Smarandache's Ratio Theorem

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Abstract.
In this paper we present the Smarandache's Ratio Theorem in the geometry of the triangle.

Smarandache's Ratio Theorem.
If the points A_1, B_1, C_1 divide the sides ||BC|| = a, ||CA|| = b, respectively ||AB|| = c of a triangle ΔABC in the same ratio $k > 0$, then

$$\|AA_1\|^2 + \|BB_1\|^2 + \|CC_1\|^2 \geq \frac{3}{4}(a^2 + b^2 + c^2).$$

Proof.
Suppose $k > 0$ because we work with distances.

$$\|BA_1\| = k\|BC\|, \quad \|CB_1\| = k\|CA\|, \quad \|AC_1\| = k\|AB\|$$

We'll apply tree times Stewart's theorem in the triangle $\Delta\ ABC$, with the segments AA_1, BB_1, respectively CC_1:

$$\|AB\|^2 \cdot \|BC\|(1-k) + \|AC\|^2 \cdot \|BC\|k - \|AA_1\|^2 \cdot \|BC\| = \|BC\|^3(1-k)k$$

where

$$\|AA_1\|^2 = (1-k)\|AB\|^2 + k\|AC\|^2 - (1-k)k\|BC\|^2$$

similarly,

$$\|BB_1\|^2 = (1-k)\|BC\|^2 + k\|BA\|^2 - (1-k)k\|AC\|^2$$

$$\|CC_1\|^2 = (1-k)\|CA\|^2 + k\|CB\|^2 - (1-k)k\|AB\|^2$$

By adding these three equalities we obtain:

$$\|AA_1\|^2 + \|BB_1\|^2 + \|CC_1\|^2 = (k^2 - k + 1)\left(\|AB\|^2 + \|BC\|^2 + \|CA\|^2\right),$$

which takes the minimum value when $k = \frac{1}{2}$, which is the case when the three lines from the enouncement are the medians of the triangle.

The minimum is $\frac{3}{4}\left(\|AB\|^2 + \|BC\|^2 + \|CA\|^2\right)$.

Open Problems on Smarandache's Ratio Theorem.

1. If the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 of a polygon in a ratio k>0, determine the minimum of the expression:

$$\|A_1A_1'\|^2 + \|A_2A_2'\|^2 + \ldots + \|A_nA_n'\|^2 .$$

2. Similarly question if the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 in the positive ratios k_1, k_2, …, k_n respectively.
3. Generalize this problem for polyhedrons.

References:

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[2] F. Smarandache, *Problèmes avec et sans… problèmes!*, pp. 49 & 54-60, Somipress, Fés, Morocoo, 1983.