FROM KEPLER PROBLEM TO SKYRMIONS

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Abstract. The classical treatment of the Kepler problem leaves room for the description of the space region of the central body by a hyperbolic geometry. If the correspondence between the empty space and the space filled with matter is taken to be a harmonic mapping, then the region of atomic nucleus, like the one of the Sun for the planetary system proper, is described by hyperbolic skyrmions. This fact makes possible the description of the nuclear matter within framework of general relativity. The classical "hedgehog" solution for skyrmions can then be classically interpreted in terms of the characterizations of intra-nuclear forces.

CLASSICAL KEPLER MOTION: THE POSITION OF CENTRAL BODY

The classical Kepler motion can be described with the Newtonian equations of motion

$$\ddot{\vec{r}} + \frac{K}{r^2}\frac{\vec{r}}{r} = \vec{0} \tag{1}$$

Here K is a constant, $\vec{r}$ denotes the position vector of the material point whose motion is described, with respect to the center of force, and a dot over a symbol means derivative with respect to time. The constant K does not depend on quantities related to the point in motion, but only in cases when electric forces are involved. We can simplify the algebra by confining the geometry to the plane of motion, where the coordinates of the point in motion are ξ and η say (**Mittag, Stephen, 1992**). Equation (1) is then equivalent to the system

$$\ddot{\xi} + K\frac{\cos\phi}{r^2} = 0, \qquad \ddot{\eta} + K\frac{\sin\phi}{r^2} = 0 \tag{2}$$

with r and ϕ the polar coordinates of the plane with respect to the attraction center. The magnitude of the rate of area swept by the position vector of the particle is then given by

$$\dot{a} \equiv \xi\dot{\eta} - \eta\dot{\xi} = r^2\dot{\phi} \tag{3}$$

This constant of motion allows us an elegant integration of the system (2) with the analytical form of the trajectory as a direct outcome. First we define the complex variable

$$z \equiv \xi + i\eta = re^{i\phi} \tag{4}$$

so that (2) can be written in the form

$$\ddot{z} + \frac{K}{r^2}e^{i\phi} = 0 \tag{5}$$

Now, use (3) to eliminate r^2, such that

$$\ddot{z} + \frac{K}{\dot{a}}e^{i\phi}\dot{\phi} = 0 \qquad \therefore \qquad \dot{z} = i\left(\frac{K}{\dot{a}}e^{i\phi} + w\right) \tag{6}$$

where $w \equiv w_1 + iw_2$ is a complex constant of integration to be determined by *the initial conditions* of the problem. The analytical equation of motion can be then extracted directly from (3) by using (6). In polar coordinates of the plane of motion the result is

$$\frac{\dot{a}}{r} = \frac{K}{\dot{a}} + w_1 \cos\phi + w_2 \sin\phi \tag{7}$$

The shape of this trajectory is best pictured by going back to Cartesian coordinates, where we have, instead of (7) the second-degree curve – a conic:

$$\left(\frac{K^2}{\dot{a}^2} - w_1^2\right)\xi^2 - 2w_1 w_2 \xi\eta + \left(\frac{K^2}{\dot{a}^2} - w_2^2\right)\eta^2 + 2\dot{a}(w_1\xi + w_2\eta) = \dot{a}^2 \tag{8}$$

The center of this conic is *not the center of the force,* but has the coordinates

$$\xi_0 = -\frac{\dot{a}w_1}{\Delta}, \qquad \eta_0 = -\frac{\dot{a}w_2}{\Delta}, \qquad \Delta \equiv \left(\frac{K}{\dot{a}}\right)^2 - w_1^2 - w_2^2 \tag{9}$$

In cases where $\Delta = 0$, the center of this trajectory is at infinity: the trajectory is a parabola. We have here the ballistic cases, where the basic motion is parabolic.

Assuming the center of the trajectory at finite distance with respect to the center of force, and referring the trajectory to this center by the translation $x = \xi - \xi_0$, $y = \eta - \eta_0$, its equation becomes

$$\left(\frac{K^2}{\dot{a}^2} - w_1^2\right)x^2 - 2w_1 w_2 xy + \left(\frac{K^2}{\dot{a}^2} - w_2^2\right)y^2 = \frac{K^2}{\Delta} \tag{10}$$

The quadratic form from the left hand side of this equation is completely characterized by the 2×2 matrix

$$\mathbf{A} \equiv \begin{pmatrix} \dfrac{K^2}{\dot{a}^2} - w_1^2 & -w_1 w_2 \\ -w_1 w_2 & \dfrac{K^2}{\dot{a}^2} - w_2^2 \end{pmatrix} \tag{11}$$

The eigenvalues of this matrix are Δ and $K^2/\dot{a}^2$, with the corresponding eigenvectors

$$\left|e_1\right\rangle = \begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix}, \qquad \left|e_2\right\rangle = \begin{pmatrix} -\sin\omega \\ \cos\omega \end{pmatrix}; \qquad w_1 \equiv w \cdot \cos\omega, w_2 \equiv w \cdot \sin\omega \tag{12}$$

Thus, the orientation of trajectory in its plane is completely defined by the initial conditions of the motion. The magnitude 'w' is proportional with the eccentricity 'e' of trajectory. Indeed the semi axes 'a' and 'b' are

$$a^2 = \frac{K^2}{\Delta^2}, b^2 = \frac{\dot{a}^2}{\Delta}; \quad \therefore \qquad e^2 \equiv \frac{a^2 - b^2}{a^2} = \left(\frac{\dot{a}}{K}\vec{w}\right)^2 \tag{13}$$

Thus, the initial conditions can actually be expressed only in terms of 'contemporary' magnitudes allowing us to *forget about the past*:

$$w_1 = \frac{K}{\dot{a}} e \cdot \cos\omega, \qquad w_2 = \frac{K}{\dot{a}} e \cdot \sin\omega \tag{14}$$

This is essentially the observation that imposed the Newtonian explanation for the real planetary motions in terms of contemporary quantities. Indeed, a force is always contemporary, and the initial conditions of the motion, whatever they might be, are then to be read in some contemporary parameters of motion: area constant and eccentricity.

As it can be seen directly from equations (13) one of the semi axes can be imaginary, for $\Delta < 0$, in which case we have to do with hyperbolic trajectories. It is only in cases where $\Delta > 0$, that we have to do with elliptic trajectories, properly representing planetary motion. Along this line of reasoning, the parabolic trajectories are all characterized by points on the circle $\Delta = 0$, i.e.

$$\mu^2 + \nu^2 = e^2, \qquad \mu \equiv e \cdot \cos\omega, \nu \equiv e \cdot \sin\omega \tag{15}$$

and the *whole interior of this circle corresponds to all possible finite motions that a material point can have* around a center of force acting with a force inversely proportional to square of distance. This would mean that a planet would have infinitely many possible initial conditions we have to choose from. Fact is that the actual motion of a planet is perceived as if it had *unique initial conditions*. Any departure from this perception has always induced arguments about some actual perturbations acting on the planet. To a certain extent this is true: the discovery of Neptune is an example. However, as the history shows, it has not the touch of universality needed for the continuity of knowledge. Thus, we have to turn to the origin of the problem, and lead the reasoning along the following lines: Kepler motion has reality only as a “snapshot”, this is undeniable; it could not have been discovered otherwise. However the planetary motion is a succession of such snapshots, which have to be put together in order to make the whole thing. First of all, we have to find the time scale of such a snapshot, and that is hard. But we have another possibility, opened by the remarks just made above: there is an a priori metric geometry of the defining parameters of the snapshot, which are the *initial conditions* of the dynamical problem describing this snapshot. This geometry defines a kinematics, and the kinematics offers us a natural way to continuously connect the snapshots in a succession representing a real trajectory.

Indeed, even superficially it can be seen at once that the mentioned freedom of the parameters defining the types of orbits, allows us to construct a Cayley-Klein (or Absolute) geometry (**Cayley, 1859**; **Klein, 1897**) characterizing the variation of those orbits. We know that an Absolute geometry is related to some conservation laws, at least as long as some realizations of SL(2, R) group structure are involved. And indeed, the absolute metric for the interior of the circle (15)

$$(ds)^2 = \frac{(1-\nu^2)(d\mu)^2 + 2\mu\nu(d\mu)(d\nu) + (1-\mu^2)(d\nu)^2}{(1-\mu^2-\nu^2)^2} \tag{16}$$

can be brought to the form of Poincaré metric

$$(ds)^2 = -4\frac{dh \cdot dh^*}{(h-h^*)^2} = \frac{(du)^2 + (dv)^2}{v^2} \tag{17}$$

by the following transformation of coordinates:

$$\mu = \frac{hh^* - 1}{hh^* + 1}, \nu = \frac{h + h^*}{hh^* + 1} \quad \leftrightarrow \quad h \equiv u + iv = \frac{\nu + i\sqrt{1 - \mu^2 - \nu^2}}{1 - \mu}, h^* = u - iv \tag{18}$$

The conservation laws for the metric (17) are

$$\omega_1 = \frac{du}{v^2}, \quad \omega_2 = 2\frac{udu + vdv}{v^2}, \quad \omega_3 = \frac{(u^2 - v^2)du + 2uvdv}{v^2} \tag{19}$$

The description we are also interested in now is the one in variables (e, ω), i.e. the eccentricity and the orientation of the orbit in its plane. In terms of these parameters the metric (16) becomes

$$(ds)^2 = \left(\frac{de}{1 - e^2}\right)^2 + \frac{e^2}{1 - e^2}(d\omega)^2 \tag{20}$$

We can rewrite this metric in a well-known form, by noticing that for elliptic trajectories ‘e’ is confined to the interval between –1 and +1, so that the change of parameter

$$e = \tanh \psi \tag{21}$$

is legitimate. With this the metric (20) becomes

$$(ds)^2 = (d\psi)^2 + \sinh^2 \psi (d\omega)^2 \tag{22}$$

The complex parameter ‘h’ from equation (18) has a direct relationship with the theory of classical potentials. In order to show this relationship we write here ‘h’ in terms of (e, ω). We have:

$$h = i\frac{\cosh\chi + \sinh\chi \cdot e^{-i\omega}}{\cosh\chi - \sinh\chi \cdot e^{-i\omega}}, \quad \chi \equiv \frac{\psi}{2} \tag{23}$$

It just happens that this equation represents a *harmonic map* from the usual space into the Lobachevsky plane *provided* χ (and therefore ψ) *is a solution of the Laplace equation in free space*.

Indeed, the problem of harmonic correspondences between space and the hyperbolic plane is described by the minimum of energy functional corresponding to the metric (17) where the differentials are transformed into space gradients (**Eells, Sampson, 1965**; **Misner, 1978**). The minimization of energy functional corresponds to Euler-Lagrange equations for the Lagrangian

$$\Lambda \equiv -4\frac{\nabla h \cdot \nabla h^*}{(h - h^*)^2} \tag{24}$$

These are

$$(h - h^*) \cdot \nabla^2 h - 2(\nabla h)^2 = 0 \tag{25}$$

and its complex conjugate. Then it is easy to see that ‘h’ from (23) verifies this equation when χ is a solution of Laplace equation, and ω does not depend on the position in space. It might, nevertheless, depend on the local time of the Newtonian dynamics.

This method can be thought of really ascribing “a spatial expanse”, in the form of harmonic surfaces in space, of the regions of space extended over the ranges of eccentricity of the Kepler motion. This is to say, that the harmonic maps from the Lobachevsky plane to space are related to the physics of Sun, in the case of planetary system, or to the physics of nucleus in the case of classical atomic model. There is no obvious sign today in physics for the first case, i.e. the solar system, but the conclusion seems to be fair for the case of atomic model.

SOME VARIATIONS ON A THEME OF SKYRMIONS

We don't need to insist again in recalling that the main scientific image of the atom is the planetary one, amended perhaps with the idea that this physical system is not plane but a spatial one – maybe spherical. This sphericity of the model, if real, is due to the noncentrality of forces or *to the space extension of the matter at the nuclear level*. If this is the case, then the region of the nucleus can be characterized by a *3D hyperbolic space*. With this statement we enter the realm of recent date of the *hyperbolic skyrmions* (**Atiyah, Sutcliffe 2001, 2004**).

Indeed, one of the dominant contemporary concepts in the theory of structure of the nuclear matter, is that of skyrmion, which represents itself a variation on the subject of harmonic applications. The Skyrmion is a soliton representing the nucleons. The history of this subject starts with the physicist Tony Skyrme, and the reader in need of following it can begin with his recollections of the beginning (**Skyrme 1988**), where the reasons and the original works are indicated. The main mathematical point of Skyrme's idea, is a certain, "almost harmonic" map, from the usual space to sphere, built after the traditional manner of the variational problem leading to the Laplace and Schrödinger equations. Let's shortly describe this manner, following by and large the recent work of Slobodeanu (**Slobodeanu 2009**).

As known, such a harmonic map is obtained by finding the functions which realize the extremum of the *energy* functional

$$E_2(\mathbf{\Phi}) = \iiint \langle d\mathbf{\Phi} | d\mathbf{\Phi} \rangle d^3\vec{x} \tag{26}$$

Here the function realizing the correspondence is generally of the nature of a matrix – this is why we even denoted it by $\mathbf{\Phi}$ – and the dot product is the one induced by the metric of the hyperbolic space. In the case of Lobachevsky plane, the metric Lagrangian (24) corresponds to the metric of the plane:

$$\mathbf{\Phi} \equiv \begin{pmatrix} h \\ h^* \end{pmatrix}; \quad \langle d\mathbf{\Phi} | d\mathbf{\Phi} \rangle \equiv -4\frac{dhdh^*}{(h-h^*)^2} \tag{27}$$

This variational principle is directly connected with the inertial field in vacuum, because it is equivalent with Einstein vacuum field equations (**Ernst 1968, 1971**; **Israel, Wilson 1973**). Consequently, those very equations can be simply considered as equally describing the states of nuclear matter, provided *this one admits a description in the hyperbolic plane*. And this is what the classical Kepler problem actually points out. One would therefore expect that based on that classical problem of motion, the solution of problem of the nuclear structure should be simply a matter of study of the harmonic maps, but this is not quite the case. The nucleus is assumed to have a particulate structure, and the harmonic principle, if not somehow amended, cannot account for that. So, it comes that the Skyrme functional is not quite as simple as the above one, but involves higher degree terms, belonging to different cohomology classes, given by the equation

$$E_4(\mathbf{\Phi}) = \iiint \langle d\mathbf{\Phi} \wedge d\mathbf{\Phi} | d\mathbf{\Phi} \wedge d\mathbf{\Phi} \rangle d^3\vec{x} \tag{28}$$

If $\mathbf{\Phi}$ is a mapping from the Euclidean space to itself, then this term assures that the resulting equations of motion are nonlinear, and the nonlinear equations always admit confined (soliton) solutions, as the structure of the nucleus seems to require.

In order to establish the place of the hyperbolic geometry in the theory of nuclear matter, and therefore of the stellar matter or of some other nature, we just need to guide our guesses. The first observation is actually the gist of the work from 2004 of Atiyah and Sutcliffe (**Atiyah, Sutcliffe 2004**), and is referring to the very significance of hyperbolic skyrmions. Namely, the theory of Euclidean skyrmions, with massive pions, leads to detailed results which are almost identical with those referring to *massless hyperbolic skyrmions*. It is as if the mass – the source of inertia in classical mechanics and general relativity – is somehow related to the curvature – again an idea of general relativity – not in the space-time but *simply in space*.

However, it seems more comfortable to work with maps from real space to itself – like always in classical mechanics – as in the cases leading to the equations of Laplace or Schrödinger. Along this line an essential observation has been made by N. S. Manton (**Manton 1987**), and presents the Euclidean skyrmions as *related to the deformations of matter*. This is an idea of classical inspiration, taking its roots from the deformation theory, where the experimental deformations are described by the so-called tensor of elongations (**Hill 1968**; **Ogden 1972**). The most general energetic functional of Skyrme can be written, according to Manton, in the form given in equation (26), where $\mathbf{\Phi}$ is now a vector dictated by the deformation of matter in the regular space:

$$\mathbf{\Phi} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{29}$$

with $\lambda_{1,2,3}$ the elongations along three reciprocally orthogonal directions. Here the interior product from (26) is simply the dot product of regular vectors. As from equation (29) we have

$$d\mathbf{\Phi} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{30}$$

it is plain that the energy functional from equation (26) is

$$E_2(\mathbf{\Phi}) = \int \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) d^3\vec{x} \tag{31}$$

As to the other term from the original theory of Skyrme, it is simply the dot product of the exterior square of the gradient:

$$d\mathbf{\Phi} \wedge d\mathbf{\Phi} = \lambda_2\lambda_3 dy \wedge dz + \lambda_3\lambda_1 dz \wedge dx + \lambda_1\lambda_2 dx \wedge dy \tag{32}$$

So the functional from equation (28) can be written in the form

$$E_4(\mathbf{\Phi}) = \int \left(\lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2\right) d^3\vec{x} \tag{33}$$

Incidentally, the indices 2 and 4 occuring in the energetic functionals of Manton are justified by the orders of the tensors entering the integrands of those functionals. Thus d**Φ** is defined by a second order tensor, while d**Φ**∧d**Φ** is defined by a fourth order tensor. Finally the Manton functional for the Skyrme model is the sum of the two contributions to the energy, i.e.:

$$E(\mathbf{\Phi}) = \int \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2\right) d^3\vec{x} \tag{34}$$

Details on this line of reasoning can be found in the original work of N. S. Manton (**Manton 1987**), and in numerous works that followed along this line (see **Manton, Sutcliffe 2004**). The interested reader is directed to those works, for our line of reasoning will take here another turn in the pursuit of hyperbolic skyrmions.

A GEOMETRY OF HYPERBOLIC SKYRMIONS

The way we see it, theory of hyperbolic skyrmions is the legitimate descendant of the classical theory describing the atomic structure based on the classical dynamics of electric forces. Indeed, according to our presentation above, this theory describes the region of atomic space normally assigned to the nucleus of the atom, and the geometry of this region is the hyperbolic geometry. In spite of the fact that Manton's idea is about Euclidean skyrmions, it contains almost explicitly a connection with hyperbolic skyrmions. This connection comes about in another work of Atiyah and Manton (**Atiyah, Manton 1993**), whereby the deformation is treated in terms of the roots of a family of cubic equations.

The idea of a family of cubic equations involved in the geometry of skyrmions is indeed germane to the problem, as Atiyah and Manton show. However, it is also germane to the problem of deformation, inasmuch as a state of deformation is described in terms of matrices, and a cubic equation is simply the characteristic equation of a 3×3 matrix. A process of deformation is then described by a family of matrices, therefore by a family of cubic equations. And a family of cubic equation is always described by a metric of constant negative curvature that generalizes the metric of hyperbolic plane (**Barbilian 1938**). Thus we can come directly to a metric describing the deformation, and then use the harmonic principle in order to describe the hyperbolic skyrmions. Let's expound along this line.

From energetical point of view the Manton's functional is actually a very special one. Indeed, in the realm of hyperelastic deformations, it is only a special instance of the so-called Mooney-Rivlin model, where the energy density is a linear combination of the two invariants of the deformation

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2; \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \tag{35}$$

This model describes correctly deformations up to 30 – 40% which, for some rubbers, are even small deformations. However, it is the only theoretical model accepting a microscopic physical description by means of a Gaussian statistics of the macromolecular chains, so that the contemporary theoretical physics almost got stuck within its limits. Another reason for this might also be the fact that the range of deformations covered by this description is quite enough for application in phenomena with industrial application. However, theoretically speaking, the physics cannot afford to say that this model universally describes the process of deformation: *it cannot be taken as a fundamental law of nature*. So much less, therefore, can we say that such a model will describe the deformations of nuclear matter. Here, we are in the hazy range of the confluence between the matter proper and space. It is to be expected, for instance, that the deformations of nuclear matter are not reversible, because of the energy dissipation in the form of particles and heat, like, for instance inside the Sun – our daily nucleus.

The general idea here would be that the irreversible deformations taking place with dissipation of energy, are dominated not by the usual invariants of deformation, but by an algebraical combination of them, appearing as an intensity of shearing deformations, as Novozhilov (**Novozhilov 1952**) has shown. This algebraical combination carries, in the theory of deformations, the name of *invariant of von Mises*. In terms of the invariants from equation (35) it can be written as

$$\frac{1}{2}(I_1^2 - 3I_2) \equiv \left(\lambda_2^2 - \lambda_3^2\right)^2 + \left(\lambda_3^2 - \lambda_1^2\right)^2 + \left(\lambda_1^2 - \lambda_2^2\right)^2 \tag{36}$$

Consequently, the Manton-like functional corresponding to this situation should be written in the algebraically homogeneous form

$$E(\mathbf{\Phi}) = \frac{1}{2}\int \left\{\left\langle d\mathbf{\Phi} \middle| d\mathbf{\Phi}\right\rangle^2 - 3\left\langle d\mathbf{\Phi}\wedge d\mathbf{\Phi} \middle| d\mathbf{\Phi}\wedge d\mathbf{\Phi}\right\rangle\right\} d^3\vec{x} \tag{37}$$

which is, obviously, out of the limits of Skyrme theory as first conceived. However, this functional suggests a consistent way of description of the nuclear matter, within the limits of general relativity, inasmuch as it generalizes the Ernst approach. This consists of reducing the Skyrme functional to its first term only, therefore to the usual energy of a harmonic application. We will work therefore on the form of the map **Φ** itself, and generalize it in a natural manner, so as to include the regular hyperbolic geometry as a special case. In order to understand this extension a short incursion in a special theory of deformation of higher orders will be necessary, suggested by the Barbilian's cubic space (**Barbilian 1938**) considered as a Cayley-Klein space.

Let's therefore make only the assumption that the deformations are represented by a 3×3 matrix, without being interested if it is indeed derivable from a gradient or not. It can represent, for instance, a variation, at a certain scale, of the metric tensor of the space within which the matter is contained. Let's denote by **x** this deformation matrix. Its eigenvalues, ($(\lambda^2 - 1)/2$ in the previous formalism of Manton) are the roots of a third degree equation – the characteristic equation of matrix. This is the circumstance allowing us to write a general deformation energy density, as based on the algebraical principle of the polarity of the binary algebraic forms. This will be explained as we proceed.

Assume, to start with, that we have a quadratic binary form – a homogeneous second degree polynomial – having the coefficients a_0, a_1 and a_2, which happens to have some physical meaning in a physical problem. Assume also that we have a set of cubic binary forms, representing the characteristic polynomials of our deformation matrices. These cubic polynomials have a *common* *invariant* with our starting quadratic polynomial (**Burnside, Panton 1960**). This invariant is a quadratic form in the coefficients of our family of cubic polynomials:

$$\frac{1}{9}\Psi \equiv a_2(X_0X_2 - X_1^2) - a_1(X_0X_3 - X_1X_2) + a_0(X_1X_3 - X_2^2) \tag{38}$$

Here X_0, X_1, X_2, X_3 are the coefficients of a generic cubic from our family, taken in its binomial form

$$X_0x^3 + 3X_1x^2 + 3X_2x + X_3 \tag{39}$$

X_0 – the coefficient of the third degree term of the polynomial, is responsible for the indetermination in the relations between roots end coefficients, and x is the generic nonhomogeneous variable of the cubic. Vanishing of the quadratic form (38) implies the *apolarity* between our starting quadratic form and every member of the family of cubics. As known, the apolarity can be extended to a projective concept, which here comes in handy, inasmuch as the ratios

$$\frac{X_1}{X_0};\quad \frac{X_2}{X_0};\quad \frac{X_3}{X_0} \tag{40}$$

can be taken as nonhomogeneous coordinates of a point in a 3D space of cubics.

From the point of view of the deformation process, let's first notice that the function Ψ from equation (38) can be taken as a potential which generalizes, in a natural manner, the function from equation (36) to a nonhomogeneous function. Indeed, that function is given by the first term only, from the right hand side of equation (38). The theory of potential works here directly, as follows. First, if the generic cubic of our family is the characteristic equation of the symmetric matrix **x**, with elements x_{ij} representing deformations, then the matrix having the entries given by equation

$$y_{ij} = \frac{\partial\Psi}{\partial x_{ij}} \tag{41}$$

represents the corresponding stresses; obviously, the opposite is also valid. Equation (41) suggests that Ψ is a potential indeed, according to the rules of using the concept of potential.

In our conditions the equations (41) should be necessarily of the form

$$y_{ij} = \alpha_0\delta_{ij} + \alpha_1x_{ij} + \alpha_2(\mathbf{x}^2)_{ij} \tag{42}$$

because they correlate 3×3 matrices. The problem is to find the coefficients α_0, α_1, α_2, which represent the physical properties of the continuum whose deformation is described by the matrix **x**. In order to solve this problem we need to have the coordinates X_j which, up to a common factor are given by the coefficients of the characteristic equation of **x**:

$$X_0 = 1,\qquad X_1 = -\frac{I_1}{3},\qquad X_2 = \frac{I_2}{3},\qquad X_3 = -I_3 \tag{43}$$

Here I_1, I_2, I_3 denote the scalar quantities specific to the matrix **x**, that in the case of a Euclidean tensor are its orthogonal invariants. In general though, they are defined by the formulas

$$I_1 = \mathrm{Tr}(\mathbf{x}) \equiv t_1, \qquad I_2 = \frac{1}{2}\left[(\mathrm{Tr}(\mathbf{x}))^2 - \mathrm{Tr}(\mathbf{x}^2)\right] \equiv \frac{1}{2}(t_1^2 - t_2)$$
$$I_3 = \frac{1}{6}\left[2\mathrm{Tr}(\mathbf{x}^3) - 3(\mathrm{Tr}(\mathbf{x}))\mathrm{Tr}(\mathbf{x}^2) + (\mathrm{Tr}(\mathbf{x}))^3\right] \equiv \frac{1}{6}(2t_3 - 3t_1t_2 + t_1^3) \tag{44}$$

where we used obvious notations for the traces of powers of the matrix. Using these we have

$$A_0 \equiv X_0X_2 - X_1^2 = \frac{1}{18}(t_1^2 - 3t_2)$$
$$A_1 \equiv X_0X_3 - X_1X_2 = -\frac{1}{9}(t_1^3 - 4t_1t_2 + 3t_3) \tag{45}$$
$$A_2 \equiv X_1X_3 - X_2^2 = \frac{1}{36}(4t_1t_3 - 4t_1^2t_2 + t_1^4 - t_2^2)$$

Therefore, in order to calculate (41) we need the derivatives of the traces of different powers of the matrix **x**. These are calculated according to the formulas

$$\frac{\partial t_1}{\partial x_{ij}} = \delta_{ij}, \qquad \frac{\partial t_2}{\partial x_{ij}} = 2x_{ij}, \qquad \frac{\partial t_3}{\partial x_{ij}} = 3(\mathbf{x}^2)_{ij} \tag{46}$$

so that, finally, we have for α_0, α_1, α_2 the following equations:

$$\alpha_0 = a_2t_1 + a_1(3t_1^2 - 4t_2) + a_0(t_1^3 - 2t_1t_2 + t_3)$$
$$\alpha_1 = -3a_2 - 8a_1t_1 - a_0(2t_1^2 + t_2) \tag{47}$$
$$\alpha_2 = 3(3a_1 + a_0t_1)$$

On the other hand the constitutive relation (42) is also inversible, i.e. we can write the deformations in terms of stresses in the form

$$\mathbf{x} = \beta_0\mathbf{e} + \beta_1\mathbf{y} + \beta_2\mathbf{y}^2 \tag{48}$$

where **e** is the identity 3×3 matrix. Here β_0, β_1 and β_2 are connected with α_0, α_1 and α_2 by the system of equations

$$\beta_0 + \beta_1\alpha_0 + \beta_2(\alpha_0^2 + \alpha_2^2I_1I_3 + 2\alpha_1\alpha_2I_3) = 0$$
$$\alpha_1\beta_1 + \left[2\alpha_0\alpha_1 - 2\alpha_1\alpha_2I_2 + (I_1^2 - I_2)\alpha_2^2\right]\cdot\beta_2 = 1 \tag{49}$$
$$\alpha_2\beta_1 + \left[\alpha_1^2 + 2\alpha_1\alpha_2I_1 + 2\alpha_0\alpha_2 + (I_3 - I_1I_2)\alpha_2^2\right]\cdot\beta_2 = 0$$

This shows that there are nontrivial states of stress, corresponding to null deformations, as in the case of vacuum for instance, or that of classical ether. These are determined exclusively by the starting binary quadratic form, through the equations

$$\beta_0 + \beta_1\alpha_0 + \beta_2\alpha_0^2 = 0$$
$$\alpha_1\beta_1 + 2\alpha_0\alpha_1\beta_2 = 1 \tag{50}$$
$$\alpha_2\beta_1 + \left(\alpha_1^2 + 2\alpha_0\alpha_2\right)\beta_2 = 0$$

where, according to (47), we take

$$\alpha_0 = 0; \quad \alpha_1 = -3a_2; \quad \alpha_2 = 9a_1 \tag{51}$$

Thus we have

$$\beta_0 = 0; \quad \beta_1 = -\frac{1}{3a_2}; \quad \beta_2 = \frac{a_1}{3a_2^3} \tag{52}$$

Our starting quadratic form should therefore represent such limit states of matter, that do not depend on the state of deformations or tensions. Generally, however, the constitutive relations (42) and (48) cannot be simultaneously inversed, because there are separately deformations that do not involve stresses, as well as stress phenomena that do not involve deformations. The matter for which these phenomena are simultaneous should therefore have special properties.

An example of such states can be given immediately. Indeed, the relation (42) shows that the stresses can all vanish for a certain nontrivial deformation processes. Indeed, the system of equations

$$\alpha_0 = \alpha_1 = \alpha_2 = 0 \tag{53}$$

has a nontrivial solution for the invariants of deformations. This solution is given by the following formulas for the traces of powers of deformation matrix as functions of the coefficients of the starting binary quadratic

$$t_1 = -3\frac{a_1}{a_0}; \quad t_2 = 3\frac{2a_1^2 - a_0a_2}{a_0^2}; \quad t_3 = 3\frac{4a_1^3 - 3a_0a_1a_2}{a_0^3} \tag{54}$$

Therefore, the eigenvalues of deformation matrix are functions independent of the state of stress. In cases where the starting quadratic form has real roots, say u ± v, these equations give

$$t_1 = 3u; \quad t_2 = 3(u^2 + v^2); \quad t_3 = 3u(u^2 + 3v^2) \tag{55}$$

Using now the formulas (43) and (44) we find easily the coefficients of the characteristic equation of the corresponding matrix in the form

$$\frac{c_0}{1} = \frac{c_1}{-u} = \frac{c_2}{u^2 - \frac{v^2}{2}} = \frac{c_3}{-u\left(u^2 - \frac{3v^2}{2}\right)} \tag{56}$$

This cubic is the center of the quadric given in equation (38), and has the remarkable property of having the roots strictly determined by ‘u’ and ‘v’:

$$u - \sqrt{\frac{3}{2}}v, \quad u, \quad u + \sqrt{\frac{3}{2}}v, \tag{57}$$

The Hessian of this reference cubic is given by the equations

$$\frac{A_0}{1} = \frac{A_1}{-2u} = \frac{A_2}{u^2 + \frac{v^2}{2}} = -\frac{v^2}{2} \tag{58}$$

Obviously, this is a polar conjugate of our starting binary quatratic. There are also a family of cubics conjugated with (58) in the sense that their Hessians are apolar with this one. This is the original *Barbilian hyperboloid*, having the equation:

$$\left(u^2 + \frac{v^2}{2}\right)(X_0X_2 - X_1^2) + u(X_0X_3 - X_1X_2) + (X_1X_3 - X_2^2) = 0 \tag{59}$$

The center of this quadric is given by

$$\frac{b_0}{1} = \frac{b_1}{-u} = \frac{b_2}{u^2 + \frac{v^2}{2}} = \frac{b_3}{-u\left(u^2 + \frac{3v^2}{2}\right)} \tag{60}$$

and obviously is a cubic having only one real root, i.e. its roots are

$$u - i\sqrt{\frac{3}{2}}v, \quad u, \quad u + i\sqrt{\frac{3}{2}}v, \tag{61}$$

Therefore we have here a kind of duality. One can say that ‘u’ and ‘v’ are functions of the physical characteristics of matter undergoing deformation, and that there are states of strain and stress that do not depend on anything else but on these physical characteristics. This is the meaning of the starting quadratic form, which represents the Hessian of a certain cubic form.

The quadric from equation (59) is the starting point of Dan Barbilian in the construction of the Riemann spaces associated with families of one-parameter cubics, as Cayley-Klein spaces (**Barbilian 1938**). The geometrical procedure used by Barbilian will be now discussed in broad strokes. First, we need to notice that Barbilian begins with the idea that the starting quadratic form is the Hessian of a cubic with real roots, therefore it has complex roots itself. Then, by performing the linear transformation of homogeneous coordinates

$$\begin{aligned} \frac{X_0'}{uX_0 + X_1} = \frac{X_1'}{u^2X_0 - X_2} &= \frac{X_2'}{\left(u^2 + \frac{v^2}{2}\right)X_0 - X_2} \\ &= \frac{X_3'}{u\left(u^2 + \frac{v^2}{2}\right)X_0 - \left(u^2 + \frac{v^2}{2}\right)X_1 - uX_2 + X_3} \end{aligned} \tag{62}$$

we can reduce (59) to a “canonical” form

$$X_0'X_3' - X_1'X_2' = 0 \tag{63}$$

showing explicitly that we are dealing here with a one-sheeted hyperboloid. Using now the Sylvester theorem, for the representation of a cubic in terms of its Hessian (**Burnside, Panton 1960**), we get the interesting result that a cubic from the family having the same Hessian can be represented by a 2×2 matrix having the entries

$$\frac{X_0'}{h^*k} = \frac{X_1'}{h} = \frac{X_2'}{k} = \frac{X_3'}{1} \tag{64}$$

where ‘h’ and ‘h^*’ are the complex roots of the Hessian, and k is an arbitrary complex factor of unit modulus. The Cayley-Klein metric of this representation, with respect to the hyperboloid from equation (63) as an Absolute, is the Barbilian metric given by

$$(ds)^2 = -4\frac{dhdh^*}{(h - h^*)^2} + \left(\frac{dk}{k} - \frac{dh + dh^*}{h - h^*}\right)^2 \tag{65}$$

The phase factor k can be interpreted in terms of the ensembles of harmonic oscillators (**Mazilu 2010**). Geometrically it is the parameter of a family of Bäcklund transformations from the Lobachevsky geometry, characterized by the last part of the metric (65), to a general 3D hyperbolic geometry characterized by the whole metric (65). It is thus to be expected that the harmonic mapping associated with the energy functional

$$E(\mathbf{\Phi}) = \frac{1}{2}\iiint\left\{-4\frac{\nabla h\nabla h^*}{(h-h^*)^2} + \left(\frac{\nabla k}{k} - \frac{\nabla h + \nabla h^*}{h-h^*}\right)^2\right\}d^3\vec{x} \tag{66}$$

will give us skyrmions, thus bridging the gap between general relativity and the theory of nuclear matter. We even have the possibility of an anzatz here.

Indeed, we have seen that the first part of the metric (65) is the one generating the energetic functional for the gravitational field in vacuum, and the variational principle associated to it, leads to Ernst's equations, equivalent to Einstein equations in vacuum. If we express the complex number 'h' as a function of the eccentricity 'e' of the orbit, by the formula (21), this metric will be given by equation (22). This is the metric of a section of hyperbolic space, used by Atiyah and Sutcliffe in the construction of hyperbolic skyrmions. Its form is, in general

$$(ds)^2 = (d\psi)^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \tag{67}$$

where θ and ϕ are usual spherical polar angles. Obviously, (22) can be obtained from (67) if we agree that ω represents the geodesic arc on the unit sphere. But (67) is also the absolute metric of the space of relativistic velocities (**Fock 1964**), and can be directly obtained from (65), and probably there are still many meanings of it. The fact that it was used in representing the skyrmions with zero mass pions, motivates us to construct for it a skyrmion with distinguished significance in the Newtonian mechanics.

From the point of view of Newtonian forces, it is quite probable that the third principle of dynamics is no more effective over the space of nuclear matter, and that the eccentricity of the electronic orbits is actually an expression of the nonequilibrium of the forces. Here the Newtonian measurements of the forces one by means of another can still be defined, but by means of a quantum definition of the measurement, in the axiomatic manner in which the spin is introduced within quantum mechanics (**Schwartz 1977**). Namely, the 2×2 matrix

$$\mathbf{Q} \equiv \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta \end{pmatrix} \tag{68}$$

has eigenvalues ±1, or any two numbers equal and opposite in sign, which is the ideal case of classical measurement of forces: equal and opposite. Consequently it can represent the third principle of dynamics ad literam, in the sense that the two eigenvalues are the values of a force and its reaction, i.e. their algebraic sum is zero. Based on this, we can build a matrix that has two different eigenvalues, representing two different forces no matter of their directions. Indeed, any 2×2 matrix of the form

$$\mathbf{M} = \lambda\mathbf{E} + \mu\mathbf{Q} \tag{69}$$

where λ and μ are real, and **E** is the 2×2 identity matrix, has two different real eigenvalues not depending on the angles. These are $(\lambda \pm \mu)$. Our ansatz is then identical with the original one of Skyrme, and amounts to an exponential expression of the matrix **M**:

$$\mathbf{M} = \exp(\psi \mathbf{Q}) \tag{70}$$

One can really see that this means that the two forces are always in the same ratio, no matter of how their system is oriented in space. The absolute metric of the matrices (70) is just the metric from equation (67), where we are to take

$$\frac{\mu}{\lambda} = \tanh \psi \tag{51}$$

Consequently the ratio of forces is practically represented by the eccentricity of the Keplerian motion which represents the atom. The "hedgehog" skyrmion from the equation (70) – which we would like to call *fundamental* – then represents the situation of forces at the nuclear level. They are in equilibrium according to the third law of Newton, only when the eccentricity of the electronic orbits is zero, i.e. when these orbits are perfect circles, as it should be according to the classical theory of Newtonian forces. This fundamental skyrmion is, according to the ideas of Atiyah and Sutcliffe, essential in the construction of any other with null mass pions.

CONCLUSIONS

There are reasons to hope that the theory of nuclear forces can be naturally unified with the general relativity. First, the existence of nucleus is quite naturally described even within the classical theory of Kepler motion. Then, this very description is quite close to a successful theory of the nuclear matter – the Skyrme theory. Only, the nuclear matter, and probably the matter in general, should be described not by an Euclidean geometry but by a Lobachevsky geometry. The way we see it now points toward the fact that the geodesics of this hyperbolic geometry represent motions of continuum matter giving out the energy, or energies, we usually notice. Then the affine parameter of those geodesics should be the temperature of the matter. It plays here the same role the time plays in the Newtonian kinematics.

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