Differentiable Structures on Real Grassmannians *

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Abstract

Given a vector space $\mathbb{V}$ of dimension n and a natural number $k < n$, the *grassmannian* $\mathbb{G}_k(\mathbb{V})$ is defined as the set of all subspaces $\mathbb{W} \subset \mathbb{V}$ such that $dim(\mathbb{W}) = k$. In the case of $\mathbb{V} = \mathbb{R}^n$, $\mathbb{G}_k(\mathbb{V})$ is the set of $k-$flats in $\mathbb{R}^n$ and is called *real grassmannian* [1]. Recently the study of these manifolds has found applicability in several areas of mathematics, especially in Modern Differential Geometry and Algebraic Geometry. This work will build two differential structures on the real grassmannian, one of which is obtained as a quotient space of a Lie group [1], [3], [2], [7].

Contents

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1 Introduction

Given a vector space $\mathbb{V}$ of dimension n and a natural number $k < n$, the *grassmannian* $\mathbb{G}_k(\mathbb{V})$ is defined as the set of all subspaces $\mathbb{W} \subset \mathbb{V}$ such that $dim(\mathbb{W}) = k$. In the case of $\mathbb{V} = \mathbb{R}^n$, $\mathbb{G}_k(\mathbb{V})$ is the set of $k-$flats in $\mathbb{R}^n$ and is called *real grassmannian* [1].

In this work we shall construct two differentiable structures on the real grassmannians, where one of them is obtained as quotient space of a Lie group.

Initially we will work basics concepts, namely, the concepts of differentiable maps, differentiable manifolds and Lie group (For more details see [1], [6], [2] or other titles).

We will assume that the reader known differential calculus, group theory, linear algebra, a naive set theory and topology.

In the end of this paper, we will see some results involving grasmannians and Lie groups, and we shall construct the differentiable structures promised on $\mathbb{G}_k(\mathbb{R}^n)$.

2 Differentiable maps

Let $f : \mathbb{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a map of an open in $\mathbb{R}^n$ into $\mathbb{R}^m$. Given $v \in \mathbb{R}^n$, the *directional derivative of f in $a \in \mathbb{X}$ in the direction of v* is given by:

$$\frac{\partial f}{\partial v}(a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} \tag{2.1}$$

for $t \in \mathbb{R}$, if exists this limit.

A map $f : \mathbb{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of the form, $f = (f_1, f_2, \ldots, f_m)$ with $f_i : \mathbb{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. Therefore if f have directional derivative in $a \in \mathbb{X}$, then

$$\frac{\partial f}{\partial v}(a) = \left(\frac{\partial f_1}{\partial v}(a), \frac{\partial f_2}{\partial v}(a), , \cdots, \frac{\partial f_m}{\partial v}(a) \right) \tag{2.2}$$

Let $\mathbb{X} \subset \mathbb{R}^n$ a open and $f : \mathbb{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a map. We saw that the map f *is differentiable in $a \in \mathbb{X}$, with differential $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$* ($T$ is a linear map), if $\forall v \in \mathbb{R}$ with $a + v \in \mathbb{X}$, $f(a+v) = f(a) + T(v) + r(v)$ and $\lim_{v \to 0} \frac{r(v)}{\|v\|} = 0$

Given a base $\{e_1, e_2, \cdots, e_n\}$ for $\mathbb{R}^n$ and $\{\widetilde{e}_1, \widetilde{e}_2, \cdots, \widetilde{e}_m\}$ for $\mathbb{R}^m$. If f is differentiable in a,

$$\frac{\partial f}{\partial e_j}(a) = \lim_{t\to 0} \frac{f(a+te_j)-f(a)}{t} := \frac{\partial f}{\partial x_j} \tag{2.3}$$

and we still

$$\frac{\partial f}{\partial x_j} = T(e_j) = \begin{pmatrix} T_1(e_j) \\ \vdots \\ T_m(e_j) \end{pmatrix}_{m\times 1} \tag{2.4}$$

Theorem 2.1 *Let $f : \mathbb{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $\mathbb{X}$ open in $\mathbb{R}^n$. Are equivalents:*

(*a*) *f is differentiable in $a \in \mathbb{X}$;*

(*b*) *Each f_i is differentiable in $a \in \mathbb{X}$;*

Therefore, the map differentiation $\mathcal{D}$ is such that:

$$\begin{aligned} \mathcal{D}: \quad \mathbb{X} \subset \mathbb{R}^n &\longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \\ a &\longmapsto T_a = f'(a) : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ & \qquad\qquad\qquad\qquad x \longmapsto f'(a)x \end{aligned} \tag{2.5}$$

The matrix $T_a\rfloor_\beta$ that represents the transformation T_a in the base β is called *Jacobian of T_a* and is given by:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}_{m\times n} \tag{2.6}$$

Some basics results.

Theorem 2.2 *$f : \mathbb{X} \subset \mathbb{R}^n$ is differentiable in $a \in \mathbb{X}$ if, and only if, each f_i is differentiable in a.*

Theorem 2.3 *If $f : \mathbb{X} \subset \mathbb{R}^n$ is differentiable in $a \in \mathbb{X}$, then it is continuous in a.*

Theorem 2.4 *If the differential of $f : \mathbb{X} \subset \mathbb{R}^n$ in $a \in \mathbb{X}$ exists, then it is only.*

3 Differentiable manifolds

In this section, we will see the concept of differentiable manifolds, which extends the notion of calculus to more general spaces. The idea is cover a set M by opens in $\mathbb{R}^n$ so that when there intersection between two of the open, the transition can be made smoothly.

Let M a set and $\mathfrak{F}$ a colection of maps one-to-one $x_\alpha : U_\alpha \longrightarrow M$ of opens $U_\alpha \subset \mathbb{R}^n$ into M such that:

(1) $\bigcup_{\alpha \in \Lambda} x_\alpha(U_\alpha) = M$;

(2) For all pair α, $\beta \in \Lambda$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \varnothing$ the sets $x_\alpha^{-1}(\mathbb{W})$ e $x_\beta^{-1}(\mathbb{W})$ are open in $\mathbb{R}^n$ and the maps $x_\beta^{-1} \circ x_\alpha$ are differentiables;

In this case we are saw that $\mathfrak{F}$ is a *differentiable structure on* M and the pair $(M, \mathfrak{F})$ is a *differentiable manifold.*

For simplicity we shall write "M is a differentiable manifold".

Some authors plus the definition with the following axiom: $\mathfrak{F}$ is maximal with respect to the postulates (1) and (2). However it is dispensable because given a differentiable structure $\mathfrak{F}$ on a set M, we can makes it maximal doing

$$\mathfrak{F}' := \{x'_\alpha : U'_\alpha \longrightarrow M; U'_\alpha \in \mathbb{R}^n is\ open\ and\ x'_\alpha\ satisfies\ (2)\ and\ does\ not\ belong\ \mathfrak{F}\},$$

and $\mathfrak{F}'$ is maximal with respect the axioms (1) and (2), and $(M, \mathfrak{F}')$ is a differentiable manifold.

The elements in $\mathfrak{F}$ are called (local) *parameterization* of M. In some moments we will identify the pair (x_α, U_α) as a elemnt in $\mathfrak{F}$.

4 Lie groups

A *Lie group* is a algebraic group that has structure differentiable manifold.

For example, the set $Gl(n, \mathbb{R})$ of the invertible matrices of order n with entries in $\mathbb{R}$, with the usual matrices multiplication is a Lie group.

4.1 Group actions

We saw that a group G *acts on a differentiable manifolds* M if exists a map $\phi : G\times M \longrightarrow M$, given by $(g, m) \longmapsto \phi(g, m)$ such that:

(i) For each $g \in G$ the map $\phi_g : M \longrightarrow M$ is a diffeomorphism, ie, is differentiable, has inverse ϕ^{-1} and ϕ^{-1} is differentiable;

(ii) If $g_1, g_2 \in G$, then $\phi_{g_1 g_2} = \phi_{g_1} \circ \phi_{g_2}$;

4.2 Properly discontinuous action

An action $G \longrightarrow M$ is called *properly discontinuous* if all point $p \in M$ has a neighborhood $U \subset M$ such that $U \cap g(U) = \varnothing, \forall g \in G$.

When G acts so that properly discontinuous in M, the action determines an equivalence relation in M, given by $p \sim q \Leftrightarrow q = gp$ for some $g \in G$.

We shall defined the set M/G as the quocient space of M by the relation $\sim$ above defined. we shall defined too the map

$$\begin{aligned} \pi : \quad & M \longrightarrow M/G \\ & p \longmapsto \pi(p) = [p] = G_p = \{gp; g \in G\} \end{aligned}$$

M/G has an estructure of differential manifold such that π is a local diffeomorphism (when restricted in a neighborhood of the point).

For each $p \in M$, choice a parametrization $x : V \longrightarrow M$ such that $x(V) \subset U$, $U \subset M$ is a neighborhood of p such that $U \cap g(U)$ is empty for all $g \neq 1$.

As $U \cap g(U) = \varnothing$, $\pi|_U$ is injective. Hence $y = \pi \circ x : V \longrightarrow M/G$ too is injective.

The family $\{(V, y)\}$ cover M/G, becouse $\{(V, x)\}$ cover M and π is surjective.

It remains to show that given two maps $y_1 = \pi \circ x_1 : V_1 \longrightarrow M/G$ and $y_2 = \pi \circ x_2 : V_2 \longrightarrow M/G$ with $y_1(V) \cap y_2(V_2) \neq \varnothing$ have that $y_1^{-1} \circ y_2$ is differentiable. Denote by π_i the restriction of π in $x_i(V_i)$, $i = 1, 2$. Take $q \in y_1(V_1) \cap y_2(V_2)$ and let $r = x_2^{-1} \circ_2^{-1} (q)$.

Consider also $W \subset V_2$ a neighborhood of r such that $\pi_2 \circ x_2(W) \subset y_1(V_1) \cap y_2(V_2)$.

Thus restricting for W have

$$y_1^{-1}|_W = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2$$

We shall show that $\pi_1^{-1} \circ \pi_2$ is differentiable in $p_2 = \pi 1(q)$. Let $p_1^{-1} \circ \pi_2(p_2)$, then p_1 and p_2 are equivalents in M. Thus $\exists g \in G$ such that $p_1 = gp_2$.

Note that $\pi_1^{-1} \circ \pi_2|_{x_2(W)}$. Hence $\pi_1^{-1} \circ \pi_2$ is differentiable in p_2.

The construction above ensure that π is a local diffeomorphism ($y = \pi \circ x$).

4.3 Transitive actions and Isotropy

We saw that an action $\phi : G \times M \longrightarrow M$ is *transitive* when for all pair $x, y \in M$ exist $g \in G$ such that $x = gy$.

If G acts on a manifold M, the *isotropy* of G in the point $x \in M$, $Iso_G(x)$ is the set of all $g \in G$ such that $gx = x$.

The proposition below is easily verified.

Proposition 4.1 *$Iso_G(x)$ is a subgroup of G.*

Theorem 4.1 *Let G a group that acts transitively on a differentiable manifold M and $x \in M$. Then $G/Iso(x)$ is a differentiable manifold diffeomorphic to M.*

The last theorem can be found in [1].

5 Differentiable structures on real grassmannians

Given a vector space $\mathbb{V}$ os dimension n finite and a natural number $k < n$, the *grassmannian* $\mathbb{G}_k(\mathbb{R}^n)$ is the set of all subspaces $\mathbb{W} \subseteq \mathbb{V}$ such that $dim\mathbb{W} = k$. If $\mathbb{V} = \mathbb{R}^n$, then $\mathbb{G}_k(\mathbb{R}^n)$ represent the $k-$planes in $\mathbb{R}^n$ contained the origin and is called real grassmannian.

We known that a set of linearly independent vectors define a vector space of dimension equal to the cardinality of that set. Too known that all vector space has base, and it is not unique.

5.1 Construction of local letters and association with $\mathbb{R}^{kn}$

Let $M_{k,n}(\mathbb{R})$ the set of the $(k \times n)$ matrices with real entries and $\mathfrak{F}_k(\mathbb{R})$ the subset of $M_{k,n}(\mathbb{R})$ compossed for matrices whose rows are linearly independent, ie, the set of the $(k \times n)$ matrices whose rows is not linear combination (by real coefficients!) of the other rows, ie, the matrices of rank k.

For each matrix A corresponds a vector space on the field $\mathbb{R}$ os dimension k, ie, a element $\mathbb{W}$ in $\mathbb{G}_k(\mathbb{R}^n)$, namely, the subspace spanned by rows vectors of the matrix A, $span\{A_1, A_2, \cdots, A_k\}$. Of course that two matrices A and B can span be the same element of $\mathbb{G}_k(\mathbb{R}^n)$, but this occurs if, and only if, $A = \Gamma B$ for one $\Gamma \in Gl(n, \mathbb{R})$ a $(k \times k)$ invertible matrix.

This observation motivates the following equivalence relation: we shall saw that X and Y in $\mathfrak{F}_k(\mathbb{R})$ are equivalents $(X \sim Y)$ is exists a invertible matrix $\Gamma \in Gl(n, \mathbb{R})$ such that $X = \Gamma Y$. Of course that $\sim$ makes a partition of the set $\mathfrak{F}_k(\mathbb{R})$. Let $\mathfrak{F} = \mathfrak{F}_k(\mathbb{R})/\sim$ the elements of $\mathfrak{F}$ are equivalences class by the relation $\sim$. Thus, each element of the grassmannian is identified by only one element of $\mathfrak{F}$.

For each ordered set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of k integer such that $1 \leq k \leq n$ and for each element in $\mathfrak{F}$ we defines the maps $\alpha : \mathfrak{F} \longrightarrow M_{k,k}(\mathbb{R})$ and $\alpha^* : \mathfrak{F} \longrightarrow M_{k,n-k}(\mathbb{R})$, where $\alpha(A)$ is the matrix whose $i-$th column is the α_i-th column of A and $\alpha^*(A)$ is the matrix composed (neatly) the columns that do not appear in $\alpha(A)$.

Let U_α the subset of $\mathfrak{F}$, formed by the matrices A for which $\alpha(A)$ is invertible. Is easily that the set U_α is a open set in $\mathfrak{F}$.

Defines the following maps:

$$\begin{aligned} \phi_\alpha : \quad & U_\alpha \longrightarrow \mathbb{G}_k(\mathbb{R}^n) \\ & A \longmapsto span\{A_1, A_2, \cdots, A_k\} \end{aligned}$$

that associate to each matrix A the subspace spanned by himself rows vectors.

We going now to verify that the colection $\{(U_\alpha, \phi_\alpha)\}$ provide a differentiable structure on grassmannian.

Note that, becaus e the partition made by the relation $\sim$ given above, each element in $\mathfrak{F}$ represents one, and only one, point in $\mathbb{G}_k(R^n)$, ie, the map ϕ_α are one-to-one (the reader may wonder if the maps are well defined, ie, if no depends of the class representative for the relao $\sim$, the proof od this fact will be omitted, but does not present great difficulties).

Note that all element in $\mathbb{G}_k(R^n)$ has a base, and thus there is a corresponding matrix in $\mathfrak{F}$, and reciprocally, each element in $\mathfrak{F}$ generates a subspace $k-$dimensional of $\mathbb{R}^n$. Thus

$$\bigcup_{\alpha \in \Lambda} \phi_\alpha(U_\alpha) = \mathbb{G}_k(\mathbb{R}^n)$$

It remains now to see that if two of the applications above, say ϕ_α and ϕ_β, cover the same subset of $\mathbb{G}_k(R^n)$, then its inverse images (in this set) are open and the composition is smooth, ie, $\phi_\beta^{-1} \circ \phi_\alpha$ is differentiable.

Clearly the inverse images of the maps are open, because they are images of the function $\det^{-1}$ that is a continuous function (note that according to the equivalence relation that we did above, $\det^{-1}$ is well defined, ie, is really a function).

We analyze the maps $\phi_\beta^{-1} \circ \phi_\alpha$. We have ϕ_α leads to a matrix A and subspaces in ϕ_β^{-1} subspace that takes a matrix equivalent to the A. In view of the relation $\sim$, the composition is the identity and is therefore differentiable.

The construction above ensures that the grassmannian $\mathbb{G}_k(R^n)$ has a structure of differentiable manifold.

5.2 Quotient of a Lie group

In this subsection we shall identify the real grassmannian as the quotient space of a Lie group by a isotropy.

Let $O(n) := \{A \in Gl(n,\mathbb{R});\ AA^t = I\}$ the set of the $(n \times n)$ orthogonal matrices. Defines the map

$$\begin{aligned} \phi: \quad & O(n) \times \mathbb{G}_k(\mathbb{R}^n) \longrightarrow \mathbb{G}_k(\mathbb{R}^n) \\ & (A, H) \longmapsto AH \end{aligned}$$

that associate each pair (A, H) the subspace spanned by rows os the matrix AH, with H a $(k \times n)$ matrix whose rows are vectors of the base of a point in $\mathbb{G}_k(\mathbb{R}^n)$.

Lemma 5.1 *$O(n)$ is a Lie group.*

Lemma 5.2 *The map ϕ is a transitive action.*

Proof. Fist we shall see that ϕ is a action of G into M. Let $A \in O(n)$, then $\phi_A(H) = AH$ for all matrix H that represents a point in $\mathbb{G}_k(\mathbb{R}^n)$. ϕ_A is defined to multiplication of matrices, that is differentiable. As $O(n)$ is a subgroup of $Gl(n,\mathbb{R})$, the matrix A is invertible hence ϕ_A is too invertible and given by $\phi_A^{-1}(X) = A^{-1}X$, that is too differentiable.

Let A and B two elements in $O(n)$, have that $\phi_{AB}(H) = (AB)H = A(BH) = \phi_A(BH) = \phi_A \circ \phi_B(H)$, showing that ϕ is a actions of G into the real grassmannian.

Now we shall see that ϕ is transitive. Let Π and Γ elements in $\mathbb{G}_k(\mathbb{R}^n)$, takes $\beta_\Pi = \{x_1, \cdots, x_k\}$ a orthonormal base for Π and $\beta_\Gamma = \{y_1, \cdots, y_k\}$ a orthonormal base for Γ. We can to complete β_Π for obtainer a orthonormal base β_Π^* for $\mathbb{R}^n$, of equal form we obtained other base β_Γ^* for $\mathbb{R}^n$. Defines the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $T(x_i) = y_i$. We can see easily the linearity of T and $T(\Pi) = \Gamma$. Let $T\rfloor_{\beta^*_\Pi} = A$ the matricial representation of T in the base β^*_Π, have that $A\Pi = \Gamma$, and the action is transitive action.

Let Π_0 the subspace spanned by vectors $e_1, \cdots, e_k$, the fist k canonical vectors of $\mathbb{R}^n$. It is not difficult to verify that the isotropy of Π_0 is given by

$$Iso(\Pi_0) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right);\ A \in O(k),\ B \in O(n-k) \right\}.$$

that is clearly diffeomorphic to $O(k) \times O(n-k)$.

Therefore the grassmannian $\mathbb{G}_k(\mathbb{R}^n)$ is identify with $O(n)/O(k) \times O(n-k)$. ■

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