Psi^A_\alpha (\Gamma_A)^{mn} \; \Rightarrow \; \Psi^A_\alpha \; = \; \Psi^{mn}_\alpha \, (\Gamma^A)_{nm}. \tag{3.5}$$

one can infer that each of the quantities Ψ^A_α for $A, 1, 2, 3, ..., 16$ is a *linear* superposition of *all* the 16 fermions in each single family. Hence, a $Cl(4)$ gauge invariant mass term corresponding to a *degenerate* mass M for all members of a single fermion family can be written as the trace over the 4×4 matrix indices as

$$M \, \delta^{\alpha\beta} \, \bar{\Psi}^{nm}_\alpha \, \Psi^{mn}_\beta \; = \; M \, (\, \bar{\Psi}_e \Psi_e + \bar{\Psi}_{\nu_e} \Psi_{\nu_e} + \bar{\Psi}_{u^r} \Psi_{u^r} + \bar{\Psi}_{d^r} \Psi_{d^r} + ... \,) \; =$$

$$M \, \delta^{\alpha\beta} \, \bar{\Psi}^A_\alpha \, \Psi^C_\beta \; < \; \Gamma_A \, \Gamma_C \; > \; = \; M \, \delta^{\alpha\beta} \, \delta_{AC} \, \bar{\Psi}^A_\alpha \, \Psi^C_\beta \tag{3.6}$$

where the scalar part of the Clifford product is $< \Gamma_A \Gamma_C > \; = \; \delta_{AC}$. The mass degeneracy can be lifted by introducing the interaction terms involving the (complex) Higgs scalars

$$\delta^{\alpha\beta} \, h_{ABC} \, \bar{\Psi}^A_\alpha \, \Phi^B \, \Psi^C_\beta \tag{3.7}$$

such that the vacuum expectation values of the Higgs scalars $< \Phi^B >_{vev}$ valued in the adjoint representation of the $Cl(4)$ gauge group will break the $Cl(4)$ symmetry and lead to a mass splitting $M + \lambda_1, M + \lambda_2, ..., M + \lambda_{16}$. The λ's are the eigenvalues of the 16×16 mass matrix $\mathcal{M}_{AC}$ appearing in

$$\delta^{\alpha\beta} \, \mathcal{M}_{AC} \, \bar{\Psi}^A_\alpha \, \Psi^C_\beta \tag{3.8}$$

and which is defined as $\mathcal{M}_{AC} = h_{ABC} \; < \Phi^B >_{vev}$.

A Hermitian 16×16 matrix $\mathcal{M}_{AC}$ has real eigenvalues and can be diagonalized by a unitary matrix. A priori there is *no* reason why the matrix $\mathcal{M}_{AC}$ is Hermitian unless one chooses judiciously the vacuum expectation values of the $< \Phi^B >_{vev}$ in the definition $\mathcal{M}_{AC} = h_{ABC} \; < \Phi^B >_{vev}$. If one has an initial massless family $M = 0$, by judiciously choosing the 16 parameters associated with the vevs $< \Phi^B >_{vev}$, $B = 1, 2, 3, ..., 16$, in order to render a Hermitian matrix $\mathcal{M}_{AC}$ one may find 16 real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_{16}$ to coincide with the fermion masses of $e, \nu_e, u^{rbg}, d^{rbg}$ and their anti-particles. To match the masses requires a Renormalization group flow of the values of the observed fermion masses, at a given scale, to the scale at which the $Cl(4)$ symmetry is broken. Some of the eigenvalues have to be degenerate since the masses of the red, blue, green up quark and anti-quark are equal. Similarly, the masses of the red, blue, green down quark and anti-quark are equal. The electron and positron mass are also equal. The neutrino is considered also to be masive. At the moment we will not introduce Majorana neutrino mass terms that lead to a very small neutrino mass for the left handed neutrino ν_{eL} via the see-saw mechanism.

Yukawa couplings among all generations of the form

$$\delta^{\alpha\beta}\ Y^{ij}\ h_{ABC}\ \bar{\Psi}^{A}_{\alpha i}\ \Phi^{B}\ \Psi^{C}_{\beta j}; \quad i,j = 1,2,.....,n_f \tag{3.9}$$

will also lead to fermionic mass terms for all fermion generations after the symmetry breaking $< \Phi^B >_{vev} \neq 0$ and a diagonalization procedure similar to the construction of the CKM quark matrix in the standard model.

Next we are going to discuss the relation to the Pati-Salam model [30]. In section **2** the $Cl(3,2)$ gauge field theory model of gravity was constructed. The $Cl(5,0)$ algebra is isomorphic to the direct sum of the algebras $Cl(4,0) \oplus Cl(4,0)$, and which in turn, is isomorphic to $M(2,\mathbf{H}) \oplus M(2,\mathbf{H})$, where $M(2,\mathbf{H})$ is the matrix algebra of the 2×2 matrices with quaternionic entries. The group $Cl(4)\times Cl(4)$ admits a correspondence with $U(4)\times U(4)$ as shown in [21], and each factor $U(4) = SU(4)\times U(1)$. A unified theory of the strong, weak and electromagnetic interactions based on the flavor-color symmetric group $SU(4)_C \times SU(4)_F$ was advanced by Pati and Salam [30]. Another version of the Pati-Salam model is based on the group $SU(4)_C \times SU(2)_L \times SU(2)_R$. Therefore, the $Cl(4)$ algebra is relevant in as much as it is connected to the $SU(4)\times U(1)$ algebra because the algebra $SU(4)$ is a key ingredient in the Pati-Salam model.

The procedure of the symmetry breaking patterns is very elaborate in general [33]. For instance, the symmetry breaking patterns for $SU(N)$ gauge theories with Higgs scalars in totally antisymmetric and symmetric representations of degree k were studied by [34] by solving the extremum conditions of the $SU(N)$ invariant Higgs potential for the fields $H_{a_1 a_2 ....a_k}$. Antisymmetric tensors $H_{[a_1 a_2 ....a_k]}$ are the ones appearing in the components of the Clifford poly-vector Φ^A associated with the Clifford gauge group $Cl(4)$.

The Pati-Salam $SU(4)\times SU(2)_L \times SU(2)_R$ group arises from the symmetry breaking of *one* of the $SU(4)$ factors in $SU(4)\times SU(4)$ given by $SU(4) \rightarrow SU(2)_L \times SU(2)_R \times U(1)_Z$, see [31], [36] and references therein. This requires taking the following vacuum expectation value (VEV) of the Higgs scalar

$$< \Phi > \ \equiv \ v_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.10}$$

Taking the VEV of the other Higgs scalar

$$< \tilde{\Phi} > \equiv v_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \quad (3.11)$$

leads to a breaking of $SU(4) \to SU(3)_c \times U(1)_{B-L}$. Therefore, an overall breaking of $SU(4) \times SU(4)$ contains the Patti-Salam (PS) model in the intermediate stage as follows

$$SU(4) \times SU(4) \to [SU(4) \times SU(2)_L \times SU(2)_R]_{PS} \times U(1)_Z \to$$
$$SU(3)_c \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R \times U(1)_Z. \quad (3.12)$$

The Higgs Potential $V(\Phi, \tilde{\Phi})$ involving quadratic and quartic powers of the fields is of the form

$$V = -m_1^2 \, Tr(\Phi^2) + \lambda_1 \, [Tr(\Phi^2)]^2 + \lambda_2 \, Tr(\Phi^4) - m_2^2 \, Tr(\tilde{\Phi}^2) + \lambda_3 \, [Tr(\tilde{\Phi}^2)]^2 +$$
$$\lambda_4 \, Tr(\tilde{\Phi}^4) + \lambda_5 \, Tr(\Phi^2 \tilde{\Phi}^2) + \lambda_6 \, Tr(\Phi \tilde{\Phi} \Phi \tilde{\Phi}) \, . \quad (3.13)$$

A further symmetry breaking

$$U(1)_{B-L} \times SU(2)_R \times U(1)_Z \to U(1)_Y. \quad (3.14)$$

requires additional Higgs fields leading to the Standard Model

$$SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{EM}. \quad (3.15)$$

There is another symmetry-breaking branch that leads to the Standard Model and which does not contain the PS model. This requires breaking one of the $SU(4)$ factors as

$$SU(4) \times SU(4) \to SU(3)_c \times SU(4) \times U(1)_{B-L}. \quad (2.34)$$

leading to a partial unification model based on $SU(4) \times U(1)_{B-L}$. which can be broken down to the minimal left-right model via the Higgs mechanism [31], [36].

To finalize we show how the Higgs fields admit a natural C-space interpretation that differs from the one in the Chamseddine-Connes spectral action model of Noncommutative geometry [37]. The C-space analog of the fermionic kinetic terms (3.3) is

$$\sum_{i=1}^{n_f} \bar{\Psi}^A_{\alpha i} \, \Gamma^M_{\alpha\beta} \, (\, \delta_{AC} \, i \, \partial_M + g \, h_{ABC} \, \mathcal{A}^B_M \,) \, \Psi^C_{\beta i}. \quad (3.16)$$

where ∂_M is the derivative with respect to a C-space poly-vector valued index $(\partial/\partial s)$, $(\partial/\partial x^\mu)$, $(\partial/\partial x^{\mu\nu})$, ... and $\Gamma^M = \mathbf{1}, \Gamma^\mu, \Gamma^{\mu\nu}$, ... are the generators corresponding to the $Cl(3,1)$ spacetime algebra.

One may notice that the Yukawa self-coupling terms (within each family) furnishing mass terms for the quarks and leptons are contained in the $h_{ABC}\bar{\Psi}^A \mathcal{A}_0^B \Psi^C, h_{ABC}\bar{\Psi}^A \mathcal{A}^B_{\mu_1\mu_2....\mu_4}\Psi^C$ pieces (after taking the VEV of the Higgs scalars) associated to the C-space fermionic kinetic terms $\bar{\Psi}_A \Gamma^M (D_M)^{AC}\Psi_C$. This is due to the fact that two sets of Higgs scalar fields can be identified with the $Cl(3,1)$ scalar part $\Phi^A = \mathcal{A}_0^A$ and pseudo-scalar part $\epsilon_{\mu\nu\rho\tau}\tilde{\Phi}^A = \mathcal{A}^A_{\mu\nu\rho\tau} = \mathcal{A}_5^A$, respectively. The kinetic terms for the Higgs fields $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ and $(D_\mu\tilde{\Phi})^\dagger(D^\mu\tilde{\Phi})$ are contained in the $F^A_{0N}F_A^{0N}$ and $F^A_{5N}F_A^{5N}$ components, respectively, associated to the $Cl(4)$ gauge fields kinetic $F^A_{MN}F_A^{MN}$ terms in C-space. As usual, the $Cl(4)$ field strength in C-space is defined as

$$F^A_{MN} = \partial_{[M}\,\mathcal{A}^A_{N]} + \mathcal{A}^B_M\,\mathcal{A}^C_N < [\Gamma_B,\,\Gamma_C]\,\Gamma^A > = \partial_{[M}\,\mathcal{A}^A_{N]} + \mathcal{A}^B_M\,\mathcal{A}^C_N\,f^A_{BC}. \tag{3.17}$$

Inserting the VEV of the Higgs scalars into their kinetic terms, after redefining the fields such that the new fields have zero VEV, yields the mass terms from the gauge fields associated to the broken gauge symmetries.

To finalize, there are models were the 16 fermions of a single family fit into the **16** dimensional *chiral* spinor representation of the $SO(10)$ gauge unification group. All the 16 fermions can be assembled into a column of 16 entries. The chirality operator in the internal group space $SO(10)$ (associated with the $Cl(10)$ algebra) must not be confused with the usual Dirac chirality operator γ_5 of the $Cl(3,1)$ spacetime algebra and which implies definite parity (left-handed or right-handed currents) in weak interactions [32]. A thorough study of the symmetry breaking patterns of $SO(10)$ and its descent into the $SU(5)$ group of Georgi-Glashow; the Pati-Salam $SU(4)_c\times SU(2)_L\times SU(2)_R$ and other groups was performed by [32]. In our Clifford algebra model based on $Cl(4)$, the 16 fermions of a single family are assembled into the 4×4 matrix entries as shown in (3.1). More work remains to be done to verify whether or not the Clifford algebraic approach to unification is feasible.

APPENDIX

We begin firstly by writing the commutators $[\Gamma_A,\Gamma_B]$. For $pq = odd$ one has [35]

$$[\,\gamma_{b_1b_2.....b_p},\,\gamma^{a_1a_2......a_q}\,] = 2\gamma^{a_1a_2......a_q}_{b_1b_2.....b_p} -$$
$$\frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[a_1a_2}_{[b_1b_2}\,\gamma^{a_3....a_q]}_{b_3.....b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!}\,\delta^{[a_1....a_4}_{[b_1....b_4}\,\gamma^{a_5....a_q]}_{b_5.....b_p]} - ...... \tag{A.1}$$

for $pq = even$ one has

$$[\,\gamma_{b_1b_2.....b_p},\,\gamma^{a_1a_2......a_q}\,] = -\frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\,\delta^{[a_1}_{[b_1}\,\gamma^{a_2a_3....a_q]}_{b_2b_3.....b_p]} -$$
$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\,\delta^{[a_1....a_3}_{[b_1....b_3}\,\gamma^{a_4....a_q]}_{b_4.....b_p]} + ... \tag{A.2}$$

The anti-commutators for $pq = even$ are

$$\{ \gamma_{b_1 b_2 \ldots\ldots b_p}, \gamma^{a_1 a_2 \ldots\ldots a_q} \} = 2\gamma^{a_1 a_2 \ldots\ldots a_q}_{b_1 b_2 \ldots\ldots b_p} -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3 \ldots\ldots a_q]}_{b_3 \ldots\ldots b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!} \delta^{[a_1 \ldots\ldots a_4}_{[b_1 \ldots\ldots b_4} \gamma^{a_5 \ldots\ldots a_q]}_{b_5 \ldots\ldots b_p]} - \ldots\ldots \tag{A.3}$$

and the anti-commutators for $pq = odd$ are

$$\{ \gamma_{b_1 b_2 \ldots\ldots b_p}, \gamma^{a_1 a_2 \ldots\ldots a_q} \} = -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 \ldots\ldots a_q]}_{b_2 b_3 \ldots\ldots b_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[a_1 \ldots\ldots a_3}_{[b_1 \ldots\ldots b_3} \gamma^{a_4 \ldots\ldots a_q]}_{b_4 \ldots\ldots b_p]} + \ldots\ldots \tag{A.4}$$

For instance,

$$[\gamma_b, \gamma^a] = 2\gamma^a_b; \quad [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] = -8 \, \delta^{[a_1}_{[b_1} \gamma^{a_2]}_{b_2]}. \tag{A.5}$$

$$[\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] = 2 \, \gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} - 36 \, \delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3]}_{b_3]}. \tag{A.6}$$

$$[\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] = -32 \, \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} + 192 \, \delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3} \gamma^{a_4]}_{b_4]}. \tag{A.7}$$

$$[\gamma_{b_1 b_2 b_3 b_4 b_5}, \gamma^{a_1 a_2 a_3 a_4 a_5}] = 2 \, \gamma^{a_1 a_2 a_3 a_4 a_5}_{b_1 b_2 b_3 b_4 b_5} - 400 \, \delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3 a_4 a_5]}_{b_3 b_4 b_5]} + 1200 \, \delta^{[a_1 a_2 a_3 a_4}_{[b_1 b_2 b_3 b_4} \gamma^{a_5]}_{b_5]}. \tag{A.8}$$

$$[\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2}] = 12 \, \delta^{[a_1}_{[b_1} \gamma^{a_2]}_{b_2 b_3]}. \tag{A.9}$$

$$[\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3}] = -24 \, \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3]}_{b_2 b_3 b_4]} + 48 \, \delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3} \gamma_{b_4]} \tag{A.10}$$

etc...

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